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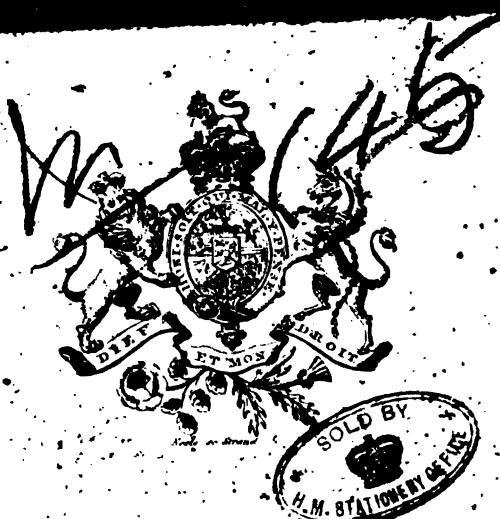
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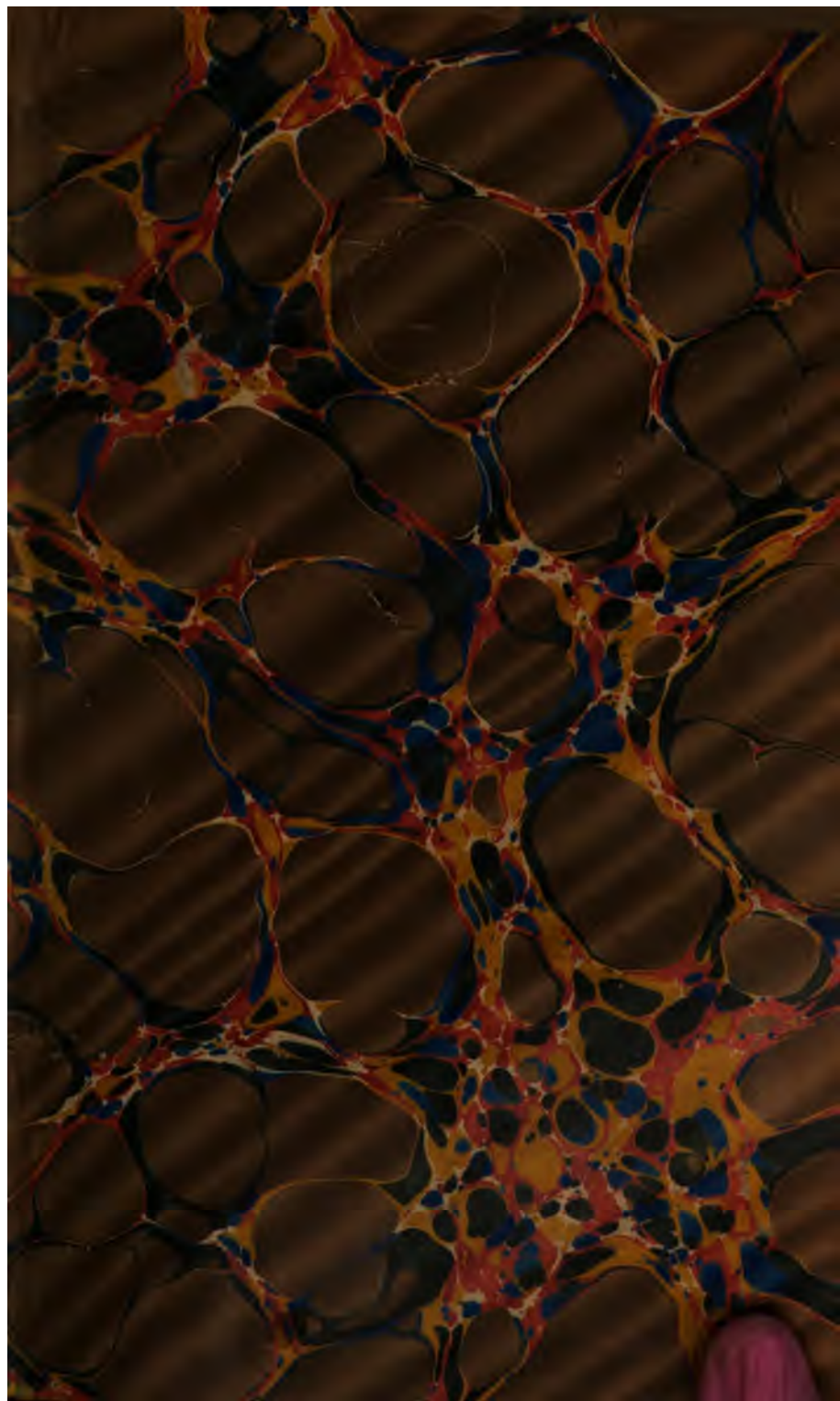
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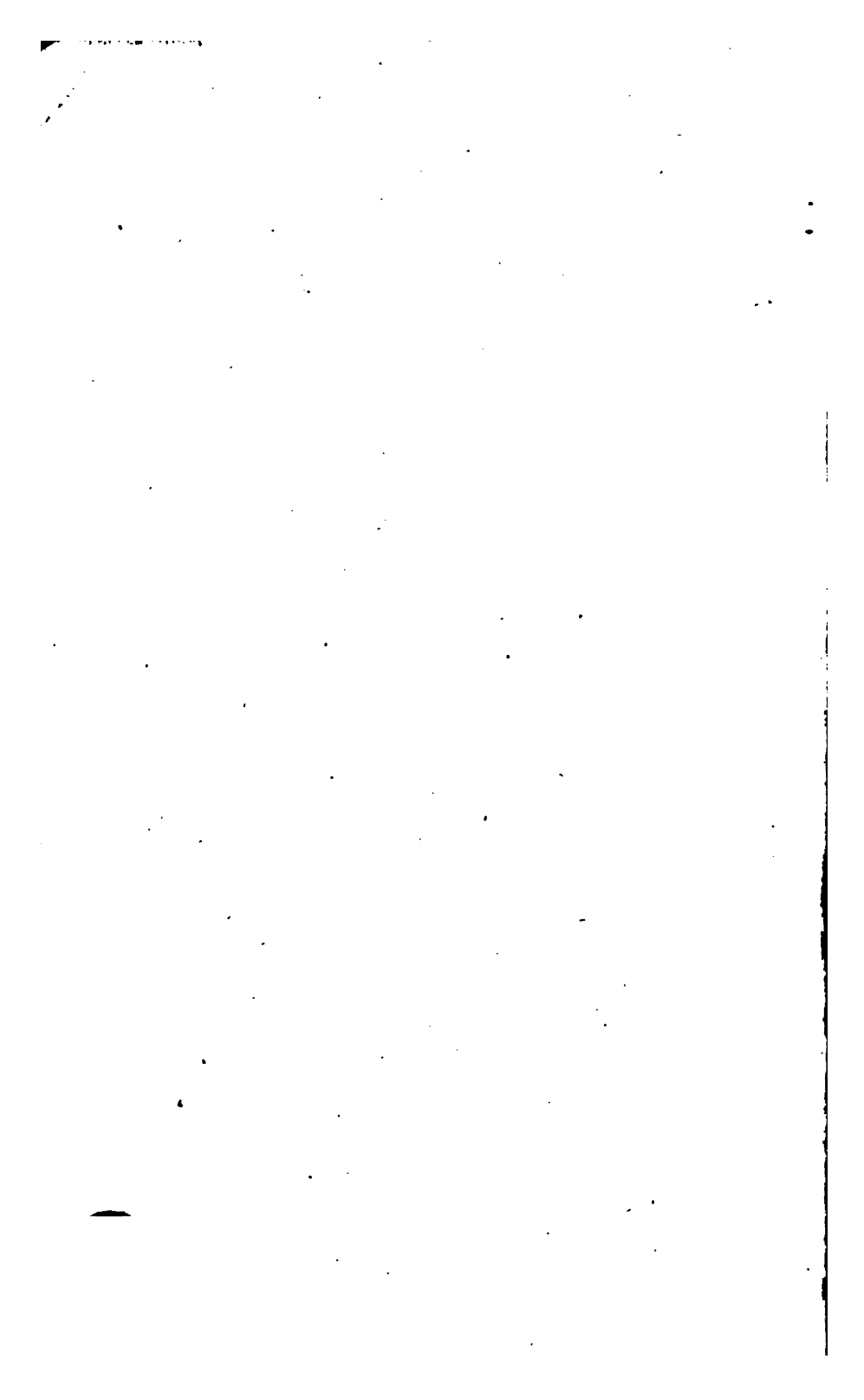
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NEW SERIES
OF THE
MATHEMATICAL
REPOSITORY.

BY THOMAS LEYBOURN,
Of the Royal Military College.

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DIRECTIONS FOR THE BINDER.

I. The Binder is desired to observe that the Vol. consists of different sets of Signatures. The first set A, B, C, D, contains the questions, and is to be placed at the beginning of the Vol. to serve as a table of Contents to the first part.

II. The first, second, and third parts of the Vol. must follow in their order. The two sets of Questions of Vol. VII. (viz. signatures A and B), which are stitched up with numbers forming the Sixth Vol. must be preserved till the Seventh Vol. is completed.

The Cambridge Problems to be placed after the third part : and the Plates to Wildbore's Demonstrations also to be placed at the end of the third part.

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CONTENTS

OF THE

SIXTH VOLUME.

PART I.

MATHEMATICAL QUESTIONS.

	PAGE
Art. I. Solutions to Questions proposed in No. xx. viz. 491 to 510.....	1
II. Solutions to Questions proposed in No. xxi. viz. 511 to 530.....	65
III. Solutions to Questions proposed in No. xxii. viz. 531 to 550.....	155
IV. Obituary. Sir John Leslie	215
V. Solutions to Questions proposed in No. xxiii. viz. 551 to 570	225
Notices relating to Mathematics.....pages 48, 150, 222	
Nugæ.....	151

PART II.

ORIGINAL ESSAYS.

	PAGE.
Art. I. Two Indeterminate Problems. By Mr. Cuxliffe..	1
I. Solution of a Problem in the Gentleman's Diary. By Mr. Barnwell.....	4
III. On Rule Surfaces. By Mr. Davies	8
IV. On Stereographic Projection. By Mr. Davies.....	41
V. Horæ Arithmeticae. By Mr. Horner.....	53
VI. Analytical Solution of the Prize Question. By Mr. Davies	57
VII. Researches in the Geometry of three Dimensions. By Mr. Davies	60
VII.* On Spherical Geometry. By Mr. Davies.....	65
VIII. Propositions in the Lunar Theory. By the Rev. Brice Bronwin	81
IX. On the Attraction of an Oblate Spheroid of Small Ellipticity. By the Rev. B. Bronwin.....	102
X. On the Expansions of Functions. By the Rev. B. Bronwin.....	106
XI. Four Indeterminate Problems. By Mr. John Davy.	113

03-27-40 #187

	PAGE
Art. XII. Solution of a Physical Problem. By Mr. Woolhouse	118
XIII. On the Cycloid. By Lieutenant Colonel Glover..	123
XIV. Some Properties of a plane triangle. By Mr. Cunliffe	124
XV. On Spherical Geometry. By Mr. Davies.....	128
XVI. Two Indeterminate Problems. By Mr. Cunliffe..	141
XVII. Supplement to the Lunar Theory. By the Rev. B. Bronwin	152
XVIII. On the Computation of Superfices and Solids bounded with lines and planes passing through points by means of the co-ordinates to each point. By Mr. Woolhouse	159
XIX. New Researches in Spherical Trigonometry. By Mr. Davies.....	165
XX. On the Deviation between two curves in Answer to a Note of Mr. Woolhouse. By Mr. Davies	189
XXI. On the Solution of a partial differential Equation. By Mr. Sutcliffe	198
XXII. On the figure of a Homogeneous mass of Fluid, revolving round on an Axis with a given uniform velocity. By the Rev. B. Bronwin	195
XXIII. On the Aberlon. By Mr. Davies.....	206
XXIV. On the Power of Machines. By Mr Marrat	209
XXV. Letter to the Editor in Reply to Mr. Davies on the deviation between two Curves. By Mr. Woolhouse	223

PART III.

MATHEMATICAL MEMOIRS,

Extracted from Works of Eminence.

	PAGE
Art. I. Pascal's Concis.....	1
II. Wildbore's Demonstrations of Lawson's Propositions	9
III. On the Theory of Elliptic Transcendents. By Mr. Ivory	26
IV. On Prop. 4, Book 4, of the Mathematical Collections of Pappus Alexandrinus, and other Propositions of a similar nature. By Matthew Stewart	53
V. Certain Properties of plane triangles which are not generally known. By C. F. A. Jacobi.....	68

THE MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XXII.

I. QUESTION 491, *by the Rev. Mr. E. C. TRWON.*

A bent lever of which the arms are a and b and the angle θ makes small oscillations in its own plane; find the time of one vibration?

II. QUESTION 492, *by the Rev. Mr. TYSON.*

Sum n terms of the series

$$1^2 \cdot 2^2 - 2^2 \cdot 3^2 + 3^2 \cdot 4^2 - 4^2 \cdot 5^2 + \&c.$$

III. QUESTION 493, *by the Rev. Mr. TYSON.*

Having given the focal length of a convex lens whereby a person may see distinctly at a given distance, to find the length of another lens, which will enable the same person to see distinctly at another given distance.

IV. QUESTION 494, *by Mr. JAMES CUNLIFFE.*

Suppose a body revolves in a given circle, the centre of force being a given point in the periphery; it is required to compare its velocity, at any given point, with that of a body revolving about the centre of the same, (or an equal) circle.

V. QUESTION 495, *by Mr. CUNLIFFE.*

Find the sum of the infinite series

$$\frac{3^2}{4^2 \cdot 6} + \frac{3^2 \cdot 5^2}{4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5^2 \cdot 7^2}{4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \&c.$$

VI. QUESTION 496, *by Mr. SAMUEL JONES, Liverpool.*

Take any two points in the opposite sides of a given trapezium, and from each of these draw two straight lines to the opposite angles; then the two points where these lines intersect and the point of intersection of the diagonals are in a straight line. Required the demonstration.

VII. QUESTION 497, *by R. N.*

Find the sum of the series

$$1 + 1 + 2 + 3 + 5 + 8 + \&c.$$

any term being the sum of the two preceding terms.

VIII. QUESTION 498, *by R. N.*

At page 84, Whewell's Dynamics, it is stated that, "The semicubical parabola with its axis vertical is a curve, the time of a body's falling down which, can be found." It is required to find the time, if possible, in finite terms.

IX. QUESTION 499, *by Mr. CUNLIFFE.*

Suppose a small body or pellet of lead at a given point, in a perfectly smooth narrow tube, of a given length, and suppose the tube to be whirled about one of its ends as a centre, with a given angular celerity, in a horizontal plane: It is required to determine the velocity and direction of the pellet when it leaves the tube; the motion being solely generated by the revolving of the tube.

X. QUESTION 500, *by R. N.*

The plane of the paper representing the horizon of a place at any time, and the positions of the earth, sun, and moon being given, it is required to find the equation to the straight line which is the intersection to the two planes passing through the moon's centre perpendicular to the lines joining the sun and moon, and the earth and moon.

XI. QUESTION 501, *by Mr. THOMAS STEPHENS DAVIES, Bath.*

Lines drawn from the angles of a triangle to the points of contact of the inscribed circle meet in the same point (G), as is well known: lines drawn to the

points of contact of the circles which touch externally meet in a point (H); and lines drawn from the centres of the circles of external contact to the points where the inscribed touches meet also in a point (K). Prove these two latter, and shew whether the three points G, H, K are in a straight line or not.

XII. QUESTION 502, by Mr. SAMUEL JONES.

If a thin rectangular bar of polished steel of a given length were placed horizontally *in vacuo*, and if a smooth magnetic sphere of a given diameter were placed at a given point on the bar, and then abandoned to the attractive force of the steel it would oscillate thereon perpetually: Find the greatest velocity it would acquire.

XIII. QUESTION 503, by S. S. S.

A country clock-maker being directed to make a pair of cycloidal pendulum cheeks, to a circle whose radius was π inches, made them by mistake to a circle only π inches radius. To remedy this defect he put a rim of brass of equal thickness round the cycloid to make their axes $= 2\pi$ inches, and believes that the cheeks are properly formed. It is required to ascertain the nature of the curve described by the pendulum vibrating between them; and to find the point at which its deviation from a cycloid is greatest.

XIV. QUESTION 504, by Mr. DAVIES.

Planes passing through a given point cut a surface of the second order: Find the loci of the centres and foci of the resulting sections.

XV. QUESTION 505, by S. S. S.

Given the radii of the four spheres of external contact with a tetrahedron, to find the tetrahedron, together with its inscribed and circumscribing spheres.

XVI. QUESTION 506, by S. S. S.

Find the nature of the curve surface (in *Gent. Diary*, No. 1284) generated by the line which joins the moving points, also the locus of the centre of gravity of these points supposing them urged by equal forces.

XVII. QUESTION 507, by Mr. DAVIES.

A series of equal circles indefinitely near to each other and whose centres are in a straight line, are put in perspective: What are the loci of the centres and foci of the representing ellipses.

XVIII. QUESTION 508, by Mr. PETER MASON.

Find the equation to that surface whose property is such, that if the tangent plane at any point cuts off portions of the axes of x , y , and z respectively equal x^2 , y^2 , and z^2 . Then $x^2 + y^2 + z^2$ shall always be equal to the same constant quantity a^2 .

XIX. QUESTION 509, by R. N.

Two bodies, P and Q, are placed on a smooth horizontal plane, and connected by a perfectly flexible inextensible thread which passes freely through a small ring in the plane, a given velocity is communicated to P in a given direction: it is required to find, the equation to the curve described by P on the plane, the length of the thread being indefinite.

XX. PRIZE QUESTION 510, by Mr. DAVIES.

If a polygon be inscribed in a conic section, all whose sides but one are parallel to given lines that side will either be parallel to a given line or touch a given conic section.

This Question for the Triangle was the Prize Question in the *Mathematical Companion* for 1813, but the Solutions there given were Algebraical: it was in the form here given the Prize Question in the *Leeds Correspondent* for July 1822, but the principle of Orthographic Projection there employed cannot properly apply except to the ellipse. It is here proposed to demonstrate the property geometrically without the intervention of any kind of projection or algebraic calculus.

THE MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XXIII.

I. QUESTION 511, by Mr. THOMAS STEPHENS DAVIES, Bath.

In what curve is the normal equal the radius of curvature?

II. QUESTION 512, by MYTHOLOGICUS.

From what point will three given spheres subtend equal conical angles?

III. QUESTION 513, by DELTA SOLIUS.

DG, FG are two lines given in position, and E a given point: what is the locus of the centre of the circle circumscribing the triangle DFG, when the base DF passes through E?

IV. QUESTION 514, by UMBRA.

A given triangle ABC has its base AC resting upon two rectangular directrices AO, CO. Find the locus of its vertex B.

V. QUESTION 515, by UMBRA.

Mr. Ferguson constructed a cylindrical dial, whose axis was vertical, and the stile perpendicular to the axis and in the plane of the meridian. Now, when the cylindrical surface is developed, what is the equation of the extremity of the shadow, supposing the sun retained its mean declination during a whole revolution.

VI. QUESTION 516, by Mr. DAVIES.

Let one of the solid angles of a regular octahedron be cut off by a plane: then the greatest and least edges thus cut off will be the extremes, and the others the means of an harmonical proportion.

VII. QUESTION 517, by Mr. DAVIES.

Two parabolas are given on the same plane, to each of which it is required to draw a tangent, so that they shall be parallel to one another, and the focal radii vectores of the points of contact shall meet in a given line.

VIII. QUESTION 518, from REYNAUD and DUHAMEL *Problèmes et développemens sur diverses parties des Mathématiques.*

"Let ABCDE be any polygon inscribed in a circle, FGHIK a polygon circumscribed whose sides touch in A, B, C, D, E of the inscribed one: if with any radius we describe a new circle concentric with the first, the sum of the squares of the distances of these summits A, B, &c. from any point M of the second circle, multiplied respectively by the tangents KI, KF, FG, GH, HI, will be always the same, that is to say, the sum of the solids, $MA^2 \cdot KI + MB^2 \cdot KF + MC^2 \cdot FG + MD^2 \cdot GH + ME^2 \cdot HI$, is constant."

IX. QUESTION 519, by Mr. DAVIES.

In any conic section let two points A, B be taken, and two tangents AC, BC to meet in C. Likewise two other lines inflected from A, B to any point D in the curve: then any line from C will be harmonically divided by these latter and the curve in E, F, G, H.

X. QUESTION 520, by SPHERIUS.

From any point in one given sphere describe a cone to envelope another given sphere, and find by considerations *purely geometrical*, the curve surface to which the plane of contact is always tangential.

XI. QUESTION 521, by Mr. DAVIES.

Let a, b, c be the three sides of a triangle, and r, r', r'', r''' the radii of the four circles which touch them, then

$$ab + ac + bc = rr' + rr'' + rr''' + r'r'' + r'r''' + r''r'''$$

That is the sum of all the rectangles made by the binary combinations of the sides is equal to the sum of all the rectangles made by the binary combinations of the four radii of the circles of contact.

XII. QUESTION 522, by M. CHAMBERLIN DE MAUGRIS, (tom. IX. Mem. Etranger.)

AO, OR are two planes any how inclined to the horizon intersecting in an angle, OR is an inflexible rod, devoid of gravity, loaded in m with a given body M fixed to it there; and the whole apparatus thus formed, acted on by gravity glides along the planes AO, OR. Determine the velocity of M, including the effect of friction.

XIII. QUESTION 523, by Mr. DAVIES.

Let the opposite sides of a hexagon inscribed in a circle be produced to meet: then of the three angles thus formed the middle one will be equal to both the others. Geometrical and analytical proofs are required.

XIV. QUESTION 524, by Mr. DAVIES.

Let A, B, C, D, E, F be the six summits of a complete quadrilateral, and G, H, K the intersections of its diagonals; from one of the intersections as G, draw GL perpendicular to the third diagonal, meeting it in L: then

$$1^{\text{mo}}. \angle DLB = \angle E \pm F,$$

$$2^{\text{do}}. \angle CLA = \angle E \pm F,$$

3^{do}. The angles DLB, CLA are bisected by GL. Analytical and geometrical demonstrations are required.

XV. QUESTION 525, by R. N.

Prop. 15. Sec. 4. Vol. II. Newton's Principia. A body is supposed to describe the logarithmic spiral in a medium whose resistance varies as the density $\propto$

(velocity)², and centripetal force varies $\frac{1}{(\text{dist.})^2}$. To find the law of resistance

and of the density. It is now required to find the law of resistance and density of the medium, when the body describes any spiral, the centripetal force varying as

$\frac{1}{(\text{dist.})^n}$, and resistance varying as density $\propto$ (velocity)^m.

XVI. QUESTION 526, by R. N.

Art. 340, page 335, vol. II. Playfair's Outlines of Natural Philosophy it is stated, that a circle whose diameter is equal to the whole height of the tide being placed perpendicular to the horizon the lowest point being at the level of low water, the tide will ascend and descend through equal arcs in equal times. Required the proof. The earth is supposed a fluid sphere and disturbed only by the action of the sun and moon.

XVII. QUESTION 527, by R. N.

A fine inextensible and flexible string passes freely through a small hole in an horizontal smooth table, two weights P and Q are attached to the string, Q hanging down, and P being on the horizontal table or plane, a given velocity is communicated to P in a given direction, required the equation of the path it describes on the plane.

XVIII. QUESTION 528, by DELTA SULUS.

Two sides of a triangle inscribed in a conic section pass constantly by the foci; to what curve is the third side always a tangent.

XIX. QUESTION 529, by PARRSONS.

All the angles but one of a Polygon circumscribing a given conic section are upon given lines: what is the locus of the remaining one?

XX. PRIZE QUESTION 530, by Mr. W. S. B. WOOLHOUSE.

AC and AB are two heavy metal and equally uniform bars of given lengths, and freely moveable about C in the plane ACB by means of a joint at C; BC rolls over a small pulley P, while AC moves freely round a pin which is fixed at A in the vertical right line PA, the distance PA being given: being placed in a given position they are allowed to descend by the force of gravity. It is required to investigate the velocities when they arrive at any other position: and also to give a method of computing their position at the instant BC quits the pulley.

THE MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XXIV.

I. QUESTION 531, by N. Y.

If on the segments AB, Ba. of the diameter of a semicircle, the semicircles ASB, asB be constructed, and the ordinate BQ drawn; it is required, first, to inscribe a circle in each of the spaces AQBBA, aQBba; secondly, to shew that these circles are equal; and thirdly, to prove that the least circle circumscribing the two equal circles, is equal to the space included between the arcs of the three semicircles.

II. QUESTION 532, by R. N.

If two sums of money S_1 and S_2 become due respectively at the end of T_1 and T_2 years; the equated time for paying both the sums is found by some authors, on the supposition that the amount of the present values of S_1 and S_2 at the expiration of the equated time must = $S_1 + S_2$, when simple interest is allowed. The rate of interest being (r): It is required to explain the cause of the difference of the results obtained thus by supposing the interest of S_1 for the time it is held after its becoming due, equal to the discount of S_2 for payment before it becomes due.

III. QUESTION 533, by Mr. THOMAS STEPHENS DAVIES, Bath.

If the rectangular ordinates of a conic section be prolonged till they be equal to the normals of the same points, what is the equation of the curve traced out by their extremities?

IV. QUESTION 534, by Mr. W. S. B. WOOLHOUSE, North Shields.*

Four given equal spheres being placed in close contact with each other, it is required to find the volume of the space enclosed between them and the three triangular planes through each three centres.

V. QUESTION 535, by Dr. BROOK TAYLOR.

Let OV, da be parallels cut by a transversal in V, v respectively. From a point O in one of them, (OV), draw four lines meeting the transversal in D, C, B, A and the other parallel in d, c, b, a. Then

$$\frac{CD}{cd} : \frac{AB}{ab} :: CV.VD : AV.VB.$$

VI. QUESTION 536, by Mr. GRIFFITH JONES, Liverpool.

A pendulum composed of three cylindrical rods of unequal lengths SA, SB, SC, lying in the same plane ($\angle ASB = \angle BSC$) is supposed to vibrate about the point S in the plane of the figure. Required the distances of the centres of gravity and oscillation from the point of suspension.

VII. QUESTION 537, by Mr. SAMUEL JONES, Liverpool.

Sum the series

$$\frac{1 \cdot 7}{2^2 \cdot 4^2 \cdot 6} + \frac{3 \cdot 11}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \frac{5^2 \cdot 5 \cdot 15}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \frac{7^2 \cdot 5^2 \cdot 7 \cdot 19}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdot 12} + \&c \text{ ad infinitum in finite terms.}$$

VIII. QUESTION 538, by M:

Let AB be the diameter of a circle, AM always equal to $\frac{1}{2}BM$; MQ, M'Q' perpendicular to AB; let AQ and BQ' (produced if necessary) meet in P: find the equation to the curve traced out by P. Trace its whole extent and shew how all its parts are connected with the circle.

IX. QUESTION 539, by GALILÆUS.

Any conoid being cut by a plane, and the section seen from either focus, it shall appear a circle.

X. QUESTION 540, by TINTORETTA.

If spheres any how scattered in space, be put in perspective, the major axes

* Mr. Woolhouse has in the Press an Elementary Treatise on the Modern Application of the Algebraic Analysis to Geometry.

of the representations all tend to one point in the picture. Determine that point, both by Geometrical and Analytical considerations.

XI. QUESTION 541, by Nemo.

A ring is cut by a plane in a given direction: what is the equation of the section?

XII. QUESTION 542, by Mr. W. S. B. WOOLHOUSE.

Suppose with an organ the natural diatonic octave C, D, E, F, G, A, B, C, have C and D tuned in the ratio of 9 to 8; and that E, F and G are tuned a perfect third major, a perfect fourth, and a perfect fifth respectively from the key note C; and that the intervals of the tetrachord G, A, B, C are made the same as those of C, D, E, F; and that the semitones are taken half way between the successive notes CD, DE, FG, GA, AB. It is required to shew the defects to the diatonic scales of each of the other eleven major keys, viz. G one sharp, D two sharps, A three sharps, E four sharps, B five sharps; F one flat, B $\flat$ two flats, E $\flat$ three flats, A $\flat$ four flats, D $\flat$ five flats, and G $\flat$ six flats.

XIII. QUESTION 543, by Mr. W. S. B. WOOLHOUSE.

Let there be assumed any two opposite hyperbolas; let there also be assumed any ellipse to which any two conjugate semi-diameters are drawn having the same inclination as the asymptotes of the hyperbolas; let this ellipse be placed with its centre any where in one of the branches of the hyperbolas in such a manner that the two said semi-diameters shall be parallel to the asymptotes—if it be intersected by the hyperbolas in four points, shew that the four normals drawn from thence will all of them pass through the same point.

XIV. QUESTION 544, by Mr. DAVIES.

Find the equation of a great circle, which passes through two given points on the surface of the sphere, in terms of the variables lat. and long. Also find the co-ordinates of the poles of that circle.

XV. QUESTION 545, by Mr. W. S. B. WOOLHOUSE.

Suppose a combustible straight line of a given length and a given uniform temperature to be lighted at one extremity and to commence burning by a flame with a given velocity. It is required to determine the time in which it will be totally destroyed, supposing the combustion to be affected by two conditional circumstances;—1st. the velocity and size of the flame when arrived at any point be proportional to the resulting temperature at that point; and 2nd. that the temperature of each point during the burning be continually increasing at a rate proportional to the magnitude of the flame and reciprocally as a given constant value added to the square of its distance.

XVI. QUESTION 546, by Mr. T. S. DAVIES.

PORISM. A circle and a point within it are given, as well as the position of a plane parallel to the picture, then another plane may be determined, in any point of which the eye being placed, the perspective representation of the point shall be the centre of the perspective representation of the circle.

XVII. QUESTION 547, by R. N.

Prove that $\frac{\sin \theta}{1 + 2 \cos \theta} = \frac{1}{4} \cot \frac{\theta}{2} - \frac{3}{4} \cot \frac{3\theta}{2}$.

XVIII. QUESTION 548, by M.

Find the integral of $\frac{ds}{1+s^2}$ between the limits of $s = 0$ and $s = \infty$.

XIX. QUESTION 549, by Mr. T. S. DAVIES.

Two bodies move in the curve of an ellipse in such a manner as to be always at the same distance from each other; it is required to find the nature of the curve to which the straight line joining the bodies is always a tangent.

Note. This question was proposed as the Prize for the last number that was published, of the Gentleman's Mathematical Companion (1827), but there was no answer to it printed.

XX. PRIZE QUESTION 550, by ZEPHYRUS.

A certain curve can be determined from the following property.

Any five points being denoted by $x_1 y_1$, $x_2 y_2$ &c in order, and the abscissas x_1 , x_2 , x_3 , x_4 , and x_5 , x_6 , being respectively in arithmetical progression, then $y_1 - 9y_2 + 16y_3 - 9y_4 + y_5 = 0$.

Solutions to these questions must come to hand (post paid) by the first day of

THE MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XXV.

I. QUESTION 551, by Mr. JOHN BAINES, *Thornhill*.

On what two days of the year in latitude $52^{\circ} 14' 19.6''$ is the sun's altitude $35^{\circ} 39' 32''$ at the middle of the time between sun-rise and noon, disregarding refraction, semidiameter and parallax?

II. QUESTION 552, by Mr. WILLIAM GODWARD, *London*.

Given the vertical angle, the rectangle of the segments of the base by the content of the inscribed circle, and the difference of the segments of the base, by a perpendicular from the vertical angle, to determine the plane triangle.

III. QUESTION 553, by Mr. J. H. SWALE, *Liverpool*.

Theorem. The diameter DE cuts the chord AB perpendicularly at F, and Q is the centre of the circle; describe semicircles upon the segments FD, FE, FQ, and draw the tangents CT, CI, CV from any point C in the periphery of the segment AEB. Then

$$\begin{aligned} 1st. & (AC + CB)^2 = 4CT^2 \\ 2nd. & (AC - CB)^2 = 4CI^2 \\ 3rd. & AC^2 + CB^2 = 4CV^2 \\ 4th. & 4AC \cdot CB = 2CT^2 - 2CV^2. \end{aligned}$$

IV. QUESTION 554, by Mr. JOHN BAINES.

Find the equation of a curve whose tangent varies directly as the absciss.

V. QUESTION 555, by Mr. T. S. DAVIES, *Bath*.

Let M, N, S, R, be the centres of four circles about AED, EBC, ABF DCF (fig. to the solution of question 524) and let L be the point where they intersect: then M, N, S, R, L are in one circumference.

VI. QUESTION 556, by Mr. HUMPHREY CLINKER, *Caer Badon*.

The logarithmic spiral rolls upon the circumference of a given circle, what is the locus of its pole?

VII. QUESTION 557, by Mr. JAMES CONLIVRE, *London*.

Let a, b , and c denote the rational numerical values of the sides of a plane triangle, and let a denote its double area; then will $a^2b^2 - A^2$, $a^2c^2 - A^2$, and $b^2c^2 - A^2$ be all rational squares: required the proof.

VIII. QUESTION 558, by Mr. J. H. SWALE, *Liverpool*.

To determine (geometrically) the position of equilibrium of a uniformly heavy inflexible rod, of a given length, resting on the edge of the brim of a given hemispherical bowl, while the lower end rests against the concave surface of the bowl.

IX. QUESTION 559, by Mr. T. S. DAVIES.

Given the base and radius of the inscribed circle, to find the locus of the vertex of the spherical triangle.

X. QUESTION 560, by Mr. T. S. DAVIES.

Find the equation of the hour-lines on the Horodical Quadrant; see Hutton's Dictionary, art. Quadrant; or Ozanam's Recreations, vol III. p. 313.

XI. QUESTION 561, by the Rev. Mr. BRUCE BROWWIN, *Dorset*.

The arch of a semicircle stands in a vertical plane, with its extremities in the same horizontal line; and convexity downwards. Two material points connected by an inextensible rod without weight are laid on the inside of the arch, one of them at its extremity. In this situation they are left to the action of gravity. Find the difference of the time of their descent through any given arch in this case, and that in which the two points form but one, namely the apparent, the length of the rod being small. Find also the pressure and the tension.

XII. QUESTION 562, by the Rev. Mr. BROWWIN.

Suppose the arch of a semicircle placed in a vertical plane, extremities parallel to the horizon, and convexity downwards. A bar of heavy metal of uniform

thickness is laid on the inside of the curve, one end at the top of the semicircle and descends along the arch by its weight. What is the difference between the time of descent through a given arch, and that of a material point descending down the same, the length of the bar being small? Required also the pressures at each end of the rod?

XIII. QUESTION 563, by Mr. W. S. B. WOOLHOUSE, London.

Find the equation to the evolute surface of any proposed loxodromic curve. And show that if any number of loxodromics having different meridional inclinations, be cut by a parallel of latitude that the centres of absolute curvature at the points where they are intersected will all of them range in the surface of a sphere.

XIV. QUESTION 564, by Mr. WOOLHOUSE.

A curve whose equation is $f(x, y) = 0$, is successively modified in its figure by the simultaneous motion of all its points. At each instant the points move all in directions perpendicular to their curve and the motions resolved parallel to a given line are all the same. It is required to determine the general equation for the curve through all its changes and also that of the curve which each point describes.

XV. QUESTION 565, by Mr. WILLIAM GODWARD.

The six edges of any irregular tetrahedron are opposed two by two, and the nearest distance of two opposite edges is called breadth; so that the tetrahedron has three breadths and four heights. It is required to demonstrate, independently of the geometry of co-ordinates, that in every tetrahedron, the sum of the reciprocals of the squares of the breadths, is equal to the sum of the reciprocals of the squares of the heights. (See Ladies' Diary, Question 1523, and Young's Differential Calculus, page 235.)

XVI. QUESTION 566, by Mr. W. S. B. WOOLHOUSE.

A heavy metal rod AB is suspended at C by a string SC connected to a fixed point S and is made to describe a double conic surface round an unknown point P in the vertical right line SR. There are given $SC = s$, $AG = GB = a$, $GC = b$, and the angular velocity round SR $= \omega$; to find the position of P and the inclination of the rod. Also $AG = GB$ and GC being given as above, there are particular determinate values of SC and GP which are required, being such that if the angular velocity be varied at pleasure, the point P will constantly be found in the vertical right line SR; that is P will always be the centre of rotation or common vertex to the two conic surfaces described by the rod.

XVII. QUESTION 567, by Mr. W. S. B. WOOLHOUSE.

Given $(a^2 - x^2) \left(y \cdot \frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} \right) - \frac{y}{x} (y^2 - 2x^2) \frac{dy}{dx} + \frac{y^4}{4x^3} - \frac{9x^2}{4} + 2y^2 - 3x^2 = 0$, to find y in terms of x without series; supposing $y = 0$ when $x = 0$.

XVIII. QUESTION 568, by the Rev. Mr. BRICE BRONWIN, Denby, near Wakefield.

Investigate the expression of LAGRANGE for the variations of the elements a, e, π of a disturbed orbit, considered as an ellipse; and make use of the very same principle and method as those by which NEWTON determined the regression and variation of the inclination.

XIX. QUESTION 569, by Mr. T. S. DAVIES.

Take a globe in any oblique position with respect to the horizon, and π -sect all its semi-diurnal arcs. Now if any parallels of equal declination D, D' be drawn, and great-circle chords be drawn through the corresponding sections of D, D' : Then as π takes consecutive values, the intersection of the resulting consecutive great-circle chords, will trace out upon the surface of the sphere a curve which it is required to determine.

XX. PRIZE QUESTION 570, by Mr. T. S. DAVIES, Bath.

If the loxodrome were detached from its fixed position and made to roll upon one of the circles of the sphere, its pole would trace out a particular kind of epicycloid. It is required to find the equations of this epicycloid—to ascertain whether in any case it can be algebraical, quadrable, or rectifiable—and to assign the conditions which must be fulfilled by any class of loci, which employed as directrices instead of the circle, shall confer upon the epicycloid the properties above mentioned either singly or simultaneously.

Solutions to these questions must come to hand (post paid) by the first day of December, 1832.

THE MATHEMATICAL REPOSITORY,

VOL. VI. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

Solutions to Questions proposed in No. XX.

1. QUESTION 491, by the Rev. E. C. TYSON.

A bent lever of which the arms are a and b , and the angle θ , makes small oscillations in its own plane; find the time of one vibration.

FIRST SOLUTION, by — MASON, of Scoulton, near Watton, Norfolk.

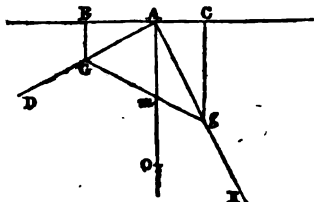
Let ϕ be the angle the arm (a) makes with the horizon, $\therefore \pi - \theta - \phi$ = the angle the arm (b) makes with it: hence, considering the weights of the arms as their lengths, we have, by the property of the lever, $a^2 \cdot \cos \phi + b^2 \cdot \cos (\theta + \phi) = 0 \dots \dots (1)$. Also $a^2 \cdot \sin \phi + b^2 \cdot \sin (\theta + \phi) = 2$, moment of the lever $\dots \dots (2)$. We now square equations (1) and (2) and add the results which give $a^4 + 2a^2b^2(\cos \phi \cos (\theta + \phi) + \sin \phi \sin (\theta + \phi)) + b^4 = 4(\text{moment})^2$, or $\sqrt{(a^4 + 2a^2b^2 \cos \theta + b^4)} = 2 \cdot \text{moment of the lever estimated in a vertical line}$. Also $\frac{1}{2}(a^2 + b^2) = \text{momentum of inertia of the same}$, and $\therefore$ the length of the simple pendulum which will vibrate in the same

time as this lever is $\frac{2}{3} \cdot \frac{a^3 + b^3}{\sqrt{(a^4 + 2a^2b^2 \cos \theta + b^4)}}$. Hence the

time of oscillation = $2\pi \sqrt{\left\{ \frac{2}{3g} \cdot \frac{a^3 + b^3}{\sqrt{(a^4 + 2a^2b^2 \cos \theta + b^4)}} \right\}}$.

SECOND SOLUTION, by the Rev. Mr. E. C. TYSON, London.

Let DAB be the bent lever of uniform thickness, $AE = a$, $AD = b$, and $\angle DAE = \theta$. Bisect AD , AE in G , g which will be the centres of gravity of the arms. Join Gg and draw Am perpendicular to the horizontal line BC and cutting Gg in m which will be the centre of gravity of the system.



Suppose o the centre of oscillation.

$$\text{tion. Then } AO = \frac{\frac{2}{3} \cdot b \cdot \frac{b}{2} + b \cdot \frac{2}{3} \cdot a \cdot \frac{a}{2}}{(a+b) \cdot Am} = \frac{1}{3} \cdot \frac{a^2 + b^2}{(a+b) \cdot Am}$$

Now suppose the lever to be at rest, then $b \cdot AB = a \cdot AC$, or if $x = \angle GAB$, we have $b^2 \cdot \cos x = a^2 \cdot \cos \{x - (x + \theta)\}$. from this equation

$$\tan x = \frac{b^2 + a^2 \cdot \cos \theta}{a^2 \cdot \sin \theta} \text{ and } \sin x = \frac{b^2 + a^2 \cdot \cos \theta}{\sqrt{(a^2 + b^2 + 2a^2b^2 \cdot \cos \theta)}} = \sin GAB.$$

$$\text{By a similar process, } \sin \angle GAC = \frac{a^2 + b^2 \cdot \cos \theta}{\sqrt{(a^2 + b^2 + 2a^2b^2 \cdot \cos \theta)}}.$$

$$\text{Now } GB = GA \cdot \sin GAB = \frac{b}{2} \cdot \frac{b^2 + a^2 \cos \theta}{\sqrt{(a^2 + b^2 + 2a^2b^2 \cdot \cos \theta)}}.$$

$$\text{Similarly } gC = gA \cdot \sin GAC = \frac{a}{2} \cdot \frac{a^2 + b^2 \cdot \cos \theta}{\sqrt{(a^2 + b^2 + 2a^2b^2 \cdot \cos \theta)}}.$$

Hence

$$AO = \frac{1}{3} \cdot \frac{a^2 + b^2}{(a+b)Am} = \frac{1}{3} \cdot \frac{a^2 + b^2}{a \cdot gC + b \cdot GB} = \frac{2}{3} \cdot \frac{a^2 + b^2}{\sqrt{(a^2 + b^2 + 2a^2b^2 \cdot \cos \theta)}}.$$

Now the time of oscillation

$$= \pi \cdot \sqrt{\frac{L}{mg}} = \pi \cdot \left\{ \frac{2(a^2 + b^2)}{3mg} \right\}^{\frac{1}{2}} \cdot \frac{1}{\{a^2 + b^2 + 2a^2b^2 \cdot \cos \theta\}^{\frac{1}{4}}} \text{ the time of one oscillation.}$$

Ingenuous Solutions were also received from Messrs. BAINES, DAVIES, and GODWARD.

II. QUESTION 492, by the Rev. Mr. TYSON.

Sum n terms of the series

$$1^2 \cdot 2^2 - 2^2 \cdot 3^2 + 3^2 \cdot 4^2 - 4^2 \cdot 5^2 + \&c.$$

FIRST SOLUTION, by ~~Wm~~ MASON, of Scoulton.

The increment of the sum of x terms is $(-1)^x \cdot (x+1)^2 \cdot (x+2)^2$, the integral of which or the sum of x terms is

$$\frac{(-1)^x}{(-2)} \cdot (x+1 \cdot x+2)^2 - \frac{(-1)^{x+1}}{(-2)^2} \Delta (x+1 \cdot x+2)^2 +$$

$$\frac{(-1)^{x+2}}{(-2)^3} \Delta^2 (x+1 \cdot x+2)^2 - \frac{(-1)^{x+3}}{(-2)^4} \Delta^3 (x+1 \cdot x+2)^2$$

$$+ \frac{(-1)^{x+4}}{(-2)^5} \Delta^4 (x+1 \cdot x+2)^2 + C = \frac{(-1)^x}{4} \times$$

$$\left\{ -2(x \cdot x+2)^2 + 1 \right\} + C; \text{ whence } C = -\frac{1}{4}, \text{ the sum is}$$

$$\frac{(-1)^x}{4} \left\{ 2(x \cdot x+2)^2 + 1 \right\} - \frac{1}{4}.$$

SECOND SOLUTION, by Mr. BAINES, Thornhill, near Wakefield.

First $1^2 \cdot 2^2 - 2^2 \cdot 3^2 + 3^2 \cdot 4^2 - 4^2 \cdot 5^2 + \&c. = 4(1^4 - 3^4 + 6^4 - 10^4 + 15^4 - 21^4 + \&c.)$. When the number of terms is odd, the n^{th} term is $= n^2 (n+1)^2$, and when n is even, the series may, by taking the differences of each two adjacent terms, be written thus, $-4(8 + 64 + 216 + 512 + 1000 + \&c.)$ in which the number of terms within the parenthesis $= \frac{1}{2}n$. Hence by the common differential method,

8, 64, 216, 512, 1000, &c.
 56, 152, 296, 488.
 96, 144, 192,
 48, 48

Putting $a = 1$, the first term, and 56, 96, 48, the first terms of the several orders of differences, $= d', d'',$ and d''' respectively, we have

$na + n \cdot \frac{n-1}{2} \cdot d' + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot d'' + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot d''' = 8n + 28n(n-1) + 16n(n-1) \cdot (n-2) + 2n(n-1) \cdot (n-2) \cdot (n-3) = 2n^4 + 4n^3 + 2n^2$, in which writing $\frac{n}{2}$ for n , and multiplying the result by -4 , we have

$$-\frac{n^2}{2} (n^2 + 4n + 4) = -\frac{n^2}{2} (n+2)^2 = \text{the sum of } n \text{ terms}$$

when n is even.—Again, for the sum when n is odd; if we write $n-1$ for n in the above expression, and add the n^{th}

term, viz. $n^2(n+1)^2$, the result will obviously be the sum in this case, that is

$-\frac{1}{2}(n-1)^2 \cdot (n+1)^2 + n^2(n+1)^2 = \frac{1}{2}(n^4-1) + 2n^2(n+1) =$ the sum when n is odd.

Other Solutions were received from Messrs. DAVIES, GODWARD, THOMPSON, and TYSON.

III. QUESTION 493, by the Rev. Mr. TYSON.

Having given the focal length of a convex lens whereby a person may see distinctly at a given distance, to find the focal length of another lens, which will enable the same person to see distinctly at another given distance.

FIRST SOLUTION, by — MASON, of Scoulton.

Let D = the distance the person can see distinctly with the naked eye, D_1 the first given distance, D_2 the second, and f the given focal length of the convex lens, x the focal length of the required lens. Now the rays proceeding from an object at the distance D_1 and passing through the first lens enter the eye as though they proceeded from the distance D . Therefore,

by Optics, we have $\frac{1}{D} = -\frac{1}{f} + \frac{1}{D_1}$ and for second lens $\frac{1}{D} =$

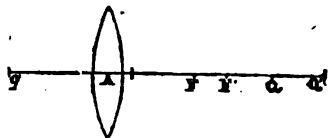
$\frac{1}{x} + \frac{1}{D_2}$; hence $\frac{1}{x} = -\frac{1}{f} + \frac{1}{D_1} - \frac{1}{D_2}$ and $x =$

$\frac{D_1 D_2 f}{(D_2 - D_1)f - D_1 D_2}$. If D_1 and D_2 be such that the value of x is negative the lens must be convex, and if it be positive it must be a concave lens.

SECOND SOLUTION, by Mr. GODWARD, Wakefield.

Let A be the lens with which a person can see distinctly at the distance AQ , and AF its focal length; also let Af' be the focal length of that lens with which the same person can see distinctly at the distance AQ' .

Then it is manifest that the rays diverging from Q and Q' and incident upon the lenses must, after refraction in each case, converge to the same point q .

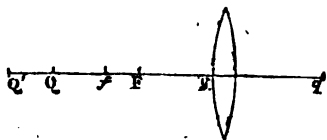


Now by Art. 183, Wood's Optics

we have Aq equal to each side of the equation $\frac{AQ \cdot AF}{AQ - AF} = \frac{AQ' \cdot AF'}{AQ' - AF'}$ which gives $AF' = \frac{AQ \cdot AF \cdot AQ'}{AQ(AF + AQ') - AQ' \cdot AF}$ the focal length required.

THIRD SOLUTION, by the Rev. Mr. TYSON, London.

Suppose the lens to be placed at E , and let QE be the distance at which he can view objects by means of the lens, whose focal length EF is given, and $Q'E$ the distance at which he wishes to view objects by means of the lens whose focal length is required. Then since he wishes to view objects with both lenses, rays diverging from Q' , Q must be made to converge to the same point q .



Let $QE = a$, $Q'E = b$, $EF = f$; $Ef = x$.

Then $QF : QE :: QE : Qq = \frac{a^2}{a-f}$, Wood's Optics, prop. 45.

Also $Q'f : Q'E :: Q'E : Q'q = \frac{b^2}{b-x}$.

And since $QF : QE :: QE : Qq$, $\therefore FE : QE :: Eq : Qq$

also $\therefore Q'f : Q'E :: Q'E : Q'q$, $\therefore Q'E : Ef :: Q'q : Eq$.

Also $QE : Q'E :: a : b$;

Therefore $FE : Ef :: \frac{ab^2}{b-x} : \frac{a^2b}{a-f}$, or $f \cdot x :: \frac{b}{b-x} : \frac{a}{a-f}$;

hence $x = \frac{abf}{ab + af - bf}$ the focal length required.

Messrs. BAINES and DAVIES answered this question.

IV. QUESTION 494, by Mr. JAMES CUNLIFFE.

Suppose a body revolves in a given circle, the centre of force being a given point in the periphery; it is required to compare its velocity, at any given point, with that of a body revolving about the centre of the same, (or an equal) circle.

FIRST SOLUTION, by Mr. GODWARD, Wakefield.

Let the force at P in direction PC = m , and deflect $SP' = CP$; then the force at P in direction P's is also = m . By Cor. 1. Prop. 7. B. I. Newton's

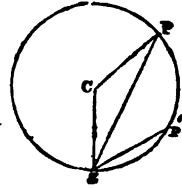
Principia, $SP^3 : SP'^3 :: m : m \left(\frac{SP'}{SP} \right)^3 =$

force at P in direction PS; also by Whewell's Dynamics, Page 29, Note, the vel.² of P

about S = $2m \left(\frac{SP'}{SP} \right)^3 \times \frac{SP}{4} = \frac{m}{2} \cdot \frac{CP^3}{SP^4}$,

and vel.² of P about C = $m \cdot CP$. Hence vel.² of P about S : vel.² of P about C :: $\frac{m}{2} \cdot \frac{CP^3}{SP^4} : m \cdot CP :: C^4 : 2SP^4$, or vel.

of P about S : vel. of P about C :: $CP^2 : SP^2 \sqrt{2}$.

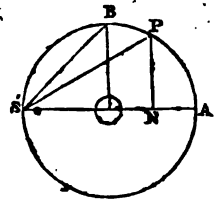


SECOND SOLUTION, by — MASON, of Scoulton.

Let S be the centre of force in the circumference of the circle APB, P the place of the body at any time, (m) the accelerating force at the distance which is considered the unit: hence

$\frac{m}{SP^3} =$ force on body at P, and, since the

velocity is always equal to that which a body would acquire in falling down $\frac{1}{2}$ of chord of curvature with the force continued



constant, we have velocity at P = $\sqrt{\frac{2m}{SP^3} \cdot \frac{SP}{4}} = \frac{\sqrt{m}}{SP^2 \cdot \sqrt{2}}$;

and if a body describe a circle rad. = $\frac{1}{2}AS$ or = SO round the

same centre of force the velocity must = $\sqrt{\frac{2m}{SO^3} \cdot \frac{SO}{2}} = \frac{\sqrt{m}}{SO^2}$.

Hence veloc. at P . veloc. in circle rad so :: $\frac{1}{SP^2 \cdot \sqrt{2}} : \frac{1}{SO^2}$

:: $\frac{SO^2}{\sqrt{2}} : SP^2 :: \frac{SO^2}{\sqrt{2}} : 2SO \cdot SN :: \frac{SO}{\sqrt{2}} : 2SN :: SB : 4SN$,

if OB and NP be perpendicular to the diameter SA and SB be joined.

Answers were also given by Messrs. BAINES and DAVIES,

V. QUESTION 495. by Mr. CUNLIFFE.

Find the sum of the infinite series

$$\frac{3^2}{4^2 \cdot 6} + \frac{3^2 \cdot 5^2}{4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5^2 \cdot 7^2}{4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \&c.$$

FIRST SOLUTION, by — MASON, of Scoulton.

The expansion of $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{3 \cdot x^4}{2 \cdot 4} + \frac{3 \cdot 5 \cdot x^6}{2 \cdot 4 \cdot 6} + \&c.$ to infinity. Multiply by dx and integrate, and $\sin^{-1} x = x + \frac{x^3}{2 \cdot 3} + \frac{3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \&c.$ to infin.

Multiply this by $\frac{x dx}{\sqrt{1-x^2}}$ and take the integrals at both sides from $x=0$ to $x=1$ and we have

$$1 = \frac{1}{2} \left(\frac{\pi}{2} \right) + \frac{1}{2^2 \cdot 4} \left(\frac{\pi}{2} \right) + \frac{3^2}{2^2 \cdot 4^2 \cdot 6} \left(\frac{\pi}{2} \right) + \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \left(\frac{\pi}{2} \right) + \&c. \text{ to infinity:}$$

$$\text{Hence } \frac{8}{\pi} - 2\frac{1}{2} = \frac{3^2}{4^2 \cdot 6} + \frac{3^2 \cdot 5^2}{4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5^2 \cdot 7^2}{4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \&c.$$

where $\pi = 3.1415926 \&c.$

SECOND SOLUTION, by Mr. GODWARD.

$$\text{Put } s = \frac{3^2}{4^2 \cdot 6} + \frac{3^2 \cdot 5^2}{4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5^2 \cdot 7^2}{4^2 \cdot 6^2 \cdot 8^2 \cdot 10} +$$

$$\text{Then } \frac{s}{2^2} = \frac{3^2}{2^2 \cdot 4^2 \cdot 6} + \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} +$$

$$= \frac{2}{\pi} - \frac{1}{2} - \frac{1}{2^2 \cdot 4} \text{ (Simpson's Fluxions, art. 435);}$$

$$\text{hence } s = \frac{8}{\pi} - 2\frac{1}{2}, \text{ as required.}$$

THIRD SOLUTION, by Mr. THOS. THOMPSON, Newcastle-upon-Tyne.

$$-\sqrt{1-x^2} = -1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{2x^6}{2 \cdot 4 \cdot 6} + \dots \text{transpose}$$

$-1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4}$, then multiply by $\frac{x}{x^2 \sqrt{1-x^2}}$ and we have
 $\frac{-x}{x^2} + \frac{x}{x^2 \sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \left\{ \frac{1}{2} + \frac{x^2}{2 \cdot 4} \right\} = \frac{3x^3}{2 \cdot 4 \cdot 6 \sqrt{1-x^2}}$
 $+ \frac{3 \cdot 5x^5}{2 \cdot 4 \cdot 6 \sqrt{1-x^2}}$ &c. but between the limits $x = 0$ and
 $x = 1$, $\int \frac{-x}{x^2} + \int \frac{x}{x^2 \sqrt{1-x^2}}$ is $= 1$, and $\int \frac{x^{2n}}{\sqrt{1-x^2}} =$
 $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \int \frac{x}{\sqrt{1-x^2}}$; therefore taking the fluents
between those limits we have $1 - A \left\{ \frac{1}{2} + \frac{1}{2^3 \cdot 4} \right\} =$
 $A \left\{ \frac{3^2}{2^2 \cdot 4^2 \cdot 6} + \frac{3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \right.$ &c. $\left. \right\}$ where $A =$ quadrantal
arc to radius 1, therefore the required sum is $\frac{4}{A} - (2 + \frac{1}{4})$.

FOURTH SOLUTION, by Mr. JOHN BAINES, *Thornhill*.

If a and b represent the semi-axes of an ellipse, and
 $1 - \frac{b^2}{a^2}$ be put $= x^2$, and $\pi = 3.141592$, it is well known that

$\frac{\pi x}{2} (1 - \frac{x^2}{2} - \frac{3x^4}{2^2 \cdot 4^2} - \frac{3^2 \cdot 5x^6}{2^2 \cdot 4^2 \cdot 6^2} - \frac{3^2 \cdot 5^2 \cdot 7x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \&c.)$
 $=$ the length of the quadrantal arc of the ellipse. Now, if
 $b = 0$, and $a = 1$, it is evident not only that $x^2 = 1$, but also
that $a = x$; also in this case, the curve obviously becomes a
straight line, coinciding with, and equal to the semi-axis major,

or 1. Hence $\frac{\pi x}{2} (1 - \frac{x^2}{2} - \frac{3 \cdot x^4}{2^2 \cdot 4^2} - \frac{3^2 \cdot 5x^6}{2^2 \cdot 4^2 \cdot 6^2} - \&c.) = 1$,

or $1 - \frac{x^2}{2} - \frac{3x^4}{2^2 \cdot 4^2} - \frac{3^2 \cdot 5x^6}{2^2 \cdot 4^2 \cdot 6^2} - \&c. = \frac{2}{\pi x}$. Divide by
 x , differentiate, and change the signs, then

$d\dot{x} \left(\frac{1}{x^2} + \frac{1}{2^2} + \frac{3^2 x^2}{2^2 \cdot 4^2} + \frac{3^2 \cdot 5^2 x^4}{2^2 \cdot 4^2 \cdot 6^2} + \&c. \right) = \frac{2dx}{\pi x^2}$;
multiply by $4x^3$, and integrate, then

$2x^3 + \frac{x^4}{4} + \frac{3^2 x^5}{4^2 \cdot 6} + \frac{3^2 \cdot 5^2 x^7}{4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5^2 \cdot 7^2 x^{10}}{4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \&c. =$

$\frac{8}{5}$. If x be now taken $= 1$, and the two first terms transposed, we shall have $\frac{8^2}{4^1 \cdot 6} + \frac{8^2 \cdot 6^2}{4^2 \cdot 6^2 \cdot 8} + \frac{8^2 \cdot 6^2 \cdot 7^2}{4^3 \cdot 6^3 \cdot 8^2 \cdot 10} + \text{&c.} = \frac{8}{5} - \frac{2}{4} = .9647909 = \text{the sum of the infinite series required.}$

Mr. DAVIES answered this Question.

VI. QUESTION 496, by Mr. SAMUEL JONES, Liverpool.

Take any two points in the opposite sides of a given trapezium; and from each of these draw two straight lines to the opposite angles; then, the points where these lines intersect and the point of intersection of the diagonals are in a straight line. Required the demonstration.

FIRST SOLUTION, by Mr. SAMUEL JONES.

Let $ABCD$ be the given trapezium, P and Q the assumed points, and S the intersection of its diagonals; join $C, P; B, Q$; intersecting in T ; and $D, P; A, Q$; intersecting in R ; then R, S , and T , are in a straight line.

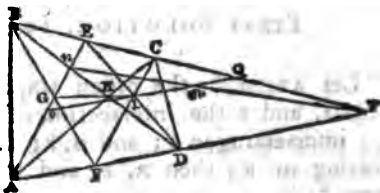
For, draw EH parallel to GP meeting AO in H ; and HI parallel to PC meeting BD in I ; also, TF parallel to AQ meeting AO in F ; and FE parallel to BA meeting BD in E ; and join $D, T; E, I; E, S$; and S, T ; then, it is manifest, from the nature of parallel lines, that the trapezia $AHIE, FCDE$, are equiangular, therefore $AH : HI :: EC : CD$; and for the same reason the triangles AHE, FOT , are equiangular, therefore $AH : HE :: FC : CT$; whence $HI : HE :: CD : CT$, and since $\angle EHI = \angle TCD$, it follows that the $\Delta s HIE, ECT$, are similar and DT is parallel to EI , hence $\angle RIS = \angle TDS$; again, $DT : EI :: DC : HI :: DS : IS$, alternately $DT : DS :: EI : IS$, wherefore the $\Delta s TDS, RIS$, are similar and $\angle DST = \angle RSI$: but DSI is a straight line, consequently RST is a straight line. *Q. E. D.*

This remarkable proposition occurred to me while studying the 40th Question in the *United States Almanac*, as proposed

by Mr. J. H. Swale, and previously to the publication of the 20th No. of the *Repository*, I was not aware that it had been noticed by any person whatever. On its being proposed in the *Repository* I was, however, immediately told that it was an old question, and had already appeared in the *Leeds Correspondent*. Conscious of not having borrowed from any one, I was extremely desirous of seeing its solution, and having procured the last number of that work, I find it is a deduction made from a question proposed therein by Mr. J. H. S. which question is, in fact, the same as that from which I deduced it, though in another work; and since its solution has not previously appeared, I claim the indulgence of laying the preceding one before the readers of the *Repository*.

SECOND SOLUTION, by Mr. JOHN BAINES.

Let ABCD represent the given trapezium, two of its sides, viz. BC, AD produced, meeting in P. Take any two points E, F in the opposite sides BC, AD, and let AE, AF, FC, BF, ED and FD be joined, then G, H and I being the points of concurrence of AE, BF; AC, BD; and FC, ED, we have only



to prove that these three points are in the same straight line. Through I draw IQ parallel to AP , intersecting AP (drawn from the intersection n of AE , BD to P ;) in m ; also draw the line mho intersecting AE in o ; then om is either parallel or not parallel to AP . If it be not parallel, conceive another line mw (not expressed in the fig.) parallel to AP , intersecting GI , joined, in t , and BD in u . By parallels $AD : DP :: wu : um$, and $oi : IQ :: wt : tm$; but $AD : DP :: wt : tm$. This analogy, it is very obvious, can only take place when the points t and u coincide with H , or when the straight line mw coalesces with am ; om is therefore parallel to AP . Hence $oi : IQ :: AD : DP :: oh : hm$, and consequently ih tends to the point G , or the three points G , H and I are in the same straight line.

Thus nearly the question was answered by Mr. GODWARD.

THIRD SOLUTION, by Mr. DAVIES.

Let CD, BA be taken for axes of co-ordinates, and their intersection O as origin. Denote the co-ordinates of A, B, C, D by $\alpha, \beta, \gamma, \delta$ and those of P, Q by x, y , respectively. Then the equations of the lines of the figure are as follow.



The line AC gives $\alpha y + \gamma x = \alpha\gamma$.. (1)
 BD .. $\beta y + \delta x = \alpha\beta\delta$.. (2)
 AQ .. $\alpha y + y, x = \alpha y, \delta$.. (3)
 DP .. $x, y + \delta x = 2x, \delta$.. (4)
 BQ .. $\delta y + y, x = \alpha\beta y, \delta$.. (5)
 CP .. $x, y + \gamma x = 2x, \gamma$.. (6)

Put x, y, x, y for the co-ordinates of the points $\alpha, \beta, \gamma, \delta$. Then taking the above equations in pairs in order, viz. (1, 2), (3, 4), (5, 6), we obtain

$$x, = \frac{\alpha\beta(\gamma - \delta)}{\gamma\beta - \alpha\delta}; y, = \frac{\alpha\gamma(\beta - \alpha)}{\beta\gamma - \alpha\delta} \dots (7)$$

$$x, = \frac{\alpha\alpha x, (y, - \delta)}{x, y, - \alpha\delta}; y, = \frac{2\delta y, (x, - \alpha)}{x, y, - \alpha\delta} \dots (8)$$

$$x, = \frac{\alpha\beta x, (y, - \gamma)}{x, y, - \beta\gamma}; y, = \frac{\alpha\gamma y, (x, - \beta)}{x, y, - \beta\gamma} \dots (9)$$

Now the inclinations of the lines RS, ST to the axis of x depend upon the equations $\frac{y, - y,}{x, - x,}$ and $\frac{y, - y,}{x, - x,}$.. (10)

But by inserting in (10) the values furnished by (7), (8), (9), we find each of the quantities (10) equal to the same quantity: $\frac{x, y, (\delta - \gamma) + y, (\beta\gamma - \alpha\delta) + \beta\gamma (\alpha - \beta)}{x, y, (\alpha - \beta) + x, (\beta\gamma - \alpha\delta) + \alpha\beta (\gamma - \delta)}$.

Hence the two lines RS, ST being equally inclined to the same line OB are one continuous line, as stated in the question.

Scholium. This proposition is only a particular case of that remarkable property discovered by Pascal, that "if a hexagon be inscribed in a conic section, and its opposite sides are produced to meet, the three points of intersection range in the same straight line". The property is familiar to every geometer when the hexagon is salient, and inscribed in a circle*: and

* See Math. Rep. Vol. II. Question 180. The solution of Mr. Ivory is copied into Bland's Geometrical Problems, p. 231.

readily inferred by central projection to belong to all the curvilinear sections of the cone. Pascal himself, it is quite clear from his *Essais pour les Coniques*, (1640), extended it to the other lines of the second order by projecting the circle and its hexagon on an oblique plane. He also, contemplated the present case, though it does not appear under what aspect he could have viewed it, to see the trapezium as the projection of a circle upon a plane. Yet he expressly in enumerating the conic sections (in his definition) mentions the "*angle rectiligne*," as one of them. It would seem that he had employed some other considerations, perhaps derived from the law of continuity, in establishing the proposition in reference to the trapezium; for however elongated the conic section might be, by the approach of one point towards the line joining two others, the property was still true by projection, and therefore, it was true also at the limit, when the conic section degenerated into straight lines forming a rectilinear angle. The subject, however, did not excite much interest, and the writings of Pascal were irrecoverably lost. The taste of the times was for other, and, doubtless, more important subjects: though, perhaps in respect to single properties of the conic sections this is the most important as well as the most elegant yet discovered.

The hint given by Newton, and followed out by Maclaurin while yet a minor, for the organic description of curves was calculated to produce a change in the method of investigating some classes of properties of the conic sections. The restoration, by Simson, of two of the porismatic propositions of Pappus of great geometrical beauty, gave new interest to enquiries connected with the properties of the trapezium. Maclaurin during a voyage to France, whilst considering the paper of Simson in conjunction with his own *Geometria Organica*, struck out the method of describing curves by means of rulers only, instead of angles, which after Newton he had employed in his book. This was in 1722. In 1727 he published his Algebra, in which (B. III. § 38. chap. 2.) he first made his method public: and in which he deduced the proposition of Pappus, and a proposition identical with Mr. Swale's (Leeds Correspondent, No. 19. p. 251.), as corollaries from his general proposition—which general proposition, is in reality, though differing from it in appearance, identical with that of Pascal. Indeed, in his "Appendix" he figures two particular cases of Pascal's proposition. Notwithstanding the ingenuity of the method, and aided, moreover, by one of the most successful auxiliaries for keeping interest alive,—the acrimonious discussion concerning priority

of discovery which took place between Maclaurin and Braikenridge,—the subject was again dropped.

It was revived within the present century by Carnot, who in his *Géométrie de Position*, brings it perpetually into use: though rather for the developement of properties of the conic sections than for their organic description. Brianchon, however has applied it very efficiently to the latter purpose as well as for an instrument of investigation: and the methods are now so generally cultivated on the continent as to have rendered "the Geometry of the Rule," an important branch of mathematical science. If fecundity of result, and power of investigation can confer importance in a science, there can be little doubt that this deserves the estimation in which it is held; inasmuch as it affords in many cases a greater facility than even the analytical methods themselves. Hence in the investigation of some classes of properties we see it almost exclusively employed by foreign writers.

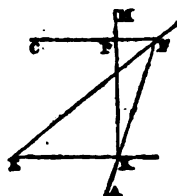
In this country, except in case of the salient hexagon inscribed in the circle, the subject does not appear to have obtained attention till about 1821, when Mr. Swale proposed in the Leeds Correspondent the following theorem. "The right lines EM , MF , EK , and the point A are given in position; and C , M , are given points in EM ; draw AC , AM intersecting EK , EF , in D , Q . Then if any line, as AKF be drawn through A meeting EK , EF , at K , F , the points D , Q and the points of concurrence of CF , MK will be in the same right line." The truth is, that "the points D , Q " are two points in the general linear locus of "the point of concurrence," as will appear further on. It is important, however, to bear in mind, in attending to the solution given to that question in the Correspondent, that the term trapezium though properly applied to *any assemblage of four lines, has not usually been so applied in this country, except in the case of their forming a salient figure*. This can only take place in reference to the present question when the line EF lies *between* EM , EK , and the point A without the angle KEF on the side opposite to M , C . The corollary from that solution, and in reference to the figure was not, then, justifiable; except the definition of the trapezium had been extended to include the case to which the figure was adapted.

In common with several other gentlemen who had sketched their figures in the same manner I had, I then obtained the same conclusion—for the one was identical with the other, and who could then help obtaining it? I did not, at that time pursue the subject to much extent: but in 1825 I published a demonstration of that corollary upon the very obvious principle of pro-

jection: viz. by first establishing it for a trapezoid, and then projecting it upon an oblique plane. See Monthly Magazine, July, September, October, 1825. In the same year I read before the Society of Enquirers, in Bristol, a paper containing several properties of the trapezium with their demonstrations, amongst which this was included. These were effected by the method of transversals—a method strictly geometrical: and were published in the Philosophical Magazine, August, 1826. Though in some of these, as well as in some of the analagous properties of the conic sections which I published in the same work, November 1826, I afterwards found myself anticipated by M. Branchon and other foreign writers; yet there has not, as far as I have discovered, been any anticipation of my general reasonings on the subject. There is, however, one proposition to which I feel disposed to attach some value in that series which has not, I think been noticed by any one before me: “that if two triangles have their sides intersecting in pairs in the perimeter of a conic section, then the three lines which join the opposite pairs of angles will pass through the same point.”

When I sat down to write this scholium, I intended to give sketches of the different methods employed in demonstrating the proposition before us, which I have met with in more recent reading: but I find it would far exceed the possible limits allowable to such a subject. Besides, I cannot foresee what methods the other correspondents of the Repository may devise. A very neat geometrical solution might obviously be made up from the investigation employed in the Leeds Correspondent, by scarcely altering more than the order of the reasoning and the structure of the diagram. I am, myself in possession of another geometrical demonstration, but as it forms a part of a course of enquiries scarcely matured for publication, I shall withhold it at present. I shall, however, give a new analytical investigation of Maclaurin's Organic theorem above referred to, which furnishes a result remarkable in the symmetry of its form, and affords an instance of the superiority of general methods over particular artifices. I give it, moreover, as well as the algebraical one at the head of this scholium, to shew with respect to this case at all events, that M. Brianchon is mistaken in supposing that no algebraical solution of the proposition is possible. Indeed, the skilful general investigation given by Mr. Lubbock in the Philosophical Magazine, Oct; 1829, had already shewn this: but on such a question it cannot be uninteresting to make an addition to the instances in which the surmise of the learned Frenchman was erroneous.

Referring the system of lines to oblique co-ordinates $ЕК$, $ЕF$ (the two directrices of the system) and denoting the co-ordinates of the three poles A , C , M by $x'y'$, $x''y''$, $x'''y'''$; and the variable values of $ЕК$, $ЕF$ by $x_{//}$, $y_{//}$; we shall have for the equation of $ЕК$



$$y - y_{//} = -\frac{y_{//}x}{x_{//}}, \text{ or } x_{//}y - xy_{//} + xy_{//} = 0.$$

But as this line also passes through A , we have

$$x_{//}y'' - xy_{//} + x'y_{//} = 0 \dots (1).$$

Again the equation of CF is $y - y'' = -\frac{y_{//} - y''}{x_{//} - x''} (x - x'') \dots (2)$

and that of $МК$ is $y - y''' = -\frac{y_{//} - y'''}{x_{//} - x'''} (x - x''') \dots (3).$

$$\text{From (2) we find } y_{//} = \frac{y''x - x''y}{x - x''} \dots (4);$$

$$\text{and from (3), we have } x_{//} = \frac{yx''' - xy'''}{y - y'''} \dots (5).$$

Substituting the values of $x_{//}$, $y_{//}$ of (4), (5) in (1), and reducing we obtain for the locus of P , the equation

$$(xx' - xy'')(x - x'')y' + (y'x - x'y'')(y - y'')x' + (yx'' - xy'')(yx'' - xy'') = 0 \dots (6)$$

The point P then is in a line of the second order.

Let now E , C , M be in one straight line, the equation of which is $y = ax$. Then $y'' = ax''$ and $y''' = ax'''$; and (6) becomes divisible by the common factor, which gives us

$x \cdot x''(y' - ax'') + y \cdot x''(x''' - x') + x'x''(y' + ax') = 0$, the equation of a straight line, which the locus in this case becomes: agreeing in that respect with Maclaurin's concluding paragraph of § 98 already referred to.

The point D , Q of Mr. Swale's theorem are points in this locus: the former when M , K , A are in a straight line, and the latter when C , Q , A , are also in one line. Hence the property in question is also established by this method.

It may also be remarked that in Maclaurin's reduction he has made the wrong letters vanish in the particular case he supposed: but so far as merely the depression of the equation is concerned the same effect is produced. It may also be stated that no method of solution of the general theorem has yet been devised, *except by analytics*, which justifies its application to the trapezium as a particular case included in that general investigation. Make, for instance, the experiment of deducing the

trapezium from Leslie's Geom. of Curve Lines, (p. 274-77) or any other work of the same kind, and it will immediately be perceived that the reasoning does not apply.

Other demonstrations were received from Messrs. CLERIOUS and N. Y.

VII. QUESTION 497. by R. N.

Find the sum of the series

$$1 + 1 + 2 + 3 + 5 + 8 + \&c.$$

any term being the sum of the two preceding terms.

FIRST SOLUTION, by — MASON, of Scoulton.

Let u_x = the x^{th} term, then by the same notation u_{x-1} and u_{x+1} are the $x-1^{\text{th}}$ and $x+1^{\text{th}}$ terms of the same series: hence by the question we have $u_{x+1} - u_x - u_{x-1} = 0$. Now a^x is a value of u_x in this equation of differences $a^x - a = 1$, $\therefore a = \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$: hence the complete value of

$$u_x = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^x + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^x; \text{ if we make } x = 0 \text{ and}$$

$$1 \text{ successively we shall find } c_1 = \frac{1}{\sqrt{5}} \text{ and } c_2 = -\frac{1}{\sqrt{5}} \therefore$$

$$u_x = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^x + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^x \right\} \text{ the } x^{\text{th}} \text{ term,}$$

which is the increment of the sum of $x-1$ terms $= \Delta s_{x-1}$.

$$\text{Hence } s_{x-1} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{x+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{x+1} \right\} + c; \text{ and}$$

$$c = -1;$$

$$\text{hence } s_x = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{x+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{x+2} \right\} - 1 \text{ the}$$

sum of x terms, which was required.

SECOND SOLUTION, by Mr. JOHN BAINES.

Since each term of a geometric series depends on the preceding only, it follows by a kind of inductive reasoning, that a series whose terms depend upon the two preceding respectively,

must be composed of two geometric series; when upon the three preceding terms, of three geometric series, &c.—Suppose the series $1 + 1 + 2 + 3 + 5 + \&c.$ composed of $a + ar + ar^2 + ar^3 \&c.$ and $b + by + by^2 + by^3, \&c.$; then we have obviously $a + b = 1$, $ax + by = 1$, $ax^2 + by^2 = 2$, and $ax^3 + by^3 = 3$. By transposition b, by, by^2 and $by^3 = 1 - a$, $1 - ax$, $2 - ax^2$ and $3 - ax^3$; but the former are in geometrical progression, hence the latter are so too, and consequently we have these two equations, viz. $(1 - a) \cdot (2 - ax^2) = (1 - ax)^2$, and $(1 - ax) \cdot (3 - ax^3) = (2 - ax^2)^2$. The former gives

$$a = \frac{1}{x^2 - 2x - 2}, \text{ and the latter } a = \frac{1}{-x^3 + 4x^2 - 3x}.$$

Equating these two values of a , we get $x^3 - 3x^2 + x + 2 = 0$. In this equation, one value of x is obviously $= 2$, but as this gives

$$a = \frac{1}{2}, \text{ and consequently } y = 0, \text{ it is of no use in the present}$$

enquiry. Wherefore depressing the equation by dividing by $x - 2 = 0$, we have the quadratic $x^2 - x - 1 = 0$, which

gives $x = \frac{1}{2}(1 \pm \sqrt{5})$, the upper sign belonging to the value of x , and the lower to y . Hence we have $a + b = 1$, and

$$\frac{a}{2}(1 + \sqrt{5}) + \frac{1}{2}(1 - \sqrt{5}) = 1. \text{ These equations solved give}$$

$$a = \frac{1}{10}(5 + \sqrt{5}), \text{ and } b = \frac{1}{10}(5 - \sqrt{5}). \text{ By a proper application of what has been already determined we easily obtain the two geometric series, viz.}$$

$$\frac{1}{10} \left\{ (5 + \sqrt{5}) + (5 + 3\sqrt{5}) + (10 + 4\sqrt{5}) + (15 + 7\sqrt{5}) + (25 + 11\sqrt{5}) + \&c. \right\},$$

and

$$\frac{1}{10} \left\{ (5 - \sqrt{5}) + (5 - 3\sqrt{5}) + (10 - 4\sqrt{5}) + (15 - 7\sqrt{5}) + (25 - 11\sqrt{5}) + \&c. \right\}$$

the sum of which is evidently = the given series. The sum of the first geometric series to n terms, by well known rules, =

$$\frac{\left(\frac{1 + \sqrt{5}}{2}\right)^n - 1}{\frac{1 + \sqrt{5}}{2} - 1} \cdot \frac{5 + \sqrt{5}}{10} = \left\{ (1 + \sqrt{5})^n - 2^n \right\} \cdot \frac{5 + 3\sqrt{5}}{10 \cdot 2^n},$$

and the latter =

$$\frac{\left(\frac{1 - \sqrt{5}}{2}\right)^n - 1}{\frac{1 - \sqrt{5}}{2} - 1} \cdot \frac{5 - \sqrt{5}}{10} = \left\{ (1 - \sqrt{5})^n - 2^n \right\} \cdot \frac{5 - 3\sqrt{5}}{10 \cdot 2^n}$$

Hence

$$\frac{1}{10 \cdot 2^n} \left\{ (5+3\sqrt{5}) \cdot \{(1+\sqrt{5})^n - 2^n\} + (5-3\sqrt{5}) \cdot \{(1-\sqrt{5})^n - 2^n\} \right\} \\ = \text{the sum of } n \text{ terms of the given series, as was required.}$$

Remark. In prosecuting my researches on the subject of this question, it has appeared sufficiently manifest, that if the scale of relation of a recurring series be $f + g$, the ratios of the two geometric series into which it is separable are the two roots of the quadratic equation $x^2 - fx - g = 0$; when the scale of relation is $f + g + h$, the ratios of the three geometric series are the three roots of the cubic equations $x^3 - fx^2 - gx - h = 0$, &c. Hence in the present example we have $x^3 - x - 1 = 0$, as above.

Mr. Herschel has given a general investigation of the proposition—“that every recurring series may be resolved into two or more geometrical series.” (*Treatise on Differences and Series*, p. 456). By a reference, however, to that work, our correspondents will learn that the method has a numerous class of exceptions, viz. when the equation from which the geometrical ratios are derived, has any number of equal roots. They will see, also, a little farther on, that whether the roots of this equation can be found or not, or even if they be imaginary, yet the recurring series can in all cases be summed by direct methods.—*Ed.*

Messrs. DAVIES and GODWARD also answered this question.

VIII. QUESTION 498. by R. N.

At page 84, Whewell's Dynamics, it is stated that, “The semicubical parabola with its axis vertical is a curve, the time of a body's falling down which, can be found”; it is required to find the time in finite terms.

SOLUTION, by — MASON, of Seoulton.

The equation of this curve $y^3 = ax^3$ the axis of y being the vertical axis of the curve. Let h be the perpendicular height from which the body begins to fall, then $\frac{dt}{ds} = \frac{1}{v} =$

$\frac{1}{\sqrt{2g} \cdot \sqrt{h-y}}$, and by the nature of the curve

$$\frac{ds}{dy} = \sqrt{\left\{ 1 + \frac{4}{9} \left(\frac{y}{a} \right) \right\}} \therefore \frac{dt}{dy} = - \frac{\sqrt{\left\{ 1 + \frac{9}{4} \cdot \frac{y}{a} \right\}}}{\sqrt{2g} \cdot \sqrt{h-y}};$$

$$\text{hence } t = \frac{\sqrt{\left\{1 + \frac{9}{4} \cdot \frac{y}{a}\right\}} \sqrt{(h-y)}}{\sqrt{2g}} + \frac{2}{3} \sqrt{\frac{a}{2g}} \left(1 + \frac{9}{4} \cdot \frac{h}{a}\right)$$

$$\times \cos^{-1} \sqrt{\frac{1 + \frac{9}{4} \cdot \frac{y}{a}}{1 + \frac{9}{4} \cdot \frac{h}{a}}} \text{ the correct integral. Let } y = 0, \therefore$$

$$\text{the time of descent to the cusp} = \sqrt{\frac{h}{2g}} + \frac{2}{3} \sqrt{\frac{a}{2g}} \\ \times \left(1 + \frac{9}{4} \cdot \frac{h}{a}\right) \cdot \cos^{-1} \frac{1}{\sqrt{\left\{1 + \frac{9}{4} \cdot \frac{h}{a}\right\}}}.$$

Messrs. BAINES, DAVIES, and GODWARD answered this question.

IX. QUESTION 499, by Mr. CUNLIFFE.

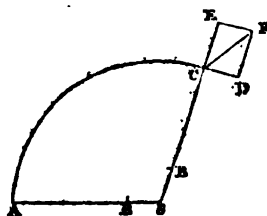
Suppose a small body or pellet of lead at a given point, in a perfectly smooth narrow tube, of a given length, and suppose the tube to be whirled about one of its ends as a centre, with a given angular velocity, in a horizontal plane; it is required to determine the velocity and direction of the pellet when it leaves the tube; the motion being solely generated by the revolving of the tube.

FIRST SOLUTION, by — MASON, of Scoulton.

Let SA be the tube, s the point round which it revolves in the plane of the paper with the uniform angular velocity w , $SA = a$, B the place of the body at first, $SB = b$, r the distance of the body from s at the time t ;

$\therefore w^2 r = \frac{d^2 r}{dt^2}$ is the force accelerating the body along the tube. Multiply by dr and integrate and we have $w^2 r^2 + c = \left(\frac{dr}{dt}\right)^2$ and $w^2 b + c = 0$; there-

fore $w^2 (r^2 - b^2) = \left(\frac{dr}{dt}\right)^2 \therefore \frac{dt}{dr} = \frac{1}{\sqrt{w}} \cdot \frac{1}{\sqrt{(r^2 - b^2)}};$



hence $t = \frac{1}{w} \cdot \text{hyp. log. } \frac{r + \sqrt{r^2 - b^2}}{b}$ and $\tau = \frac{1}{w} \cdot$

h. l. $\frac{a + \sqrt{a^2 - b^2}}{b} =$ the time at which the body quits the tube as at c. Hence the angle $\angle \text{asc} = \tau \cdot w = \text{hyp. log. } \frac{a + \sqrt{a^2 - b^2}}{b}$, also velocity of body at c in direction sc =

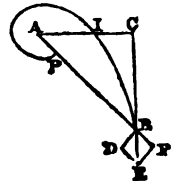
$w \sqrt{a^2 - b^2}$. Let ec, in sc produced, represent this velocity, and the velocity of the body at c in common with the point c = $w \cdot a$. Let cd perpendicular to sc represent this, $\therefore$ cf the diagonal of the rectangle ed represents the actual velocity of the body in direction and quantity, $\therefore$ velocity =

$w \sqrt{2a^2 - b^2}$, and the $\tan$ of $\angle \text{ecf} = \frac{\text{ef}}{\text{ce}} = \frac{a}{\sqrt{a^2 - b^2}}$:

hence the direction of the motion of the body at c makes with as the angle $\tau w + \tan^{-1} \frac{a}{\sqrt{a^2 - b^2}}$.

SECOND SOLUTION, by Mr. JOHN BAINES.

Let AB represent the given tube, P the given point in it at which the pellet is situated at the commencement of motion, and PIB the curve described by the pellet in moving along the tube, till it quits it at B. Also, let nc be a tangent to the curve at B, meeting AC, drawn from A, at right angles at C. Put AP = a, AB = b, the angular velocity at an unit's distance from A = c, the distance from A passed over by the pellet in any variable time $t = x$, and the corresponding velocity = v ; then $c^2 x = f =$ the centrifugal force, and $v dv = f dx = c^2 x dx$, $\therefore v^2 = c^2 x^2 + \text{const.}$; but when $x = a$, $v = 0$, $\therefore$ the correct integral is $v = c \sqrt{x^2 - a^2}$, which suppose to be represented by BF, and complete the parallelogram BFED, having BE, the tangent CB produced for its diagonal. But the velocity of the pellet in the direction BD = cx ; hence by compounded motion, the velocity in the direction of the tangent, which is that in which it quits the tube, is $= c \sqrt{2x^2 - a^2}$, and when $x = b$, we have $c \sqrt{2b^2 - a^2} =$ the velocity of the pellet when it is quitting the tube.—Again it is evident that the direction of the body at the time of leaving the tube = $\angle \text{ABC} =$



$\angle EBF$. By trig. $BE : \text{rad} :: BF : \cos \angle EBF = \sqrt{\frac{b^2 - a^2}{2b^2 - a^2}}$.

$$\therefore \sin \angle EBF = \sin \angle ABC = \frac{b}{\sqrt{(2b^2 - a^2)}}.$$

Cor. 1. When $\frac{b}{\sqrt{(2b^2 - a^2)}} = 1$, or $a = b$, the pellet quits the tube at right angles as it ought.

Cor. 2. The time of quitting the tube from the commencement of motion is easily obtained; for $dt = \frac{dx}{v} = \frac{dx}{c\sqrt{(x^2 - a^2)}}$, which being integrated between the limits $x = a$, and $x = b$, gives $t = \frac{1}{c} \cdot \text{h. l.} \left(\frac{a + \sqrt{(b^2 - a^2)}}{b} \right)$.

Messrs. DAVIES and GODWARD answered this question.

X. QUESTION 500, by R. N.

The plane of the paper representing the horizon of a place at any time, and the positions of the earth, sun, and moon being given, it is required to find the equation to the straight line which is the intersection of the two planes passing through the moon's centre perpendicular to the lines joining the sun and moon, and the earth and moon.

FIRST SOLUTION, by — MASON, of Scoulton.

Let the centre of the earth be the origin of co-ordinates, the rational horizon the plane of xy , the intersection of the meridian with this plane the axis of x . Then the axis of y is the intersection of the prime vertical with the same, and the line drawn to the zenith or the intersection of the prime vertical and the meridian is the axis of z . Let a, b, c be the co-ordinates of the sun's centre a, b, c , those of the moon's centre. Hence

$x + \frac{a,c - ac}{bc - b,c} y + \frac{ab - a,b}{bc - b,c} z = 0$, is the equation to the plane passing through the centres of the earth, sun, and moon. Now the intersection mentioned in the question being perpendicular to each of the lines drawn from the moon's centre to that of the earth and sun, is necessarily perpendicular to the plane passing through these lines, that is to the plane whose

equation is given above. The equations to this perpendicular are $y - b = \frac{ac - ac'}{bc - b'c'}(x - a)$ and $z - c = \frac{ab - a'b}{bc - b'c'}(x - a)$.

Cor. This line is that which joins the horns of the moon: hence the times of the year and of the day being given, the positions of the centres of the sun and moon are given with respect to the horizon of a given place and the meridian, and also with respect to the prime vertical: hence a, b, c as well as a', b', c' are given, and consequently the line is given. If we suppose $z = 0$ in the equation to the line; we shall have the co-ordinates of the point of intersection of this line with the horizon, and the sine of inclination of this line to the horizon is

$$\pm \frac{ab - a'b}{\sqrt{\{(ab - a'b)^2 + (bc - b'c')^2 + (ac - a'c')^2\}}}.$$

SECOND SOLUTION, by Mr. DAVIES, Bath.

Let E, S, M be the centres of the earth, sun, and moon; and let ooo, x, y, z, x', y', z' be the co-ordinates of these points referred to the horizon, prime vertical, and meridian as co-ordinate planes. Then the line in question is obviously drawn at right angles to the plane ESM through the point M .

The equation of the plane is

$$(z, y, -z, y')x + (x, z, -x, z')y + (y, x, -y, x')z = 0 \dots (A),$$

and the line through x', y', z' at right angles to (A) has, as well known, the three following equations

$$\left. \begin{aligned} (y - y')(y, z' - y', z) &= (x, z, -x, z')(x - x') \\ (z - z')(x, y, -x, y') &= (x, y, -x, y')(x - x') \\ (y - y')(x, y, -x, y') &= (x, z, -x, z')(z - z') \end{aligned} \right\} (B),$$

any two of which equations involve the third.

If, however, by the position of the bodies we are to understand their astronomical positions referred to the equinoctial and meridian, then the coefficients of these equations must be otherwise expressed: Thus let α, α' be the distances of the sun and moon estimated in right ascension from the meridian of the place; δ, δ' their respective declinations; and r, r' their distances from the earth's centre. Let also λ be the latitude of the place of observation. Denote moreover, by $x''y''z''$ and $x''y''z''$ the co-ordinates of the sun and moon referred to the equator, prime meridian, and secondary meridian as co-ordinate planes. Then we have

$$\left\{ \begin{aligned} x' &= r \cos \delta \cos \alpha, \\ y' &= r \cos \delta \sin \alpha, \\ z' &= r \sin \delta, \end{aligned} \right\} \quad \left\{ \begin{aligned} x'' &= r'' \cos \delta'' \cos \alpha'', \\ y'' &= r'' \cos \delta'' \sin \alpha'', \\ z'' &= r'' \sin \delta'', \end{aligned} \right\} (C).$$

Again, the inclinations of the new and old co-ordinate-axes are expressed in the annexed tablet.

Axes	$x,$	$y,$	$z,$
x'	$90^\circ - \lambda$	90°	$180^\circ - \lambda$
y'	90°	0°	90°
z'	λ	90°	$90^\circ - \lambda$

From this, and the formulæ in general use for transformations, we obtain

$$x, \sin \lambda + z, \cos \lambda = x' = r, \cos \delta, \cos \alpha, \dots (1)$$

$$y, = y' = r, \cos \delta, \sin \alpha, \dots (2)$$

$$z, \sin \lambda - x, \cos \lambda = z' = r, \sin \delta, \dots (3)$$

in virtue of equations (c).

To find $x,, y,, z,,$ we may proceed thus:

Multiply (1) by $\cos \lambda$, and (3) by $\sin \lambda$, and add the results: then

$$x, = r, \{ \cos \delta, \cos \lambda \cos \alpha, + \sin \delta, \sin \lambda \} \dots (4)$$

Multiply (1) by $\sin \lambda$, and (3) by $\cos \lambda$, and subtract: then

$$x, = r, \{ \cos \delta, \sin \lambda \cos \alpha, - \sin \delta, \cos \lambda \} \dots (5)$$

and $y,$ is given already by equation (2).

In precisely the same manner we learn that

$$x,, = r,, \{ \cos \delta,, \sin \lambda \cos \alpha,, - \sin \delta,, \cos \lambda \} \dots (6),$$

$$y,, = r,, \cos \delta,, \sin \alpha,, \dots (7)$$

$$z,, = r,, \{ \cos \delta,, \cos \lambda \cos \alpha,, + \sin \delta,, \sin \lambda \} \dots (8).$$

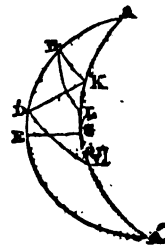
The values of $x,y,z,, x,,y,,z,,$ furnished by (2, 4, 5, 6, 7, 8) inserted in the equations (B) give a solution of the problem.

Scholium. The coefficients of $r,$ $r,,$ in the values of $x,y,z,$ $x,,y,,z,,$ may be thus exhibited. Describe a spherical lune $\Delta EA'A$ whose angle Λ is $= \alpha,$, measured by the intercepted portion EQ of the equator. Take $\Lambda D = E'D' = \delta,$; $QL = Q'L' = \lambda$; join $DL, D'L'$; and lastly draw DK perpendicular to ΔE meeting AQ in K . Denote $DL, D'L'$, and $\angle K$ by $K', K,$ and v , respectively. Then

$$\cos K' (= D'L') = \cos \delta, \sin \lambda \cos \alpha, - \sin \delta, \cos \lambda$$

$$\cos v, (= \angle K) = \cos \delta, \sin \alpha,$$

$$\cos K, (= DL) = \cos \delta, \cos \lambda \cos \alpha, + \sin \delta, \sin \lambda$$



and doing the like with another lune whose angle is $\alpha_{''}$, and taking $\delta_{''}$ instead of δ , we have also

$$\cos \kappa_{''} = \cos \delta_{''} \sin \lambda \cos \alpha_{''} - \sin \delta_{''} \cos \lambda$$

$$\cos \nu_{''} = \cos \delta_{''} \sin \alpha_{''}$$

$$\cos \kappa_{''} = \cos \delta_{''} \cos \lambda \cos \alpha_{''} + \sin \delta_{''} \sin \lambda.$$

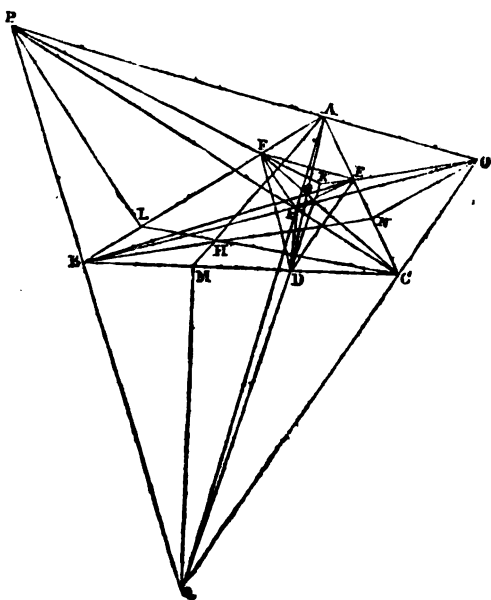
XI. QUESTION 501, by Mr. THOMAS STEPHENS DAVIES, Bath.

Lines drawn from the angles of a triangle to the points of contact of the inscribed circle meet in the same point (G), as is well known: lines drawn to the points of contact of the circles which touch externally meet in a point (H); and lines drawn from the centres of the circles of external contact to the points where the inscribed touches meet also in a point (K). Prove these two latter, and shew whether the three points G, H, K are in a straight line or not.

SOLUTION, by Mr. DAVIES, the Proposer.

LEMMA 1. If the sides of a triangle ABC be divided in D, E, F so that the solids of the alternate segments be equal, (that is, so that $BF \cdot AE \cdot CD = FA \cdot EC \cdot DB$): then the lines AD, BE, CF intersect in one point G. Converse of Simpson's Geometry, Prop. 23. Book IV.

LEMMA 2. If three sides of one triangle DEF be respectively parallel to three sides of another, PGO; then lines joining each pair of equal angles, will pass through the same point. This is too simple to need demonstration.



We come next to the Proposition itself.

1. Let R be the centre of the inscribed circle, and D, E, F the points of contact. Then by common properties of tangents to the circle, and Lem. 1. we find that AD, BE, CF pass through the same point.

2. Let O, P, Q be the centres of the escribed circles (the circles of external contact): then by well known properties of the figure we find that $BL \cdot AN \cdot DM = LA \cdot NC \cdot MB$; and hence (Lem. 1.) the lines CL, AM, BN intersect in one point.

3. Since PO and FE are both perpendicular to AQ they are parallel. In the same manner, $OQ \parallel DE$ and $PQ \parallel FD$. Hence the $\Delta s PGO$ and FED have their sides respectively parallel: and, therefore, by Lem. 2. the lines PF, OE, QD meet in one point.

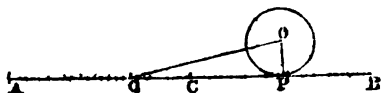
The determination whether the points G, H, K be in one line or not, being discussed in my Paper on the Analytical Values of the different parts of a Triangle, in the number* of the Repository, need not be further noticed here.

XII. QUESTION 502, by SAMUEL JONES.

If a thin rectangular bar of polished steel of a given length were placed horizontally *in vacuo*, and if a smooth magnetic sphere of a given diameter were placed at a given point on the bar, and then abandoned to the attractive force of the steel, it would oscillate thereon perpetually. Find the greatest velocity it would acquire.

SOLUTION, by Mr. SAMUEL JONES, the Proposer.

Let AB be the given bar, c its middle point, O the centre of the sphere, P its point of contact with the bar at the commencement of the motion, and Q any variable point in AB . Put $AC = CB = a$; a vertical section



of the bar $= A$; the radius of the sphere plus half the thickness of the bar $= r$; the mass of the sphere $= m$; also $PQ = x$; and $OP = z$; then, $OQ^2 = r^2 + x^2$; and as $A dx$ is an element of the bar, the attractive force of this element on a particle at O is $\frac{A dx}{r^2 + x^2}$, whence the horizontal value of this force in

* This Paper is necessarily deferred to our next Number. — Ed.

the direction OP is $\frac{Ax dx}{\{r^2 + x^2\}^{\frac{3}{2}}}$, the integral of which gives

$c - \frac{A}{\sqrt{r^2 + x^2}}$ for the attractive force of the segment PA on

a particle O ; but when $x = 0$, then $c = \frac{A}{r}$; therefore the mo-

tive force of the segment AP is $A \times \left\{ \frac{a+z}{r} - \frac{a+z}{\sqrt{(a+z)^2 + r^2}} \right\}$;

in like manner $A \times \left\{ \frac{a-z}{r} - \frac{a-z}{\sqrt{(a-z)^2 + r^2}} \right\}$ is the mo-

tive force of the segment BP ; but the difference of these is the

motive force of the sphere, which divided by its mass gives

$\frac{A}{m} \times \left\{ \frac{2z}{r} + \frac{a-z}{\sqrt{(a-z)^2 + r^2}} - \frac{a+z}{\sqrt{(a+z)^2 + r^2}} \right\}$ which

is the accelerative force of the sphere on the bar at the point P in direction PC .

Now

$v dv = f dz = \frac{A}{m} \times \left\{ \frac{2z dz}{r} + \frac{(a-z) \cdot dz}{\sqrt{(a-z)^2 + r^2}} - \frac{(a+z) \cdot dz}{\sqrt{(a+z)^2 + r^2}} \right\}$,

the integrals of which give

$\frac{v^2}{2} = \frac{A}{m} \times \left\{ \frac{z^2}{r} + \sqrt{(a-z)^2 + r^2} - \sqrt{(a+z)^2 + r^2} + c \right\}$;

but when $z = b$; then $v = 0$; whence the correct integrals give

$0 = \sqrt{\frac{2A}{m}} \times \left\{ \frac{a^2 - b^2}{r^2} + \sqrt{a^2 + r^2} - \sqrt{a^2 + r^2} - \sqrt{a^2 + r^2} + \sqrt{a^2 + r^2} \right\}^{\frac{1}{2}}$;

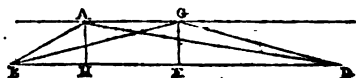
and this when $z = 0$, gives

$\sqrt{\frac{2A}{m}} \times \left\{ \sqrt{a^2 + r^2} - \sqrt{a^2 + r^2} - \frac{b^2}{r} \right\}^{\frac{1}{2}}$,

which is the greatest velocity the sphere would acquire.

SECOND SOLUTION, by Mr. DAVIES, Bath.

Let A be the centre of the sphere in one of its positions; BD the line of contact, prolonged to the ends of the bar, B, D ; E its middle, and H the point of contact corresponding to the position A . Then if M



be the attractive force of the bar at the distance unity upon the sphere, we have

$$\frac{d^2x}{dt^2} = M \left\{ \frac{1}{AB} - \frac{1}{AD} \right\} \dots\dots\dots(a).$$

Let now $BE = ED = a$, $AH = b$, and $HE = x$: multiply (a) by $2dx$, substitute these values in it, and integrate; we get

$$\frac{dx}{dt} = v = \left\{ 2M \log. C \times \frac{-a-x + \sqrt{a-x}^2 + b^2}}{a+x + \sqrt{a+x}^2 + b^2} \right\}^{\frac{1}{2}} \dots(b).$$

which expresses the velocity at A in the direction AG.

This sphere being supposed to be actuated solely by the attraction of the bar, if we put a = the initial value of x , this will be one of the limits between which the integration is to be performed. The other limit is easily found from $dv = 0$, which gives $AB = AD$, or $x = 0$.

Let us now suppose A to represent the initial position of the ball; and denote the angles ABH, ADH, and GBE = GDE, by 2β , 2δ , and 2α respectively. The integrals between $x = a'$ and $x = 0$ become

$$v = \left\{ 2M \cdot \log \{ \cot \delta \cot \beta \tan^2 \alpha \} \right\}^{\frac{1}{2}} = \text{the greatest velocity.}$$

W. W. R.

XIII. QUESTION 503, by S. S. S.

A country clock-maker being directed to make a pair of cycloidal pendulum cheeks to a circle whose radius was r inches, made them by mistake to a circle only m inches radius. To remedy this defect he put a rim of brass of equal thickness round the cycloid to make their axes equal to $2r$ inches, and believes that the cheeks are properly formed. It is required to ascertain the nature of the curve described by the pendulum vibrating between them; and to find the point at which its deviation from a cycloid is greatest.

SOLUTION, by Mr. T. S. DAVIES, Bath

Let $AB = 2r$ = diameter of generating circle, and $a = 2a'$

$$MR = MH \cot \theta = (2r \cos \theta + a) \cot \theta,$$

$$HR = MH \operatorname{cosec} \theta = (2r \cos \theta + a) \operatorname{cosec} \theta$$

$$BR = BM + HR = 2r\theta + (2r \cos \theta + a) \operatorname{cosec} \theta \\ = 2r\theta + 2r \cot \theta + a \operatorname{cosec} \theta$$

$$RQ = RM + MQ = (2r \cos \theta + a) \cot \theta + 4r \sin \theta + a\theta$$

$$RS = RQ \cos \theta.$$

$$\text{Hence } x, = BR - RS = 2r\theta + 2r \cot \theta + a \operatorname{cosec} \theta \\ - \{(2r \cos \theta + a) \cot \theta + 4r \sin \theta + a\theta\} \times \cos \theta$$

$$y, = QR \sin \theta = \{(2r \cos \theta + a) \cot \theta + 4r \sin \theta + a\theta\} \sin \theta$$

or by reduction

$$x, = (2r - a \cos \theta) \theta - (2r \cos \theta - a) \sin \theta \quad \left. \vphantom{x,} \right\} (5) \\ y, = 2r(1 + \sin^2 \theta) + a \cos \theta + \theta \sin \theta. \quad \left. \vphantom{y,} \right\}$$

These are the values of the co-ordinates of the sought curve in terms of the variable θ . We may put them to the test in one or two simple cases.

Thus—let $\theta = 0$; then $x, = 0$, and $y, = 2r + a$.

$$\text{let } \theta = 90^\circ; \text{ then } x, = 2r\theta + a, \text{ and } y, = 4r + \frac{a \cdot \pi}{4}.$$

All which we readily infer from the nature of the figure should be the case.

Again, if in the equations (5) we put $a = 0$, $\theta = \theta'$ and $r = r + a'$ we find

$$x,, = 2(r + a') (\theta' - \frac{1}{2} \sin 2\theta') \quad \left. \vphantom{x,,} \right\} \dots (6) \\ y,, = 2(r + a') (1 + \sin^2 \theta') \quad \left. \vphantom{y,,} \right\}$$

which is the equation of the involute to the cycloid upon the axis $2r + 2a'$ viz. of that which we are required to compare with the curve found in (5).

In all cases of deviation we must of course suppose an independent variable in one curve equal to a corresponding variable in the other: and we can always take that variable as independent which may best suit the object we have in view. In cases, however, where both co-ordinates are expressed in terms of a third variable, it will almost always happen that the said third variable is the most convenient one to be equated in the two curves: and so it is in the case before us. We shall, therefore consider that $\theta = \theta'$, and seek the distance of the points Q, Q'.

Then $(x, - x,,)^2 + (y, - y,,)^2 = D^2$, or by inserting the values of $x,$, $x,,$, $y,$, $y,,$ from (5), (6) we have

$$a^2 \{ (1 + \cos \theta)^2 (\sin \theta - \theta)^2 + \{ (1 - \cos \theta) + (\theta - \sin \theta) \sin \theta \}^2 \} = D^2 \dots (7).$$

or differentiating for the maximum, we have

$$\theta^2 - \frac{1}{4} \operatorname{cosec} \theta \{ 4 + 10 \cos \theta + 4 \cos^3 \theta - 4 \sin \theta \cos \theta \} \theta \\ - \frac{1}{4} (10 \cos \theta - 2 \sin \theta) = 0 \dots \dots \dots (8).$$

From (7) we learn that the deviation is independent of r , the radius of the generating circle: and that for any value of θ , it varies directly as the breadth of the rim.

Also, since the tangents to the two cycloids upon BA , BT , and to the curve (5) are all parallel, it will follow that the line QQ' makes with x , y an angle dependent alone upon θ : or which comes to the same thing, when θ is determined, in any point of the curve, the several corresponding points Q' , Q'' , Q''' . . . of the several involutes whose breadths are a , a'' , a''' , &c. . . will have a straight line for their locus.

Of course, too, the value of θ for the maximum deviation is independent of both r and a ; and may be calculated from equation (8). When once calculated the general question is *numerically* answered: but this is a work of more labour than I feel disposed to undertake. Evidently no method is so applicable to it as "Double Position."

These curious results may be partly justified by other independent investigations, but it is unnecessary here to enter further into detail upon the subject.

We may also remark that the curve of the border falls wholly *within* the cycloid upon diameter $2r + a$: and therefore that the involute of this curve falls *within* the involute of the cycloid.

For by (5) and (6) we have

$$x_1 = 2r\theta - a \cos \theta \cdot \theta - r \sin 2\theta + a \sin \theta \\ x_{11} = 2r\theta + a\theta - r \sin 2\theta - a \sin \theta \cos \theta \\ \therefore x_{11} - x_1 = a(1 + \cos \theta)(\theta - \sin \theta).$$

Now, as $\sin \theta < \theta$, the factor $\theta - \sin \theta$ is positive through the semicycloid AGE , and minus through the other. Also $1 + \cos \theta$ is essentially positive. Hence in both semicycloids the value of x_{11} is greater in its own direction than x_1 , or the curve extends to a less distance along the axis of x than the cycloid does.

In the same manner it will be found that $y_{11} - y_1$ is always positive. But we may infer the latter at once from the circumstance just established, viz. that in the similar triangle HMP , $HM'P'$ the point M is within the cycloid traced by M' ; and hence $M'P' > MP$, or $y_{11} - y_1$ is always positive.

When $BT = r - m$, then the reverse of this takes place (that

is, if a rim of equal thickness were *taken from* the cycloid) the evolute sought would fall without the corresponding evolute of the cycloid.

To ascertain whether we have chosen the most convenient independent variable let us try the two other pairs of hypothesis; viz.

$$x, - x_{//} = 0, \text{ and } d(y, - y_{//}) = 0 \dots \dots (9)$$

$$d(x, - x_{//}) = 0, \text{ and } (y, - y_{//}) = 0 \dots \dots (10).$$

Let also in both cases $\theta' = n\theta$.

Then the equations (9) give the two following

$$2(r+a')(\pi\theta - \frac{1}{2}\sin 2n\theta) = (2r-a\cos\theta)\theta - (2r\cos\theta - a)\sin\theta \dots (11)$$

$$2n(r+a')\sin 2n\theta = 2r\cos 2\theta + \theta \dots \dots \dots (12).$$

Multiply (11) by $2n$, and add it to (12): then

$$4(r+a')n^2\theta - 2(r-a\cos\theta)n\theta - r\sin 2\theta + 2r\cos 2\theta + a(\cos\theta - \sin\theta),$$

$$\text{or } x = \frac{2r-a\cos\theta}{4(r+a')} \pm$$

$$\left\{ \frac{(2r-a\cos\theta)^2 - (r\sin 2\theta - 2r\cos 2\theta - a\cos\theta + a\sin\theta) \cdot 4(r+a')}{16(r+a')^2} \right\}^{\frac{1}{2}}$$

which substituted in (12) gives an equation of which it would be hopeless to attempt the numerical reduction.

The equations (10) also give two others even more intractable than the last: viz.

$$2r(1 + \sin^2\theta) + a(\cos\theta + \theta\sin\theta) = 2(r+a')(1 + \sin^2n\theta)$$

$$2r + a\theta\sin\theta - r\cos 2\theta = 2(r+a')\sin 2n\theta.$$

XIV. QUESTION 504, by Mr. T. S. DAVIES, Bath.

Planes passing through a given point cut a surface of the second order: Find the loci of the centres and foci of the resulting sections.

SOLUTION, by Mr. DAVIES, the Proposer.

This question was inadvertently printed in two places at the same time, and solutions were given both by Petrarch and myself, in the Gentleman's Diary, 1830, p. 36—8. It was not my intention to propose the question for the loci of the foci in either place, but of the "centres" merely; chiefly because the foci of the sections will form the subject of a chapter in my Geometry of Three Dimensions. This will enable me to dispense with any partial or premature discussion of my views on this matter, as well as enable me to present my investigations in a more systematic form than I could give them were I to lay

down a series of lemmas preparatory to a single proposition of the system. I shall however add another investigation of the locus of the centres, different from those given in the Gentleman's Diary.

Let the given surface of the second order have its origin transposed to the given point, and referred to planes parallel to their conjugate diameters. Its equation will be of the form

$$Ax^2 + By^2 + Cz^2 + 2Ax + 2By + 2Cz + D = 0 \dots (1)$$

Let $y = ax, z = bx \dots \dots \dots (2)$

be the equation of one of the lines drawn from the origin to cut the surface. Then the middle of its intercept, (denoted by x, y, z) will be one of the centres in question. Insert the values of y, z , from (2) in (1), and we have

$$(A + Ba^2 + Cb^2)x^2 + 2(A + Ba + Cb)x + D = 0 \dots (3)$$

$$\text{or } x = -\frac{A + Ba + Cb}{A + Ba^2 + Cb^2} \pm \sqrt{R} \dots (4)$$

$$\text{and } x = -\frac{A + Ba + Cb}{A + Ba^2 + Cb^2} \dots (5)$$

But since x, y, z , is a point in the line (2) we have

$$a = \frac{y}{x}, \text{ and } b = \frac{z}{x} \dots \dots \dots (6)$$

Write the values of a, b from (6) in equation (5), and reduce. This gives for the locus sought, the equation

$$Ax^2 + By^2 + Cz^2 + Ax + By + Cz = 0 \dots (7)$$

This equation it is obvious represents a complete surface of the second order, of which the place of the centres form only a part. It passes through the origin, and through the centre of (1). It would be easy to shew from the equation that these points are the opposite vertices of one of its diameters.

An elegant solution of the same question by M. Durrande may be seen in the Eleventh Volume of *Gergonne's Annales des Mathematiques*: and one still more elegant, in Mr. J. R. Young's useful and well timed "*Analytical Geometry*," just published.

In my solution of this question in the Gentleman's Diary, for "parallel" read "conjugate," at the top of page 38.

XV. QUESTION 505, by S. S. S.

Given the radii of the four spheres of external contact with a tetrahedron, to find the tetrahedron, together with its inscribed and circumscribing spheres.

Not Answered.

XVI. QUESTION 506, by S. S. S.

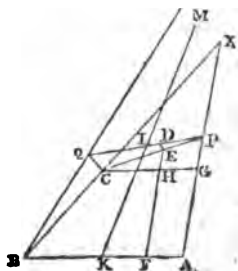
Find the nature of the curve surface (in Gentleman's Diary, No. 1243) generated by the line which joins the moving points, also the locus of the centre of gravity of these points supposing them urged by equal forces.

QUESTION 1243, from the GENTLEMAN'S DIARY, 1828.

"Two straight lines are given in space, along which two points move uniformly: supposing that the initial position of the points is given, as also the ratio of their velocities, it is required to construct the straight line which joins them at the instant when their distance from each other is the least possible."

FIRST SOLUTION, by Mr. WM. GODWARD, Wakefield.

Let APX and BQ be the lines in which the points move, A and B being the initial and P and Q any other cotemporary positions of the moving points. Let BX be the projection of BQ upon the plane BAX , draw QC perpendicular to BX , and join QP , CP . From any point D in QP , draw DE parallel to QC meeting CP in E ; and draw EF and CG respectively parallel to PA and BA , intersecting each other in H .



As it will make no difference, in the present inquiry, whether we consider the equal forces impulsive, or merely instantaneous; we will take for granted that the latter is the proposer's meaning, and shall therefore consider the points to move with an equal uniform velocity.

Put $AB = a$, $BX = b$, $\angle QBC = c$; $AP = BQ = x$, $AF = HQ = y$, and $DE = z$. Then $BC = x \cos c$, and $QC = x \sin c$ hence $CX = b - x \cos c$, and by sim. triangles $b : a :: b - x$

$\cos c : CG = \frac{a}{b} (b - x \cos c)$. Also by sim. triangles $CG : HC$

$(:: CP : PE) :: QC : DE$, or in symbols $\frac{a}{b} (b - x \cos c) : z$

$:: x \sin c : z$. Whence the equation of the surface is $z =$

$$\frac{bxy \sin c}{a(b - x \cos c)}.$$

Again, bisect AB in K , PQ in L , and through L draw the straight line KLM which will be the locus of the centre of gravity of the moving points, as is manifest from the well known property that "when any number of bodies move uniformly in straight lines in different planes, their centre of gravity also moves uniformly and in a straight line."

SECOND SOLUTION, *by Mr. DAVIES.*

This question supposes two given bodies to move with given velocities along given lines: and it requires us to determine the surface traced out in space by the line joining these bodies, and to assign the locus of their centre of gravity.

The general mechanical principle—"that when each body in the system moves in a straight line, their common centre of gravity also moves in a straight line or is at rest"—conducts us to an immediate solution of the first part of the question, and is itself a solution of the second.

The second granted, let us in the first place suppose the centre of gravity fixed. Then, the generated surface must be a plane. For a line passing through a fixed point and rectilinear directrix generates that surface.

We may learn from this, that except the two directrices of the question, be in one plane, that the centre of gravity cannot be at rest.

In the next place, consider the centre of gravity, moving in a straight line: then the generatrix moves upon three straight lines, not all in the same plane; and hence the surface in which it moves is the *hyperboloid of one sheet*.

We see from this, also, that if the centre of gravity be in motion, it does not move in the same plane as either of the directrices of the bodies, except the lines themselves be in the same plane.

* * * The analytical difficulty of the question is too inconsiderable to detain us, and presents moreover no particular novelty in its results: and hence, as I have drawn up a long scholium, containing a history of the various superficial loci generated by right lines, that I have been able to discover, I shall not dwell upon the subject here, further than to refer to Art. III. Part II. page 8, of the present number of the Repository.

Mr. BAINES answered this Question.

XVII. QUESTION 507, *by Mr. DAVIES.*

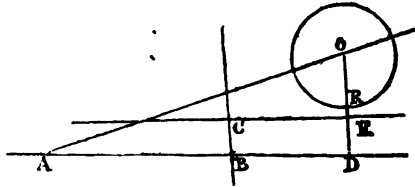
A series of equal circles indefinitely near to each other and

whose centres are in a straight line, are put in perspective: What are the loci of the centres and foci of the representing ellipses?

SOLUTION, by Mr. DAVIES, the Proposer.

Let AO be the line in which the centres of the circles are situated; AD the directing line; CE , the ground line, or trace of the picture upon the original plane; O , the centre of one of the circles; and B, C , the centres respectively of the directing and ground lines. Refer the whole system to the rectangular co-ordinates, x, y, z coinciding respectively with BC, BD , and a perpendicular, from B , to the plane of the paper.

Let the common radius of the circles, OR , be denoted by r ; the height of the eye $= c$; the distance between the ground-line and directing-line, $BC = b$; and the angle $OAD = \alpha$. Finally put xy , the co-ordinates of the line AO ; $x'y'$, those of the circle (O) ; xyz , those of the visual cone raised upon the circle (O) ; $\tilde{x}\tilde{y}\tilde{z}$, those of the intersection of the cone and picture.



Now, the equation of the line AO is

$$y = (x + a) \tan \alpha \dots (1).$$

That of the circle whose centre is O , is

$$(y' - y)^2 + (x' - x)^2 = r^2 \dots (2)$$

Combining equations (1) and (2) we get

$$(x' - x)^2 + (y' - a + x' \tan \alpha)^2 = r^2 \dots (3)$$

The equation of the cone of rays is formed by inserting in (3)

$$x' = -\frac{cx}{c-z}, \text{ and } y' = -\frac{cy}{c-z},$$

which gives

$$\left\{ \frac{cx}{c-z} + x \right\}^2 + \left\{ \frac{cy}{c-z} + x \tan \alpha \right\}^2 = r^2 \dots (4).$$

But the equation of the picture is

$$y = b \dots (5).$$

The value of y substituted in (4) gives the equation of the projection of the section made by the plane (5) in the cone of rays (4), on the plane of xz : and because of the parallelism

of (4) to xy , the projection is similar and equal to the section itself. The curve upon xy will, therefore, be identical with that upon the picture, and its equation may therefore be considered as that of the delineation, referred to its own plane.

The equation, then, of the section in question is

$$\{cx + (c-z)x\}^2 + \{cb + (c-z)(x+a)\tan\alpha\}^2 = r^2\{c-z\}^2 \dots (6).$$

Which reduced and arranged in the usual manner, becomes

$$Ax^2 + Bxz + Cz^2 + Dx + Ez + F = 0 \dots (7).$$

Where $A = c^2$

$$B = -2cx,$$

$$C = x^2 + x + a)^2 \tan^2 \alpha - r^2$$

$$D = 2c^2x,$$

$$E = 2c\{r^2 - x^2 - b \cdot x + a\} \tan \alpha - x + a)^2 \tan^2 \alpha\}$$

$$F = c^2\{b + x - a\} \tan \alpha)^2 - r^2\}$$

... (8).

Now the co-ordinates of the centre of the curve (7) are

$$\left. \begin{aligned} \bar{x} - c &= \frac{2AE - BD}{B^2 - 4AC} \\ \bar{z} &= \frac{2CD - BE}{B^2 - 4AC} \end{aligned} \right\} \dots (9).$$

Or making in (9) the substitutions furnished by (8) we get

$$\bar{x} - c = - \frac{bc(x + a) \tan \alpha}{r^2 - x + a)^2 \tan^2 \alpha} \dots (10)$$

$$\bar{z} = - \frac{bx(x + a) \tan \alpha}{r^2 - x + a)^2 \tan^2 \alpha} \dots (11).$$

The two equations (10), (11), contain three simultaneous variables, $\bar{x}$, $\bar{z}$, x . If, therefore, we drop the marks from $\bar{x}$, $\bar{z}$, and eliminate x , between the equations, we shall obtain the equation of the centres of all the sections. To effect this, divide (11) by (10), which gives

$$\frac{\bar{z}}{\bar{x} - c} = \frac{x}{c}, \text{ or } x = \frac{c\bar{z}}{\bar{x} - c} \dots (12).$$

Insert the value of x , from (12) in eq. (10) and perform the obvious reductions. The result will be

$0 = r^2 \cos^2 \alpha (\bar{x} - c)^2 - (c\bar{z} - a\bar{x} + ac)^2 \sin^2 \alpha + bc \sin \alpha \cos \alpha (\bar{z} - a)$ which is of the second degree, and represents *any one of the conic sections*, according to the values of a , b , c , r , and α .

This subject will be resumed in a more ample manner and by a different process on a future occasion.

XVIII. QUESTION 508, by Mr. PETER MASON.

Find the equation to that surface whose property is such, that if the tangent plane at any point cuts off portions of the axes x , y , and z respectively equal to x' , y' , and z' . Then $x'^2 + y'^2 + z'^2$ shall always be equal to the same constant quantity a^2 .

FIRST SOLUTION, by Mr. MASON.

Let $z + Ax + By + C = 0$ be the equation to the plane surface, $x y z$ being the co-ordinates of the point of contact with the required surface, and let c and c_1 be the distances along the axes of x and y at which this plane cuts them $\therefore \sqrt{(a^2 - c^2 - c_1^2)}$ = distance along axis of z at which it cuts it. Hence $z + A(x - c) + By = 0$, and $z + Ax + B(y - c_1) = 0$, and hence $A = -\frac{zc_1}{(x - c)c_1 + cy}$ and $B = -\frac{zc}{(x - c)c_1 + cy}$: hence the equation to the tangent plane is $z - \sqrt{(a^2 - c^2 - c_1^2)} - \frac{zc_1(x - c) + zc(y - c_1)}{(x - c)c_1 + cy} = 0$, $\therefore \frac{zcc_1}{(x - c)c_1 + cy} =$

$$\sqrt{(a^2 - c^2 - c_1^2)} \cdot (a), \text{ or } \frac{x}{c} - 1 + \frac{y}{c_1} = \frac{z}{\sqrt{(a^2 - c^2 - c_1^2)}} \cdot (b).$$

Now the point of contact $x y z$ remains fixed whilst c and c_1 independently of each other change by indefinitely small quantities: hence the differential coefficients of equation (b) with respect to c and also c_1 , are each $= 0$, from which we find

$$c = -\frac{ax^{\frac{1}{2}}}{\sqrt{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})}} \text{ and } c_1 = -\frac{ay^{\frac{1}{2}}}{\sqrt{(x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}})}};$$

these values being put in equation (a), we have $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$, the equation required.

Cor. If $x^{\frac{m}{n}} + y^{\frac{m}{n}} + z^{\frac{m}{n}} = a^{\frac{m}{n}}$, the equation of the surface touched by the plane is $x^{\frac{m}{m+n}} + y^{\frac{m}{m+n}} + z^{\frac{m}{m+n}} = a^{\frac{m}{m+n}}$ which may be shewn as above.

SECOND SOLUTION, by Mr. T. S. DAVIES, Bath.

Generally. Let the given equation of condition be

$$\left(\frac{x'}{a}\right)^n + \left(\frac{y'}{b}\right)^n + \left(\frac{z'}{c}\right)^n = 1 \dots (1)$$

then the equation of tangent plane will be $\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 1 \dots (2)$

From (1) and (2) we have $\frac{x}{x'} + \frac{y}{y'} + \frac{z}{c\left(1 - \frac{x'^n}{a^n} - \frac{y'^n}{b^n}\right)^{\frac{1}{n}}} = 1 \dots (3)$

The partial differentials of (3) relative to x' , and y' give

$$\left\{ \begin{array}{l} 1 - \frac{x'^n}{a^n} - \frac{y'^n}{b^n} \left(= \frac{z'^n}{c^n} \right) = \left(\frac{z}{x} \cdot \frac{1}{cb} \right)^{\frac{n}{n+1}} x'^n \dots (4) \\ 1 - \frac{x'^n}{a^n} - \frac{y'^n}{b^n} \left(= \frac{z'^n}{c^n} \right) = \left(\frac{z}{y} \cdot \frac{1}{cb} \right)^{\frac{n}{n+1}} y'^n \dots (5) \end{array} \right.$$

Hence from (4) and (5) we get $x'^n = \frac{z'^n}{c^n} \cdot \left(\frac{x}{z} \cdot cb \right)^{\frac{n}{n+1}} \dots (6)$

$$y'^n = \frac{z'^n}{c^n} \cdot \left(\frac{y}{z} \cdot ca \right)^{\frac{n}{n+1}} \dots (7)$$

These values of x'^n and y'^n inserted in (1) gives us

$$1 = \frac{1}{(c^n z)^{\frac{n}{n+1}}} \left\{ \left(\frac{x}{a} \right)^{\frac{n}{n+1}} + \left(\frac{y}{b} \right)^{\frac{n}{n+1}} + \left(\frac{z}{c} \right)^{\frac{n}{n+1}} \right\} \times z'^n \dots (8)$$

This value of z'^n substituted in (6) and (7) gives the value of x'^n , y'^n : and the values of x' , y' , z' thus obtained, inserted in the terms of equation (2) furnish the following

$$\left. \begin{array}{l} \frac{z}{z'} = \left(\frac{z}{c} \right)^{\frac{n}{n+1}} \left\{ \left(\frac{x}{a} \right)^{\frac{n}{n+1}} + \left(\frac{y}{b} \right)^{\frac{n}{n+1}} + \left(\frac{z}{c} \right)^{\frac{n}{n+1}} \right\}^{\frac{1}{n}} \\ \frac{y}{y'} = \left(\frac{y}{b} \right)^{\frac{n}{n+1}} \left\{ \left(\frac{x}{a} \right)^{\frac{n}{n+1}} + \left(\frac{y}{b} \right)^{\frac{n}{n+1}} + \left(\frac{z}{c} \right)^{\frac{n}{n+1}} \right\}^{\frac{1}{n}} \\ \frac{x}{x'} = \left(\frac{x}{a} \right)^{\frac{n}{n+1}} \left\{ \left(\frac{x}{a} \right)^{\frac{n}{n+1}} + \left(\frac{y}{b} \right)^{\frac{n}{n+1}} + \left(\frac{z}{c} \right)^{\frac{n}{n+1}} \right\}^{\frac{1}{n}} \end{array} \right\} \dots (9)$$

Whence, ultimately (by 9, 1)

$\left(\frac{x}{a} \right)^{\frac{n}{n+1}} + \left(\frac{y}{b} \right)^{\frac{n}{n+1}} + \left(\frac{z}{c} \right)^{\frac{n}{n+1}} = (1)^{\frac{n}{n+1}}$ which is the general equation of the system of curve surfaces enveloped by the said variable tangent plane.

Cor. 1. If $a = b = c$; then $x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} + z^{\frac{n}{n+1}} = a^{\frac{n}{n+1}}$;
in which form the question was proposed in the Cambridge Senate House, 1829.

Cor. 2. If also $n = 2$; then $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ which is the solution of the question before us.

XIX. QUESTION 509, by R. N.

Two bodies P and Q are placed on a smooth horizontal plane, and connected by a perfect flexible inextensible thread which passes freely through a small ring in the plane, a given velocity is communicated to P in a given direction : it is required to find the equation to the curve described by P on the plane, the length of the thread being indefinite.

FIRST SOLUTION, by — MASON, of Scoulton.

Let x and y be the co-ordinates of the body P at any time, the ring being the origin, and x measured along the first portion of the string ; let also r = distance of P from the ring at time t , θ the angle between r and axis of x ; x , the distance of Q from the ring ; then $r + x$, = length of the string is constant. Let T = tension of the string.

$$\text{Then } \frac{d^2x}{dt^2} = - \frac{T}{P} \cos \theta \dots (1), \quad \frac{d^2y}{dt^2} = - \frac{T}{P} \sin \theta \dots (2)$$

and $\frac{d^2x}{dt^2} = - \frac{T}{Q}$ or $\frac{d^2r}{dt^2} = \frac{T}{Q} \dots (3)$, are the equations of the motions of P and Q. Now $x = r \cos \theta$, $y = r \sin \theta$; find $\therefore$ the

values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in functions of r and θ and multiply

(1) by $\sin \theta$ and (2) by $\cos \theta$ and subtracting we have $\sin \theta$

$$\cdot \frac{d^2x}{dt^2} - \cos \theta \cdot \frac{d^2y}{dt^2} = 0 \dots (4), \text{ also multiply (1) by } \cos \theta$$

and (2) by $\sin \theta$ and add and we have $P \cos \theta \cdot \frac{d^2x}{dt^2} + P \sin \theta$

$$\cdot \frac{d^2y}{dt^2} = - Q \cdot \frac{d^2r}{dt^2} \dots (5). \text{ Now equation (4) becomes by}$$

putting for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ their values in r and θ , $\frac{d^2\theta}{dt^2} + \frac{2dr}{dt}$

• $\frac{d\theta}{dt} = 0 \dots (6)$, and (5) becomes $Q \cdot \frac{d^2 r}{dt^2} = Pr \left(\frac{d\theta}{dt} \right)^2 - P$

• $\frac{d^2 y}{dt^2} \dots (7)$. From (6) $\frac{k^2}{r^2} = \frac{d\theta}{dt}$ and $k^2 = au \sin \alpha$, a being the original value of (r) , u the velocity communicated to P and α the $\angle$ that velocity makes with (a) produced,

By (7) $\frac{P+Q}{P} \frac{d^2 r}{dt^2} = \frac{a^2 u^2 \sin^2 \alpha}{r^3} \therefore \frac{P+Q}{P} \left(\frac{dr}{dt} \right)^2 = \frac{P+Q}{P}$

$u^2 \cos^2 \alpha = a^2 u^2 \sin^2 \alpha \left\{ \frac{1}{a} - \frac{1}{r^2} \right\}$. Put for dt its value from () and we have

$$\sqrt{\frac{P+Q}{P}} \cdot \frac{dr}{r d\theta} = \sqrt{\left\{ \frac{P+Q \cos^2 \alpha}{P a^2 \sin^2 \alpha} \cdot r^2 - 1 \right\}}.$$

Hence by integrating $\theta =$

$$\sqrt{\frac{P+Q}{P}} \cdot \left\{ \sec^{-1} \frac{r}{a} \sqrt{\frac{P+Q \cos^2 \alpha}{P \sin^2 \alpha}} - \sec^{-1} \sqrt{\frac{P+Q \cos^2 \alpha}{P \sin^2 \alpha}} \right\}$$

which is the equation to the elliptic spiral which has been fully treated on by Cotes.

This question might have been solved more briefly from the consideration that P moves round a *fixed point* towards which action of the string always tends, and $\therefore P$ describes equal areas in equal times round this point, and also the principle of *vis viva*. But I thought it better to do it from first principles.

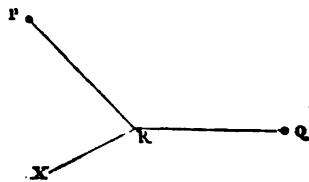
SECOND SOLUTION, by Mr. W. S. B. WOOLHOUSE, North Shields.

The motion of Q must evidently be always in the direction QR of the thread. Assume $PR = x$ and the $\angle PRX$ of PR with any fixed line $RX = \phi$. Then $x \frac{d\phi^2}{dt^2}$

• $\frac{PQ}{P+Q} =$ the tension of the thread, is the impressed force acting

on both bodies. But $Q \frac{d^2 x}{dt^2}$ is the effective force on Q

$$\therefore x \frac{d\phi^2}{dt^2} \cdot \frac{PQ}{P+Q} = Q \frac{d^2 x}{dt^2} \text{ and } x \frac{d\phi^2}{dt^2} \cdot \frac{P}{P+Q} = \frac{d^2 x}{dt^2} \text{ or}$$



$$rx \frac{d^2 p}{dt^2} = \frac{d^2 x}{dt^2} \left(r \text{ denoting } \frac{p}{p+q} \right) \dots (1)$$

Again since the tension of the thread, which alone acts on P, is towards the fixed point R it must be subject to the laws of central forces and consequently describe sectorial areas about that point proportional to the times, $\therefore x^2 d\phi = h dt$ (h being twice the area described in an unit of time) and $\frac{d\phi}{dt} = \frac{h}{x^2} \dots (2)$

Which substituted in (1) gives $\frac{r h^2}{x^3} = \frac{d^2 x}{dt^2}$. Multiply by $x dx$

and integrate and $rh^2 \left(\frac{1}{c^2} - \frac{1}{x^2} \right) = \frac{dx^2}{dt^2} \dots (3)$. $\therefore \frac{dx}{dt} =$

$\frac{h}{c} \sqrt{r} \cdot \frac{\sqrt{(x^2 - c^2)}}{x}$. Divide equation (2) by this and $\frac{d\phi}{dx} =$

$\frac{1}{\sqrt{r}} \cdot \frac{c}{x\sqrt{(x^2 - c^2)}}$, $\therefore d\phi = \frac{1}{\sqrt{r}} \cdot \frac{cdx}{x\sqrt{(x^2 - c^2)}}$. Which in-

tegrated gives $\phi - c' = \frac{1}{\sqrt{r}} \cdot \sec^{-1} \frac{x}{c}$. Where the value of c is determinable from the original motion of the bodies, by means of equation (3); and the value of c' may thence be found from their original position by means of the last equation itself.

When x is infinite we have $\phi - c' = \frac{1}{\sqrt{r}} \cdot \frac{\pi}{2}$ a finite value and therefore the curve has an asymptote.

OTHERWISE.

Assume $PR = x$ and the angular velocity of RP round $R = w$. Then xw^2P is the centrifugal force of P and therefore $xw^2 \frac{PQ}{P+Q}$ is the tension of the thread. Now as this tension which acts on P is directed to the fixed point R the body P must be subject to the laws of central forces round R , and consequently describe sectorial areas round that point proportional to the lines, $\therefore x^2 w = \text{const.} = h$ (twice the area described in an unit of time) and $w = \frac{h}{x^2}$, therefore the tension of the

thread $= \frac{h^2}{x^3} \cdot \frac{PQ}{P+Q}$ which divided by P gives the accelerating

force on P towards $R = \frac{h^2}{x^3} \cdot \frac{Q}{P+Q} = \frac{m}{x^3}$ (m being $= \frac{Q}{P+Q} h^2$).

Hence P is drawn towards R by a force which is reciprocally as the cube of the distance x ; and since $\frac{m}{h^2} = \frac{Q}{P+Q} < 1$, the curve described by P round R is the same as Species V. Whewell's Dynamics, p. 45; whose equation as there determined is

$$x = \frac{c}{\cos \gamma (\phi - c')} = c \sec \gamma (\phi - c').$$

Where ϕ denotes the polar angle with any axis, and $\gamma =$

$$\sqrt{1 - \frac{m}{n^2}} = \sqrt{\frac{P}{P+Q}}.$$

Mr. DAVIES also furnished a Solution to the Question.

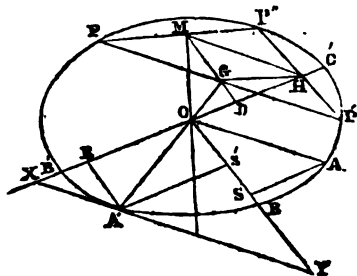
XX. PRIZE QUESTION 510, by Mr. DAVIES.

If a polygon be inscribed in a conic section, all whose sides but one are parallel to given lines, that side will either be parallel to a given line or touch a given conic section.

This Question for the Triangle was the Prize Question in the Mathematical Companion for 1813, but the Solutions there given were Algebraical: It was, in the form here given, the Prize Question in the Leeds Correspondent for July, 1822, but the principle of Orthographic* Projection there employed cannot properly apply except to the ellipse. It is here proposed to demonstrate the property geometrically without the intervention of any kind of projection or algebraic calculus.

FIRST SOLUTION, by Mr. W. S. B. WOOLHOUSE.

In the case of the ellipse let PP' and $P'P''$ represent two chords which are drawn parallel to two given lines or, (which amounts to the same,) to two given semidiameters OA and OB . Also let OA' bisecting PP' at G be the conjugate semidiam. to OA ; and $B'OC'$ bisecting $P'P''$ at H the conjugate diam. to OB . Bisect $P''P$ at M and draw MGD and the ordinate $A'R$ both parallel to $P'P''$ or OB ; and also draw



* See (*) at the end of Mr. Davies's Solution.

As and A's' parallel to OB' or OC'; join MH and GH and let the tangent XA'Y at the point A' meet the semidiams. OB' and OB at X and Y.

Then the sides of the triangle PP'P'' being bisected at the points G, H and M, it follows that MB = PG; and by the property of the ellipse.

$$OA^2 : OA'^2 :: OA'^2 - OG^2 : PG^2 = MH^2$$

Also by sim. Δs $OA^2 : OS^2 :: MH^2 : MD^2$

and $OR^2 : OA'^2 :: OR^2 - OD^2 : OA'^2 - OG^2.$

By combining these proportions we get,

$$OR^2 : OS^2 :: OR^2 - OD^2 : MD^2.$$

Which analogy shews the point M to be situated on the arc of an ellipse whose semi-conjugate diameters are OR and OS.

By a property of conic sections $XA' \cdot A'Y = OA'^2$; and since the triangles XOY and ASO are similar, we hence infer that $OS' \cdot s'Y = OS^2$. But by another principle of conic sections $OS' \cdot s'Y : A's'^2 :: OB^2 : OA'^2$. Substituting therefore for $OS' \cdot s'Y$ its equal, OS^2 , and inverting the proportion, we derive, $A's'^2 = OR^2 : OS^2 :: OB'^2 : OB^2$.

$$\therefore OR : OS :: OB' : OB.$$

It therefore appears that the ellipse which is the locus of M is not only concentric, but likewise similar and similarly situated with the given one; and consequently that PP'' is a tangent to this ellipse at the point of bisection M.

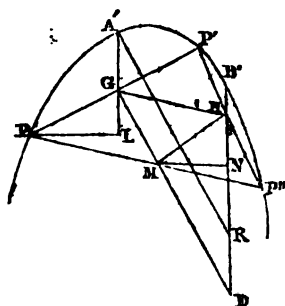
Hence the inverse of this principle, viz. "If two concentric, similar and similarly situated ellipses be given; and any tangent PMP'' be drawn to the interior; and P''P' drawn parallel to a given line. Then will PP' be parallel to another given line." For P''P' being parallel to a given line, the parallel semidiameter OSB, intersecting the interior ellipse at s and the exterior one at B, is given. Therefore SA drawn parallel to OC', which is an ordinate to the exterior ellipse and a tangent to the interior, is given; and consequently the semidiam. OA is given. Therefore PP' is parallel to a given diameter OA.

The foregoing demonstration applies equally to the hyperbola, the only difference being a change of signs in the former proportions.

But in the case of the parabola it does not apply so elegantly, since all the diameters are parallel to each other.

In the annexed figure let PP' and $P'P''$ be any two chords of a given parabola which are drawn parallel to two given right lines and consequently whose diameters $A'G$ and $B'HD$ are given.

As before bisect PP'' at M and draw GMD and AR parallel to $P'HP''$ meeting $B'HD$ at B and R ; join MH and GH and demit the perps. PL and MN on the diameters $A'G$ and $B'HD$.



Then, the sides of the triangles PGL and MHN being respectively parallel and $MH = PG$, they are both similar and equal, $\therefore PL = MN$. Supposing P to denote the parameter to the axis we have by a well known property of the parabola, $P \cdot A'G = PL^2$ which is therefore equivalent to $P \cdot RD = MN^2$. Since R is a fixed point and MD ordinately applied to the diameter $B'D$, the latter equation shews the locus of M to be an equal parabola with a coincident axis, and passing through R ; the distance of the vertices or foci being equal to $B'R$, because B' is transferred to R in direction of the axis; and since PMP'' is ordinately situated to a diameter passing through M , it is a tangent at that point.

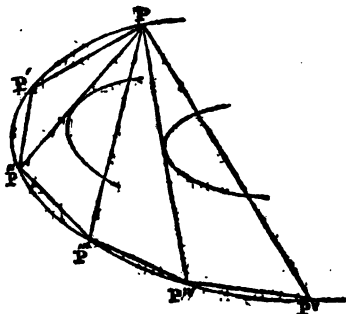
This property may be inverted as before. Thus, "if any chord PP'' touches the periphery of an equal parabola with a coincident axis, and $P'P'$ be drawn parallel to a given line, or ordinately to a given diameter; then will PP' be parallel to a given line or ordinately applied to another given diameter."

For the diameter $B'D$ being given, its intersection R with the interior parabola is given; and therefore RA' which is parallel to the same given line with $P'P'$ is given. Therefore the point A' and the diameter $A'G$ are given; and consequently PP' is ordinately applied to a given diameter $A'G$, and therefore parallel to a given right line, viz. the tangent at the given point A' or any line parallel to it.

The above principle together with its inversion, which are now generally established, may be successfully employed in demonstrating the property enunciated in the question.

Let $PP'P''P'''P'''\dots P$ represent the polygon, all of whose sides except the last are drawn parallel to given lines; and join P with each of the other angular points. By the principle now laid down; since PP' and $P'P''$ are drawn parallel to given lines, PP'' will touch a given, similar and si-

ilarly situated conic section; and, by the inverse of the same principle, PP'' touching this conic section and $P''P'''$ being drawn parallel to a given line, PP''' is parallel to a given line. Again PP''' being parallel to a given line and $P'''P''''$ being also parallel to a given line, PP'''' will touch a given conic section; and PP'''' touching this conic section and $P''''P''$ being drawn parallel to a given line, PP'' will be parallel to a given line, &c. &c.



Hence it is obvious that PP' is parallel to a given line; PP'' touches a given conic section; PP''' is parallel to a given line; PP'''' touches a given conic section; PP'' is parallel to a given line, &c. &c. alternately.

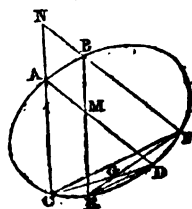
It therefore appears that when the number of sides of the polygon is even, the last side will be parallel to a given line; and when the number of sides is odd the last side will touch a given concentric, similar and similarly situated conic section. The position of the line or the magnitude of the said conic section may be ascertained or constructed by means of the foregoing principle and its inversion.

SECOND SOLUTION, by Mr. DAVIES.

LEMMA I. When in any given conic section the proximate chords bisect one another, they are tangents to another conic section, concentric, similar and similarly placed with the given one. This is an immediate corollary from what is shewn in Robertson's *Sectiones Conicæ*, p. 266; and Dr. Hugh Hamilton's *Conic Sections*, p. 165.

LEMMA II. If any two triangles, ACD , BEF , be inscribed in a conic section so that the sides AC , AD be parallel respectively to the sides BE , BF : then the lines CD , EF being drawn, these also will be parallel to one another.

For, let AC , BF intersect in N , and AD , BE in M . Then by Conics,



$$AN \cdot NC : FN \cdot NB :: BM \cdot ME : AM \cdot MD.$$

But by parallels, we have $AN = BM$, and $AM = BN$. Hence, also,

$$CN : NF :: EM : MD.$$

This proportion combined with the parallelism of the same lines, establishes the parallelism also of CF , ED .

LEMMA III. Generally, the lines CD , EF are unequal, and therefore also the segments into which CD , EF mutually divide one another in G . As, however, A , B approach each other more closely, the lines CF , ED , also, approach more closely: and the point G comes nearer to the middle both of CD and EF . At the limit ED is equal to CF , and the proximate chords CD , EF whilst they coalesce, also bisect one another in G .

Remark. Whatever be the distance of the points A , B in the case when AC , AD are conjugate chords, the bases of the triangles become diameters, and therefore bisect each other in the centre of the circumscribing conic section. The inequality of the segments, save in this case, and that of the ultimate proximity of the bases, is universal.

DEMONSTRATION OF THE PROPOSITION.

(A). *Let the inscribed polygon be the TRIANGLE.* In the figure of (Lem. II.) let AC , AD be parallel to the given lines. Then, (Lem. III.) the consecutive bases bisect one another; and hence, (Lem. I.) they touch a conic section, concentric, similar and similarly situated, with the given one. The third side of the triangle, then, touches a given conic section. *Q. E. D.*

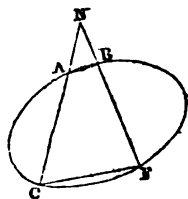
(B). *Let the inscribed figure be a QUADRILATERAL, three of whose sides CA , AB , BF are parallel to given lines. The fourth side CF shall be parallel to a given line.*

For, let CA , BF meet in N . Then by conics

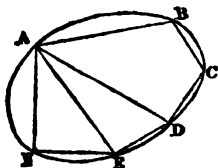
$CN \cdot NA : FN \cdot NB$ is a given ratio; and because the triangle ANB is given in species, $AN : NB$ is a given ratio. Hence also

$$CN : NF \text{ is a constant ratio.}$$

This combined with the parallel motion of CN , NF shews that CF is also parallel to a fourth given line. *Q. E. D.*



(c). *Let the figure be a PENTAGON, four of whose sides, AB, BC, CD, DE are parallel to four given lines. Then EA shall touch a conic section.*



For it may be divided by the diagonal AD into a quadrilateral and a triangle. The side AD of the quadrilateral ABCD will, by case (B), be parallel to a given line; and the base EA of the triangle ADE, will, therefore, and by (A) touch a given conic section. Q. E. D.

(d). *The HEXAGON is divisible into two quadrilaterals; and it may be shewn that the side EF is parallel to a given line.*

We thus see that polygons of an even number of sides (n) are divisible into a certain number of quadrilaterals ($= \frac{n}{2} - 1$), and hence that the last side is parallel to a given line.

We see also that polygons of an odd number of sides (n) is divisible into a number of quadrilaterals ($\frac{n-1}{2} - 1$) and one triangle: and hence that the last side touches a conic section concentric, similar and similarly placed with the given conic section. The proposition is thus fully demonstrated, as was required, without the intervention of any kind of projection or algebraic calculus.

Note. The frequent mistakes that have been made in transferring properties from the circle to the conic sections by means of perspective induced me to draw up a solution of the question on the principles of scenographic projection: more especially where parallel lines enter into the statement. The length to which that paper necessarily extends renders it unsuitable to the present section of the Repository; and, hence, I shall reserve it for a future communication to the section of Original Memoirs.

I have also investigated the proposition by means of the Analytical Geometry, considering the first angle of the polygon as a variable parameter, to be eliminated between the last side and its differential. I have not deemed it necessary, however, to insert it here, though the symmetry of the results may perhaps induce me to annex it to the perspective solution already mentioned.

(*) Mr. Davies desires us to state—that, speaking from memory at the time he wrote the note, (p. 49.) he inadver-

tently put down *Orthographic* for *Scenographic Projection*. The consequence of which is that the demonstration is not correct even for the ellipse; the learned author of that solution having, by some means, unaccountably overlooked the simple fact, that the common centre of two concentric circles seen perspective is not the centre of the two elliptic representations—that the ellipses are not, indeed, concentric—and that the several lines which form the sides of the polygon, when thus projected, cease to be parallel to their original positions. *En.*

NOTICES.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON.

For the years 1819, 1820, 1821, 1822, 1823, 1824, 1825, 1826, 1827, 1828, 1829, and 1830.

In these volumes are contained the following papers relating to Mathematics.

The Vol. for 1819, contains—1. On the laws which regulate the absorption of polarized light by double refracting crystals. By Dr. Brewster.—2. On the action of crystallized surfaces upon Light. By the same.—3. Remarks on the probability of error in physical observations; and on the density of the earth; considered especially with regard to the reduction of experiments on the pendulum. By Thomas Young, M. D.—4. On the Anomaly in the variation of Magnetic Needles as observed on Shipboard. By William Scoresby, Jun. Esq.—5. On the irregularities observed in the direction of the Compass-Needles of his Majesty's Ships, *Isabella* and *Alexander*, in their late Voyage of Discovery, and caused by the attraction of the Iron contained in the Ships. By Captain Edward Sabine.—6. Observations on the Dip of the Magnetic Needle, and on the intensity of the Magnetic Force. By the same.—7. The Results of Observations made at the observatory of Trinity College, Dublin, for determining the obliquity of the Ecliptic, and the Aberration of Light. By Dr. Brinkley.—8. On some New Methods of investigating the sums of several classes of Infinite Series. By Charles Babbage, Esq.—9. A New Method of solving Numerical Equations of all Orders by Continuous Approximation. By W. G. Horner, Esq.—10. An Account of

Experiments for determining the Variation in the Length of the Pendulum vibrating seconds at the principal Stations of the Trigonometrical Survey of Great Britain. By Captain H. Kater,

The Vol. for 1820 contains, 1. On the Action of Crystallized Bodies on Homogeneous Light, and on the causes of the Deviation from Newton's Scale in the Tints which many of them develop on Exposure to a polarized Ray. By J. F. W. Herschel, Esq. 2. On the Errors in Longitude as determined by Chronometers at Sea, arising from the action of the Iron in Ships upon Chronometers. By George Fisher, Esq. 3. A Sketch of an Analysis and Notation applicable to the Estimation of the Value of Life-Contingencies. By Benjamin Gompertz, Esq.

The Vol. for 1821 contains, 1. An Account of the comparison of various Standards of British Linear measure. By Captain H. Kater. 2. An Account of a Micrometer made of Rock-Crystal. By G. Dolland, F.R.S. 3. The Bakerian Lecture. On the best kind of Steel and Form for a Compass-Needle. By Captain H. Kater. 4. An Account of Experiments to determine the Times of Vibration of the Pendulum in different Latitudes. By Captain Edward Sabine. 5. On the Aberrations of Compound Lenses and Object Glasses. By J. F. W. Herschel, Esq. 6. On the mean Density of the Earth. By Dr. Charles Hutton. 7. Calculations of some Observations of the Solar Eclipse, September 7, 1820. By Mr. Charles Bunker. 8. An Account of the Re-measurement of the Cube, Cylinder, and Sphere, used by the late Sir George Shuckburgh Evelyn, in his enquiries respecting a Standard of Weights and Measures. By Capt. Kater. 9. An Account of Observations made with the Eight Feet Astronomical Circle, at the Observatory of Trinity College, Dublin, since the beginning of the year 1818, for investigating the effects of Parallax and Aberration on the Places of certain fixed Stars; and also the comparison of these with former Observations for determining the effects of Lunar Nutation, &c. By Dr. Brinkley. 10. On the effects produced in the Rates of Chronometers by the Proximity of Masses of Iron. By Peter Barlow, Esq.

The Vol. for 1822, contains, 1. The Bakerian Lecture. An Account of Experiments to determine the amount of the Dip of the Magnetic-Needle in London. By Captain Sabine. 2. On the Concentric Adjustment of a Triple Object-Glass. By W. Hyde Wollaston, M.D. 3. Observations of a Comet seen at Valparaiso in 1821. By Captain Basil Hall. 4. Elements of Captain Hall's Comet. By Dr. Brinkley. 5. On a

Dérangement in the Mural Circle at the Royal Observatory. By John Pond, Esq. Astronomer Royal. 6. **On the Finite extent of the Atmosphere.** By W. H. Wollaston, M.D. 7. **On the Expansion in a Series of the Attraction of a Spheroid.** By James Ivory, M. A. 8. **On the late Extraordinary Depression of the Barometers.** By Luke Howard, Esq. 9. **On anomalous Magnetic Action of Hot Iron between the White and Blood-red Heat.** By Peter Barlow, Esq. 10. **Observations for ascertaining the length of the Pendulum at Madras.** By John Goldingham, Esq. 11. **Of the Geographical situation of the Presidencies Calcutta, Madras, and Bombay, in the East Indies.** By the same. 12. **Of the Difference of Longitudes taken by Chronometer, and by Corresponding Eclipses of Jupiter's Satellites, with some supplementary information relative to Madras, Bombay, and Canton; as also the latitude and longitude of Point de Galle and the Friar's Hood.** By the same.

The Vol. for 1823, contains, 1. **Corrections applied to the great Meridional Arc, extending from Lat. $8^{\circ} 9' 38''.39$ N. to $18^{\circ} 3' 23''.64$ N. to reduce it to the Parliamentary Standard.** By Lieut. Col. Lambton. 2. **On the Changes which have taken place in the Declination of some of the principal fixed Stars, and Appendix to the same.** By John Pond, Esq. Astronomer Royal. 3. **On the Parallax of a Lyrae.** John Pond, Esq. Astronomer Royal. 3. **Observations on the Heights of Places in the Trigonometrical Survey of Great Britain, and upon the Latitude of Arbury Hill.** By B. Bevan, Esq. 5. **On the Chinese Year.** By J. F. Davis, Esq. 6. **Experiments for ascertaining the Velocity of Sound at Madras.** By John Goldingham, Esq. 7. **Details of Experiments made by Captain Basil Hall and Lieut. Henry Foster, with an invariable Pendulum in London; at the Galapagos Islands in the Pacific Ocean, near the Equator; at San Blas de California, on the N. W. Coast of Mexico; and at Rio Janiero in Brazil. With an Appendix, containing the Second Series of Experiments in London, on their return.** 8. **An Account of Experiments made with an invariable Pendulum at New South Wales.** By Major General Sir Thomas Brisbane, K. C. B. 9. **Observations and Experiments on the Daily Variation of the Horizontal and Dipping Needles under a reduced directive Power.** By Peter Barlow, Esq. 10. **On the Diurnal Variation of the Horizontal Needle when under the influence of Magnets.** By S. H. Christie, Esq. M. A. 11. **On Astronomical Refractions.** By J. Ivory, A. M. 12. **On Certain Changes which appear to have taken place in the Position of some of the principal fixed Stars.** By John Pond, Astronomer Royal.

The Vol. for 1824, contains, 1. On the North Polar Distances of some of the principal fixed Stars. By Dr. Brinkley. 2. On the Figure requisite to maintain the Equilibrium of a Homogeneous Fluid Mass that revolves on an Axis. By James Ivory, A.M. 3. On a Finite and exact Expression for the Refraction of the Atmosphere, nearly resembling that of the Earth. By Thomas Young, M.D. 4. A Comparison of Barometrical Measurement, with a Trigonometrical Determination of a Height at Spitzbergen. By Captain Edward Sabine. 5. Experimental Inquiries relative to the Distribution and Changes of the Magnetic Intensity in Ships of War. By George Harvey, Esq. 6. A short Account of some Observations made with Chronometers, in two Expeditions sent out by the Admiralty for ascertaining the Longitude of Madeira and Fal-mouth. By Dr. Tyarks. 7. Of the effects of the Density of Air on the rates of Chronometers. By George Harvey, Esq. 8. An Account of Experiments on the Velocity of Sound, made in Holland. By Doctors G. H. Moll and A. Vanbeek, of the University of Utrecht. 9. Catalogue of nearly all the principal fixed Stars between the Zenith of Cape Town and the South Pole, reduced to the 1st of January, 1824. By the Rev. Fearon Fallows. 10. Remarks on the Parallax of α Lyrae. By Dr. Brinkley.

The Vol. for 1825, contains—1. On the effects of Temperature on the Intensity of Magnetic Forces; and on the Diurnal Variation of the Terrestrial Magnetic Intensity. By Samuel Hunter Christie, Esq. of the R. M. Academy. 2. A general method of Calculating the Angles made by any Planes of Crystals and the Laws according to which they are formed. By the Rev. William Whewell. 3. Explanation of an Optical deception in the appearance of the Spokes of a Wheel seen through a vertical aperture. By Dr. P. M. Roget. 4. On a new Photometer, with its application to determine the relative intensities of Artificial Light. By Wm. Richter, Esq. 5. The Description of a Floating Collimator. By Capt. Henry Kater. 6. An Experimental enquiry into the radiant heating effects of caloric from Terrestrial Sources. By Baden Powell, Esq. 7. On the Temporary Magnetic effect induced in Iron Bodies by rotation. By Peter Barlow, Esq. 8. On the Magnetism of Iron arising from its rotation. By S. H. Christie, Esq. 9. Some Account of the Transit Instrument made by Mr. Dollond, and lately put up at the Cambridge Observatory. By Robert Woodhouse, Esq. 10. An Account of the Repetition of M. Arago's Experiments on the Magnetism manifested by various substances during the act of rotation. By C. Babbage,

and J. F. W. Herschel, Esqrs. 11. On the Magnetism developed in Copper and other Substances during rotation. By S. H. Christie, Esq. 12. On the Annual Variations of some of the fixed Stars. By J. Pond, Esq. 13. On the Nature of the Function expressive of the Law of Human Mortality, and on a new mode of determining the value of Life Contingencies. By Benjamin Compertz, Esq.

The Vol. for 1826, contains—1. Observations on the apparent Distances and Positions of 458 double and triple Stars, made in the years 1823, 1824, and 1825; together with a re-examination of 95 Stars of the same description, the distances and positions of which were communicated in a former Memoir. By James South, Esq. 2. An Account of the Construction and Adjustment of the New Standard Weights and Measures of the United Kingdom of Great Britain and Ireland. By Captain Kater. 3. On the Transit Instrument of the Cambridge Observatory; being a Supplement to a former Paper. By Robert Woodhouse, Esq. 4. Account of a series of Observations, made in the summer of the year 1825, for the purpose of determining the difference of the Meridians of the Royal Observatories of Greenwich and Paris. By J. F. W. Herschel, Esq. 5. On the Magnetic Power of the more refrangible Solar Rays. By Mrs. Somerville. 6. On the Constitution of the Atmosphere. By John Dalton, Esq. 7. On the Mathematical Theory of Suspension Bridges, with Tables for facilitating their Construction. By Davis Gilbert, Esq. 8. On the Magnetic influence of the Solar Rays. By S. H. Christie, Esq. 9. On a method of expressing by Signs the Action of Machinery. By Charles Babbage, Esq. 10. On the Parallax of Fixed Stars. By J. W. Herschel, Esq. 11. A Formula for expressing the decrement of Human Life. By Dr. Thomas Young. 12. Results of the Application of Captain Kater's Floating Collimator to the Astronomical Circle at the Observatory of Trinity College, Dublin. By Dr. Brinkley. 13. On the manner of facilitating Observations of Distant Stations in Geodetical Operations. By Lieut. Drummond. 14. Some Experiments relative to the passage of radiant heat through glass screens. By Baden Powell, Esq. 15. On the discordances between the Sun's observed and computed right ascension, as determined at the Blackman-street Observatory, in the year 1820 and 1821; with experiments to shew that they did not originate in instrumental derangement. Also a description of the seven feet Transit by which the Observations were made. By James South, Esq. 16. On the Electric and Magnetic Rotations. By Charles Babbage, Esq. 17. On the progressive

compression of Water by high degrees of force, with some trials of its effects upon other Fluids. By J. Perkins, Esq. 18. On the figure of the Earth. By George Biddell Airy, Esq. 19. Account of Experiments made with an invariable Pendulum at the Royal Observatory at Greenwich and at Port Bowen. By Lieut. Henry Foster. 20. Experiments on the Diurnal Variation of the Magnetic Needle at the Whale-Fish Islands, Davis's Straits. By the same. 21. Magnetical Observations at Port Bowen, &c. A. D. 1824-5, comprehending Observations on the Diurnal Variation and Diurnal Intensity of the Horizontal Needle, made at Woolwich and at different stations within the Arctic Circle. By Capt. Parry and Lieut. Foster. 22. Abstract of the Daily Variation of the Magnetic Needle. By Lieut. Foster. 23. Observations for determining the dip of the Magnetic Needle. By Capt. W. E. Parry. 24. Observations on the Diurnal changes in the position of the Horizontal Needle, under a reduced directive power, at Port Bowen, 1825. By Lieut. Henry Foster. 25. A comparison of the Diurnal changes of intensity in the Dipping and Horizontal Needles at Port Bowen. By the same. 26. Account of the repetition of Mr Christie's Experiments on the Magnetic properties imparted to an Iron plate by rotation, at Port Bowen, in May and June, 1825. By Lieut. Foster, with Mr. Christie's remarks thereon. 27. Observations to determine the amount of Atmospheric Refraction at Port Bowen, in 1824-5. By Capt. Parry, Lieut. Foster, and Lieut. J. C. Ross.

The Vol. for 1827, contains—1. On the expediency of assigning specific names to all such functions of simple elements as represent definite physical properties; with the suggestion of a new term in Mechanics; illustrated by an investigation of the machine moved by recoil, and also by some observations on the Steam Engine. By Davies Gilbert, Esq. 2. Remarks on a Correction of the Solar Tables required by Mr. South's observations. By G. B. Airy, Esq. 3. On the mutual action of the particles of Magnetic Bodies, and on the law of variation of the Magnetic Forces generated at different distances during rotation. By S. H. Christie, Esq. 4. Corrections to the reductions of Lieut. Foster's observations on Atmospheric Refractions at Port Bowen; with Addenda to the Tables of Magnetic Intensities at the same place. By Lieut. Foster. 5. Correction of an Error in a Paper published in the Philosophical Transactions, entitled "On the Parallax of Fixed Stars. By J. F. W. Herschel, Esq. 6. On the derangement of certain Transit Instruments by the effect of Temperature. By Robert Woodhouse, Esq. 7. On the Rules and Principles for deter-

mining the dispersive ratio of Glass; and for computing the radii of Curvature for Achromatic Object Glasses submitted to the test of experiment. By Peter Barlow, Esq. 8. On the secondary deflections produced on a magnetised needle by an iron shell, in consequence of an unequal distribution of magnetism in its two branches, By Peter Barlow, Esq. 9. On the difference of the Meridians of the Royal Observatories of Greenwich and Paris. By Thomas Henderson, Esq. 10. Theory of the Diurnal variation of the Magnetic Needle, illustrated by experiments. By S. H. Christie, Esq. 11. Considerations of the objections raised against the geometrical representation of the square roots of negative quantities. By the Rev. John Warren. 12. On a new series of Periodical Colours produced by the grooved surfaces of metallic and transparent bodies. By Dr. Brewster. 13. On the reduction to a vacuum of Captain Kater's convertible Pendulum, By Capt. E. Sabine. 14. On the Geometrical Representation of the powers of quantities whose indices involve the square root of negative quantities. By the Rev. John Warren. 15. Astronomical Observations made at the Observatory at Paramatta. By Charles Rumker, Esq.

The Vol. for 1828, contains—1. Experiments to ascertain the ratio of Magnetic Forces acting on a Needle suspended horizontally in Paris and in London. By Capt. Edward Sabine. 2. On the Resistance of Fluids to Bodies passing through them. By James Walker, Esq. 3. On the Corrections in the Elements of Delambre's Solar Tables required by the Observations made at the Royal Observatory, Greenwich. By G. B. Airy, Esq. 4. Experiments to determine the difference in the length of Second's Pendulum in London and in Paris. By Capt. Edward Sabine. 5. On Captain Parry's and Lieut. Foster's Experiments on the velocity of sound. By Dr. Gerard Moll. 6. An Account of a series of Experiments made with a view to the construction of an Achromatic Telescope with a fluid concave lens instead of the usual lens of flint glass. By Peter Barlow, Esq. 7. A Catalogue of nebulae and clusters of Stars in the Southern Hemisphere, observed at Paramatta, in New South Wales. By James Dunlop, Esq. 8. Account of the Trigonometrical Operations in the years 1821, 2, 3, for determining the difference of Longitude between the Observatories of Paris and Greenwich. By Capt. Henry Kater. 9. Description of a Vertical Floating Collimator, and an account of its application to Astronomical Observations with a circle and with a zenith sector. By Capt. Kater. 10. On the Height of the Aurora Borealis above the surface of the Earth, and

especially of one seen on the 29th of March, 1826. By John Dalton, Esq. 11. A comparison of the changes of magnetic intensity throughout the day in the dipping and horizontal needles at Treurenberg Bay in Spitzbergen. By Capt. Henry Foster. 12. Experiments relative to the effect of temperature on the refractive index and dispersive power of expansible fluids, and on the influence of these changes in a telescope with a fluid lens. By Peter Barlow, Esq. 13. On the laws of the deviation of the magnetic needles towards iron. By S. H. Christie, Esq. 14. Description of a sounding-board in Attercliffe Church, invented by the Rev. John Blackburn. 15. On the magnetic influence of the Solar Rays. By S. H. Christie, Esq.

The Vol. for 1829, contains—1. On a method of comparing the Light of the Sun with that of the Fixed Stars. By Dr. William Hyde Wollaston. 2. An account of the preliminary experiments and ultimate construction of a Refracting Telescope 7·8 inches aperture, with a fluid concave lens. By Peter Barlow, Esq. 3. On the dip of the Magnetic Needle in London, in August, 1828. By Capt. Henry Sabine. 4. Experiments to determine the difference in the number of vibrations made by an Invariable Pendulum in the Royal Observatory at Greenwich, and in the house in London in which Capt. Kater's Experiments were made. By Capt. Edward Sabine. 5. Experiments on the Modulus of Torsion. By Benjamin Bevan, Esq. 6. On a Differential Thermometer. By the late Dr. Wollaston. 7. Experiments on the friction and abrasion of the surfaces of solids. By George Rennie, Esq. 8. An attempt to rectify the inaccuracy of some logarithmic formulæ. By John Thomas Greaves, Esq. 9. On the Reflection and Decomposition of Light at the separating surfaces of media of the same and of different refracting powers. By Dr. Brewster. 10. On the reduction to a vacuum of the vibrations of an Invariable Pendulum, By Capt. Ed. Sabine.

The Vol. for 1830, contains:—1. Account of the Levellings carried across the Isthmus of Panamá to ascertain the relative height of the Pacific Ocean at Panamá, and of the Atlantic at the mouth of the river Chagris. By John Augustus Lloyd, Esq. 2. On the Law of partial polarization of light by Reflexion. By Dr. Brewster. 3. On the production of regular double refraction in the molecules of bodies by simple pressure: with observations on the origin of doubly refracting structure. By the same. 4. Experiments on the Influence of the Aurora Borealis on the magnetic needle. By the Rev. J. Farquharson. 5. On the progressive improvements that have been made in the

efficiency of Steam Engines, in Cornwall; with investigations of the best method of imparting great angular velocities. By Davies Gilbert, Esq. 6. On the Laws of the polarization of light by Refraction. By Dr. Brewster. 7. On the action of the second surfaces of transparent plates upon light. By the same. 8. Observations made at the Cape of Good Hope, with the invariable pendulum (No. 4. Jones) for the purpose of determining the Compression of the Earth. By the Rev. F. Fal-lows. 9. On some properties in achromatic object-glasses applicable to the improvement of the Microscope. By J. J. Lister, Esq. 10. On the Pendulum. By J. W. Lubbock, Esq. 11. On the Theoretical Investigations of the velocity of Sound, as corrected from M. Dulong's recent experiments, compared with the results of Dr. Moll and Dr. Van Beek. By Dr. Simmons. 12. Experiments made to determine the difference of the Number of Vibrations made by an Invariable Pendulum at Greenwich and Altona. By Captain Edward Sabine. 13. Experiments to ascertain the Corrections for Variations of Temperature within the limits of the natural Temperature of the Climate of the South of England, of the Invariable Pendulum recently employed by British Observers. By the same. 14. On the Phenomena and Laws of Elliptic polarization as exhibited in the action of metals upon light. By Dr. Brewster. 15. Researches in Physical Astronomy. By J. W. Lubbock, Esq. 16. On the Error in Standards of Linear Measure, arising from the thickness of the bar on which they are traced. By Capt. Kater.

II. ROYAL SOCIETY OF EDINBURGH.

We gave the contents of the first five volumes and a half of the Transactions of this learned body, at page 195, Vol. II. we shall continue it to the tenth, which is the last completed volume.

Vol. VI. Part II.—1. Of the Solids of Greatest Attraction, or those which, among all that have certain Properties, attract with the greatest Force in a given Direction. By Mr. Playfair. 2. An Account of a very extraordinary Effect of Rarefaction, observed at Ramsgate. By the Rev. S. Vince. 3. New series for the quadrature of the Conic Sections. By Mr. Wallace. 4. On the Progress of Heat when communicated to Spherical Bodies from their Centres. By Mr. Playfair. 5. Demonstration of the Fundamental Property of the Lever. By Dr. Brewster.

Vol. VII.—1. Comparison of the North Polar Distances of 38 principal fixed stars, on the 1st of January, 1800, as determined by Observations made at Greenwich, Armagh, Palermo, Westbury, Dublin, and Blackheath. By S. Groombridge, Esq. 2. On the Optical properties of Sulphate of Carbon, Carbonate of Barytes, and Nitrate of Potash, with inferences respecting the structure of Doubly Refracting Crystals. By Dr. Brewster. 3. On a new species of Coloured Fringes produced by the reflexion of light between two plates of parallel glass of equal thickness. By Dr. Brewster. 4. Biographical Account of the late Dr. John Robison. By Professor Playfair.

Vol. VIII.—1. On the action of Transparent Bodies upon the differently coloured Rays of Light. By Dr. Brewster. 2. Description of a New Darkening Glass for solar observations, which has the property of polarising the whole of the transmitted light. By the same. 3. On the lines that divide each semidiurnal arc into six equal parts. By W. A. Cadell, Esq. 4. On some optical properties of Muriate of Soda, Fluide of Lime, and the Diamond, as exhibited in their action upon polarised Light. By Dr. Brewster. 5. On a new optical and mineralogical property of Calcareous Spar. By the same. 6. Elementary demonstration of the Composition of Pressures. By Dr. Thomas Jackson, St. Andrew. 7. On the effects of Compression and Dilatation in altering the polarising structure of Doubly Refracting Crystals. By Dr. Brewster. 8. On the laws which regulate the distribution of the polarising force in plates, tubes, and cylinders of glass that have received the polarizing structure. By Dr. Brewster. 9. On certain impressions of cold transmitted from the higher atmosphere, and an instrument to measure them. By Professor Leslie. 10. Method of determining the time with accuracy from a series of altitudes of the Sun taken on the same side of the meridian. By Major Gen. Sir Thomas Brisbane.

Vol. IX.—1. Memoir on the Repeating Circle. By Major-Gen. Sir Thomas Brisbane. 2. Memoir relating to the Naval Tactics of John Clerk, Esq. of Eldin. By Professor Playfair. 3. On circular polarization, as exhibited in the optical structure of the Amethyst, with remarks on the distribution of colouring matter in that mineral. By Dr. Brewster. 4. An examination of some Questions connected with games of Chance. By Charles Babbage, Esq. 5. Observations on the mean Temperature of the Globe. By Dr. Brewster. 6. Method of determining the latitude by a sextant or circle, with simplicity and accuracy, from circum-meridian observations, taken near noon.

By Major Gen. Sir Thomas Brisbane. 7. Description of a Magnetimeter, by a new Instrument for measuring Magnetic Attraction, and finding the dip of the needle. By William Scoresby, Esq. 8. Description of some remarkable Atmospheric Reflexions and Refractions observed in the Greenland sea. By the same gentleman. 9. Account of a remarkable structure of Apophyllite, with remarks on the optical peculiarities of that mineral. By Dr. Brewster. 10. On the application of Analysis to the Discovery of Local Theorems and Porisms. By Charles Babbage, Esq. 11. Observations on the Errors in the sea-rates of Chronometers, arising from the magnetism of their balances; with suggestions for removing the source of Error. By William Scoresby, Esq. 12. Description of a Monochromatic Lamp for microscopic purposes, with remarks on the absorption of the prismatic rays by Coloured Media. By Dr. Brewster. 13. On the absorption of light by Coloured Media, and on the Colours of the prismatic spectrum exhibited by certain flames; with an account of a ready mode of determining the absolute dispersive power of any medium. By J. F. W. Herschel, Esq.

Vol. X.—1. Astronomical Observations made at Paramatta and Sydney. By Sir T. Brisbane. 2. Remarkable case of magnetic intensity in altering the rate of a Chronometer. By Geo. Harvey, Esq. 3. Investigation of Formulæ for finding the Logarithms of Trigonometrical Quantities from one another. By W. Wallace, Esq. 4. A proposed improvement in the solution of a case in Plain Trigonometry. By the same. 5. On a new species of Double Refraction accompanying a Remarkable structure in the mineral called Analcime. By Dr. Brewster. 6. On the forms of Crystallisation of the mineral called Sulphato-tri-Carbonate of Lead. By William Hardinger, Esq. 7. On an anomalous case of vision with regard to colours. By George Harvey, Esq. 8. Observations before and after the superior conjunction of Venus and the Sun, with the Mural circle, at Paramatta. By Sir Thomas Brisbane. 9. Observations on Two Comets discovered at Paramatta, in 1824. By Mr. Rumker and Mr. Dunlop. In which are added the elements of these Orbits. By Mr. George Innes and Mr. James Gordon. 10. On the construction of Meteorological Instruments so as exactly to determine the indications during absence, at any given instant, or at successive intervals of time. By Henry H. Blackadder, Esq. 11. Results of Thermometrical Observations made at Leith Fort, every hour of the day and night during the years 1824, 5. By Dr. Brewster. 12. On the Refractive power of the two new fluids in minerals with ad-

ditional observations on the nature and properties of their Substances. 13. Description of a new Register Thermometer, without any Index. By H. H. Blackadder, Esq. 14. On a New Photometer. By William Riechtie, Esq.

III. TRANSACTIONS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

Established Nov. 15, 1819, and composed of Fellows and Honorary Members. The Fellows must be Graduates of the University of Cambridge; and no member of that University can be elected an honorary member. Three Volumes of its Transactions are completed and they contain the following Papers relating more or less to pure and mixed Mathematics,

Vol I. contains—1. On Isometrical Perspective. By Professor Farish. 2. On certain remarkable instances of deviation from Newton's scale on Tints developed by Crystals, with one axis of double refraction, on exposure to polarized light. By Mr. Herschel. 3. On the rotation impressed by plates of rock-crystal on the planes of polarization of the rays of light, as connected with certain peculiarities of Crystallization. By Mr. Herschel. 4. Observations on the notation employed in the Calculus of Functions. By Mr. Babbage. 5. On the Reduction of certain classes of Functional Equations to Equations of Finite Differences. By Mr. Herschel. 6. On the Position of the Apse of Orbits of great eccentricity. By Mr. Whewell. 7. On the regular Crystallization of Water and upon the form of its primary crystals by Dr. E. D. Clarke. 8. On the application of Hydrogen Gas, to produce a moving power in Machinery; with a description of an Engine which is moved by the pressure of the atmosphere upon a vacuum produced by explosions of Hydrogen Gas and Atmospheric Air. By the Rev. W. Cecil. 9. On a remarkable peculiarity in the Law of Extraordinary Refraction of differently coloured Rays exhibited by certain varieties of Apophyllite. By J. F. W. Herschel, Esq. 10. Notice of the Astronomical Tables of Mahommed Abibe ker el Farsi. By the Rev. Professor Lee. 11. On double Crystals of Fluor Spar. By W. Whewell, Esq.

Vol. II. contains—1. On the distribution of the Colouring matter, and on certain peculiarities in the structure and optical properties of the Brazilian Topaz. By Dr. Brewster. 2. On the Rotatory motion of Bodies. By W. Whewell, Esq. 3. A new demonstration of the Parallelogram of the Forces. By Joshua King, Esq. 4. On an Apparatus for grinding Telescopic Mirrors and Ob-

ject Lenses. By the Rev. W. Cecil. 5. On the use of Silvered glass for the mirrors of Reflecting Telescopes. By G. B. Airy, Esq. 6. Account of some Experiments made in order to determine the Velocity of Sound in passing through the Atmosphere. By Dr. Gregory. 7. On the angle made by two planes or two straight lines, referred to their oblique Co-ordinates. By W. Whewell and J. W. Lubbock, Esqrs. 8. On the Figure assumed by a Fluid Homogeneous Mass, whose particles are acted on by their mutual attraction, and by small extraneous Forces. By G. B. Airy, Esq. 9. On the Determination of the General Term of a New Class of Infinite Series. By C. Babbage, Esq. 10. On the principles and Construction of the Achromatic Eye-Pieces of Telescopes, and on the Achromatism of Microscopes. By G. B. Airy, Esq. 11. On a peculiar Defect in the Eye, and a mode of correcting it. By the same. 12. A general demonstration of the Principle of Virtual Velocities. By the Rev. J. Power. 13. On the Forms of the Teeth of Wheels. By G. B. Airy, Esq. 14. On the Influence of Signs in Mathematical Reasoning. By C. Babbage, Esq. 15. On La Place's Investigation of the Attraction of Spheroids differing little from a Sphere. By G. B. Airy, Esq. 16. On the Classification of Crystalline Combinations and the Canons by which their Laws of Derivation may be investigated. By the Rev. W. Whewell. 17. Reasons for the selection of a Notation to designate the planes of Crystals. By the same. 18. Experiments on Percussion. By B. Bevan, Esq. 19. Computed and Observed Variations of Magnetic Intensity. By R. W. Rothman, Esq.

Vol. III. contains— 1. On the Spherical Aberration of the Eye Piece of Telescopes. By G. B. Airy, Esq. 2. On Algebraic Notation. By Thomas Jarrett, Esq. 3. On the Disturbances of Pendulums and Balances, and on the Theory of Escapements. By G. B. Airy, Esq. 4. On the pressure produced on a flat plate, when opposed to a Stream of Air issuing from an Orifice. By the Rev. Robert Willis. 5. On the Calculation of Annuities, and on some Equations in the Theory of Chances. By J. W. Lubbock, Esq. 6. On the Longitude of the Cambridge University. By G. B. Airy, Esq. 7. On the Extension of Bode's Empirical Law of the Distances of the Planets from the Sun, to the Distances of the Satellites from their Primaries. By James Challis, Esq. 8. On the Focus of a Conic Section. By Pierce Morton, Esq. 9. Mathematical Exposition of some Doctrines in Political Economy. By the Rev. W. Whewell. 10. On the Vowel Sounds, and on Reed Organ-pipes. By the Rev. R. Willis. 11. On the Theory of the small vibrating

motions of Elastic Fluids. By James Challis, Esq. 12. On the Comparison of the Various Tables of Annuities. By J. W. Lubbock, Esq. 13. On a correction requisite to be applied to the length of a Pendulum consisting of a Ball suspended by a fine wire. By G. B. Airy, Esq. 14. On an Antient Observation of a Winter Solstice. By R. W. Rothman, Esq. 15. On the Crystals of Boracic Acid. By W. H. Miller, Esq. 16. On certain conditions under which a perpetual motion is possible. By G. B. Airy, Esq. 17. On the general equations of the motions of Fluids, both incompressible and compressible; and on the pressure of Fluids in Motion. By James Challis, Esq. 18. On the Crystals found in Slags. By W. H. Miller, Esq. 19. On the Improvement of the Microscope. By H. Coddington, Esq. 20. On the general properties of Definite Integrals. By R. Murphy, Esq.

IV. MEMOIRES OF THE CLASS OF MATHEMATICAL AND PHYSICAL SCIENCES OF THE NATIONAL INSTITUTE OF FRANCE.

Vol. VII. for 1805-6, contains—1. Memoire upon the Orbit of the Comet of 1770. By M. Buckhardt. 2. Researches upon the Temperature of Water at its maximum of density. By Count Rumford. 3. Analysis of Triangles traced upon the surface of a spheroid. By A. M. Legendre. 4. Note upon the Planet discovered by M. Harding. By M. J. C. Buckhardt. 5. Second correction of the Elements of the New Planet. By Buckhardt. 6. Upon the Comets of 1784 and 1762. By the same. 7. Memoire upon the Affinities of Bodies for Light, and upon the Refracting Forces of different Gases. 8. Analysis of the Labours of the Class of Mathematical Sciences of the Institute during 1805 and 1806. By Delambre. 9. Second Memoir upon the measurement of Altitudes by the Barometer. By Raymond. 10. Observations upon the passage of Mercury over the Sun, and upon the Planet Pallas. 11. Upon the Total Eclipse of the Sun, June 16, 1806. By Jerome de la Lande.

Vol. VIII. for 1806-7, contains—1. Observations on the Nebulæ in Andromeda. By Messier. 2. Observations on the Dispersion of the light of Lamps by means of screens of roughed glass, stuffs, &c. with the description of a new lamp. By Count Rumford. 3. Experiments upon the freezing of liquor, in porcelain vases, open and shut. By the same. 4. Memoir upon the construction of new Tables of Jupiter and Saturn,

calculated according to the new division of the day and of the circumference of the circle. 5. Memoire upon the influence of humidity and of heat in Atmospheric Refractions. By Biot. 6. Analysis of the Labours of the Class. By Delambre.

Voli IX. 1807-8, contains—1. Analysis of the Labours of the Class during 1808. By Delambre. 2. Report upon the Reflecting Sextant of M. Lenoir. By Buckhardt. 3. Report to the Emperor of the Progress of Science from 1789 to 1808. 4. Distribution of the Mathematical Prizes. 5. Memoir upon the theory of the variations of the elements of the Planets; and in particular the variation of the Major Axes of their Orbits. By Lagrange. 6. Third Memoire upon the measurement of Altitudes by the Barometer. By Raymond. 7. Memoire upon the general theory of the Arbitrary Constants in Mechanical Problems. By Lagrange. 8. Supplement to the same Memoire. By the same. 9. General Formulæ for the perturbation of Superior Orders. By Buckhardt. 10. Memoires upon several means of perfecting the tables of the Moon, By the same.

Vol. X. 1808-9, contains—1. Analysis of the Labours of the Class. By Delambre. 2. Prize for 1810. 3. Distribution of the Prizes. 4. Researches upon the extraordinary Refractions which are observed very near the Horizon. By Biot. 5. Memoire upon the motion of light in Diaphanous Media. By Laplace. 6. Second Memoire upon the theory of the variation of Arbitrary Constants in Mechanical Problems, in which is simplified the application of the general formulæ to these problems. By Lagrange. 7. Memoire upon the Approximations of formulæ which are functions of very high numbers, together with their application to Probabilities. By Laplace. 8. Researches upon several kinds of Definite Integrals. By Legendre. 9. Fourth Memoire upon the measure of Altitudes by the Barometer. By Raymond. 10. Examination of the Different methods of laying down (*orienter*) a chain of Triangles. By Buckhardt. 11. Supplement to the Memoire upon approximating to the formulæ which are functions of high numbers. By Laplace.

Vol. XI. 1810, contains—1. Memoire upon Definite Integrals. By Laplace. 2. History of the Class. By Delambre. 3. Report made to the Class, by a Commission. 4. Memoire upon some new Phenomena in Optics. By Malus. 5. Memoire upon the Axe of Refraction of Crystals of organised substances. By the same. 6. Upon the method of Least Squares, for finding the most probable mean between the results of several observations. By Legendre.

Vol. XII. 1811, contains—1. Memoire upon the Distribution of Electricity upon the surface of Conducting Bodies. By Poisson. 2. Memoire upon a remarkable modification suffered by rays of light in their passage through diaphanous bodies. By Arago. 3. Memoire upon new relations which exist between reflexion and polarization of light in Crystalline bodies. By Biot. 4. A History of the Class. 5. Second Memoire upon the Distribution of Electricity upon the surface of Conducting Bodies. By Poisson.

Vol. XIII. 1812, contains—1. History of the Labours of the Class. 2. Memoires upon a new kind of oscillation which the molecules of light undergo in traversing certain crystals. By Biot. 3. Memoire upon a new application of the Theory of the oscillations of light. By the same. 4. Result of Meteorological Observations made at Clermont-Ferrand, from the month of June 1806, to the end of 1813. By Raymond. 5. Memoire upon Elastic Surfaces. By Poisson.

* * The History of the Class in this year contains the lives of Malus and Lagrange.

Vol. XIV. for 1813, 14, 15, contains—1. History of the Class. 2. Barometric Levelling of *Monts-Dores* and *Monts-Domes*. By Raymond. 3. Application of the Levelling in the Department of Puy-de-Dome to the Physical Geography of this part of France. By the same. 4. Demonstration of the General Theorem of Fermat upon Polygonal Numbers. By A. L. Cauchy. 5. Observations on the nature of the Forces which separate the Luminous Rays in the Crystals of Double Refraction. By Biot. 6. Demonstration of a Theorem from which we may deduce all the laws of ordinary and extraordinary Refraction. By Ampere. 7. Memoire upon the motion of Fluids in Capillary tubes, and the influence of temperature upon this motion. By Girard. 8. Extract from some new Researches upon Algebra, and upon the Theory of Numbers. By Poinsoit.

V. JOURNAL DE L'ÉCOLE POLYTECHNIQUE.

Only two *CAMEIRS* of this work have been published since our last report, viz. XVIII. and XIX. Their contents are as follow :—

CAMEIR XVIII.—1. Memoire containing the application of the theory expounded in No. 17*. of the Journal of the Poly-

technique School, to the integration of partial differential equations of the first and second order. By M. Ampere. 2. Continuation of the Memoire upon Definite Integrals, inserted in the two preceding volumes of this Journal. By M. Poisson. 3. Memoire upon the application of Algebra to the Theory of Numbers. By M. Poinso. 4. Upon the imaginary roots of equations. By M. Cauchy. 5. Upon the manner of expressing functions in a series of periodical quantities, and of the use of this transformation in the solution of problems.

CASSEIR XIX.—1. Memoire upon the distribution of heat in bodies. By M. Poisson. 2. Addition to the preceding memoire and to the memoire upon the manner of expressing functions by series of periodic quantities. By the same. 3. Memoire upon the general principles of Dynamics, and especially upon a new principle of general Mechanics. By M. J. Binet. 4. Memoire upon the *Courbes de Raccordement*. By M. Brianchon. 5. Memoire upon a particular kind of motion in Fluids. By M. Augustin Cauchy. 6. Memoire upon the integration of Linear equations of partial differentials. By M. Poisson. 7. Second Memoire upon the distribution of heat in solid bodies. By M. Poisson. 8. Completion of the memoire upon definite integrals, and upon the summation of series inserted in this Journal. 9. Memoire upon the integration of linear equations of partial differentials having constant coefficients. By M. Augustin Cauchy.

. We accidentally omitted in the list of Mathematical Papers in the Philosophical Transactions for 1829, the following ones.

1. Considerations on the objections raised against the Geometrical Representation of the square roots of negative quantities. By the Rev. John Warren. 2. On a new series of periodical colours produced by the grooved surfaces of metallic and transparent bodies. By Dr. Brewster. 3. On the geometrical representation of the powers of quantities, whose indices involve the square roots of negative quantities. By the Rev. John Warren. These come between the articles numbered 4 and 5 in our list, and in Part 2 of the Transactions for that year. Ed.

ARTICLE II.

Solutions to Questions proposed in No. XXI.

I. QUESTION 511, by Mr. T. S. DAVIES, Bath.

In what curve is the normal equal to the radius of curvature?

FIRST SOLUTION, by — MASON, of Scoulton, Norfolk.

Let x and y be the co-ordinates of any point in the curve, then $y' - y = -\frac{dx}{dy}(x' - x)$ is the equation to the normal and $\sqrt{y^2 - (x' - x)^2} =$ length of the normal,

$$\therefore y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \mp \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

the sign being — or + according as the curve is concave or convex to the axis of x . Hence

$$y \cdot \frac{d^2y}{dx^2} = \mp \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}, \text{ or}$$

$$y \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx dy} = \mp \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

$$\therefore \frac{dy}{dx} \cdot \frac{d^2y}{dx} \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \mp \frac{dy}{y}; \text{ hence by integra-}$$

tion h. l. $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \text{h. l. } \frac{c^2}{y^2}$ or h. l. $\frac{y^2}{c^2}$, according as we use the upper or lower sign;

$$\therefore \frac{dy}{dx} = \pm \frac{\sqrt{(c^2 - y^2)}}{y} \text{ and } \frac{dy}{dx} = \pm \frac{\sqrt{(y^2 - c^2)}}{c};$$

$$\text{hence } x = c' \mp \sqrt{(c^2 - y^2)} \text{ and } x = c \cdot \text{h. l. } \frac{y \pm \sqrt{(y^2 - c^2)}}{c}.$$

If we suppose $y = c$ when $x = 0$, then $x^2 + y^2 = c^2$, the
VOL. VI. PART I. 1

equation to a circle: and $\frac{x}{c} = h, l, \frac{y \pm \sqrt{(y^2 - c^2)}}{c}$ or $e^{\frac{x}{c}} =$

$\frac{y}{c} \pm \sqrt{\left\{\left(\frac{y}{c}\right)^2 - 1\right\}}$. Whence $e^{\frac{x}{c}} + e^{-\frac{x}{c}} = 2 \cdot \frac{y}{c}$ which is the equation to the catenary. The curves required $\therefore$ are the Circle and the Catenary.

SECOND SOLUTION, by Mr. DAVIES, the Proposer.

This question should have been, "In what curve has the radius of curvature a *given ratio* to the normal?" The particular case above proposed is obviously a property of the circle, and frequent solutions by integrating the differential equation have shewn that it belongs alone to that curve. Dr. Lardner in his valuable treatise on the Differential and Integral Calculus, has *proposed* one more general still, but a slight inspection of the differential expression will convince us that when the radius of curvature is "*a given function*" of the normal, the integration, very few cases excepted, transcends all known processes. For the *given ratio*, the solution may be as follows.

The general condition is expressed by the equation

$$F \left\{ -\frac{ds^3}{d^2y dx} \right\} = \frac{y ds}{dx} \dots\dots (1).$$

Let F denote a constant factor, n , in which case it becomes as above proposed. Then

$$y d^2y = -n(dx^2 + dy^2) \dots\dots (2).$$

Let $y'' = \frac{dy'}{dx}$, and $dy' = \frac{dy}{dx}$: then also $y'' = \frac{y' dy'}{dy}$. Mak

ing these substitutions in (2) and reducing, we obtain

$$-n \cdot \frac{dy}{y} = \frac{y' dy'}{1 + y'^2}.$$

Integrating this, we have

$$\log(1 + y'^2)^{\frac{1}{2}} = \log\left(\frac{c'}{y}\right)^n, \text{ or}$$

$$1 + y'^2 = \left(\frac{c'}{y}\right)^{2n} \dots\dots\dots (3)$$

$$\therefore dx = \frac{y^n dy}{\sqrt{(c'^{2n} - y^{2n})}} \dots\dots\dots (4).$$

This is the differential equation of the curve sought, and will

represent different families, according to the values of n . We may just notice one or two of them.

1. Let $n = 0$; then $dy = 0$, and $y = c_{//}$. This represents a line parallel to the axis of x at the distance $c_{//}$.

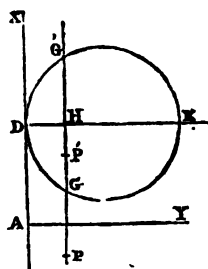
The radius of curvature is *infinite*, and $c_{//}$ finite; hence rad. curv. $\times 0 = c_{//}$ as it should be.

2. Let $n = \frac{1}{2}$: then $x = \int \frac{y^{\frac{1}{2}} dy}{\sqrt{c_1 - y}} = \int \frac{y dy}{\sqrt{c_1 y - y^2}} = \pm \sqrt{c_1 y - y^2} + c_{//} - \frac{c_1}{2} \text{ vers. } \sin^{-1} \frac{2y}{c_1}$. This curve may be thus constructed.

Let AY, AX be rectangular co-ordinates. Make $AD = c_{//}$, and draw DL parallel to AY . Make $DC = c_1$ and describe the circle DGK . Let $DH = AL = y$; then $GH = \sqrt{c_1 y - y^2}$

and arc $DG = \frac{c_1}{2} \text{ vers. } \sin^{-1} \frac{2y}{c_1}$. Make GP

$= DG$: then P is a point in the curve whose radius of curvature is equal to half the normal.



3. Let $n = -\frac{1}{2}$: then $x + c_{//} = \int \frac{y^{-\frac{1}{2}} dy}{\sqrt{c^{-1} - y^{-1}}} = \int \frac{c^{\frac{1}{2}} dy}{\sqrt{y - c_1}} = \frac{1}{2} \sqrt{c_1 y - c_1^2}$ which is the equation to the parabola, the co-ordinates of whose vertex are $x' = -c_{//}$ and $y' = c_1$, and its parameter $= \frac{c_1}{4}$. Hence,

If the radii of curvature of a parabola be prolonged beyond the curve, till the prolongation be half the radius, then their extremities will trace out a straight line perpendicular to the axis of the curve.

4. Let $n = 1$: then $x + c_{//} = \sqrt{c_1^2 - y^2}$ which denotes the circle.

5. Let $n = -1$: then the differential equation becomes

$$\frac{dy^2}{dx^2} + 1 = \frac{y^2}{c_1^2}.$$

This equation has already been discussed at pp. 229, 230* of

* The talented author of the first of those solutions has made a remark which seems to imply that the straight line is not a "proper solution"

the last volume of the Repository, and the *integral* solution found to be a *catenary*. In this respect the result of the enquiry agrees with that found at p. 507-8, of *tome X^{eme}* of the *Mémoires des Savans Etrangers*, by M. Meusnier, by a different process.

6. Let $n = 2$: then $dx = \frac{y^2 dy}{\sqrt{(c^2 - y^2)}}$, which is the differential equation of the *elastic curve*.

This question was answered by Messrs. BAINES, GODWARD, and THOMPSON.

II. QUESTION 512, by MYTHOLOGICUS.

From what point will three given spheres subtend equal conical angles.

FIRST SOLUTION, by Mr. T. S. DAVIES, Bath.

The solution of this question is a locus, and hence, there should be another condition annexed to its hypothesis to render it determinate. We shall, therefore, take *four* spheres instead of *three*.

of that question. The singular solution of a differential equation it is true, is not always a solution of a problem which gave rise to that equation—nevertheless it often is—and sometimes the only one. In that case, I think he will allow, upon reconsideration, that it is a proper solution of the problem: in *this* case, however, it is not a solution of the problem at all, for the value $n = -1$. As then, the radius of curvature is infinite, and n finite as well as c . Hence it would require infinite $\times$ finite = finite; which is a manifest absurdity. Indeed, the general equation (4) might in like manner be put under the form,

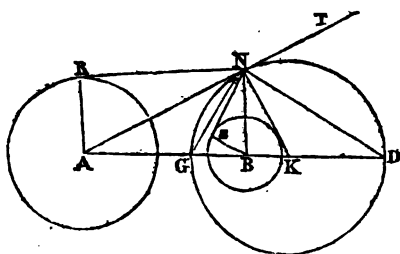
$$\frac{dy^2}{dx^2} + 1 = \frac{c^{2n}}{y^{2n}}.$$

Which for all values of n has $y = c$ for a singular solution: though the only case in which the straight line can solve the problem above is when $n = 0$. This, however, is neither the time nor the place for entering upon the discussion of so important a topic as the nature and interpretation of singular solutions.

I must also here take the opportunity of correcting the dubious phraseology of the remark upon the singular solution at the foot of p. 231, of the place referred to. It should have been thus: "Though the *integral* equation can never represent a right line, yet, if we seek for the *singular solution* of the *differential* equation, we shall find it $y = c$, and that singular solution is the straight line already mentioned."

I. The locus of the point N from which two circles subtend equal angles, is a circle described on the *diameter of similitude** of those two circles, GD .

Let NR , NS be drawn to touch the two circles: and join AR , BS , AN , BN , GN , DN and prolong AN to T . Then, because (A), (1) subtend equal angles at E , the angles ANR , BNS are equal, and ARN , BSN are right angles: hence the triangles ANR , BSN are similar, and $AN : NB :: AR : BS$.



Again, because D and G are the *poles of similitude*, we have

$$AG : GB :: AR : BS, \text{ and } AD : DB :: AR : BS.$$

Hence, *ex equali*,

$$AG : GB :: AN : NB, \text{ and } AD : DB :: AN : NB.$$

It follows, therefore, (by means of Simson's Eucl. VI. 3, and A.) that GN bisects the angle BNC , and DN bisects the angle BNT . Hence GND is a right angle, standing on the line GD : and the locus of the point N is that circle. Q. E. D.

* These terms not yet being familiarly used amongst us, and some of them being new, or used in a different sense from that which Monge assigned them, it may be necessary to define them here.

DEF. 1. The *circle of similitude* is a circle from any point of which two given circles subtend equal angles. The existence, position, and magnitude of this circle is shewn in (I).

DEF. 2. The *axis of similitude* is a line drawn through the centres of two circles and indefinitely prolonged.

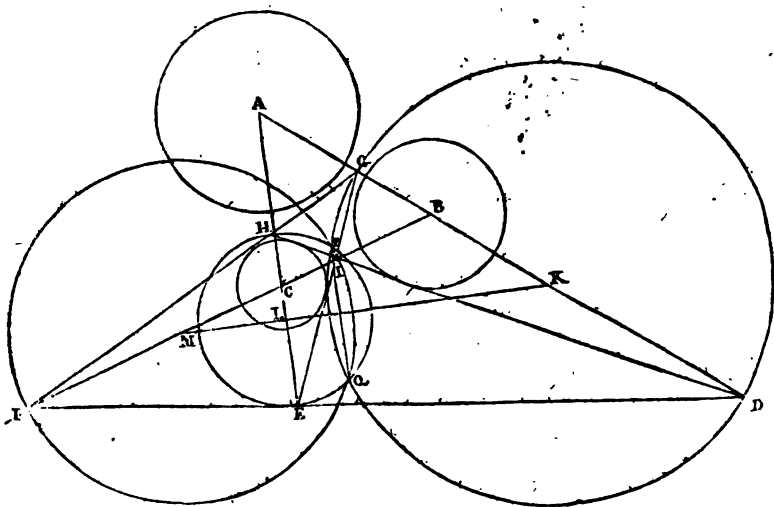
DEF. 3. The *diameter of similitude* is the diameter of the circle in Def. 1.

DEF. 4. The *radius of similitude* and *centre of similitude* are the radius and centre of the circle of similitude.

DEF. 5. The *poles of similitude* are the points in which the circle of similitude and axis of similitude intersect one another. They are internal or external.

The "*centre of similitude*" is the term employed by Monge to designate the points which we have called "*poles of similitude*." It was my wish to avoid any change of term, but enquiries concerning the figures in question have led me to results that render that change imperious: and hence I have not scrupled to change the name, even though consecrated by the authority of that illustrious mathematician, which was originally framed under the views resulting from a less extended enquiry than it was natural for succeeding Geometers to pursue. It would be incompatible with the length of a note to enter into further detail at present: but I hope to establish, hereafter, the propriety of the definitions as given above.

II. Let (κ) , (L) , (M) be the circles of similitude of the three circles (A) , (B) , (C) : then, they all intersect in the same points, P , Q .



For let us suppose (κ) , (L) to intersect in those two points. Then at P and Q ,

$$T_{(A)} = T_{(B)} \text{ and } T_{(A)} = T_{(C)}^* \text{ by (I).}$$

Hence $T_{(B)} = T_{(C)}$: which is a property of the circle (M) . Hence P and Q are also in the circle M ; or these circles pass through P and Q . *Q. E. D.*

III. The *sphere of similitude* (κ) of two spheres (A) and (B) is the locus of all the points from which those two spheres subtend equal conical angles. (Fig. 1.).

This is evident, because the sphere of similitude is the locus of the circle of similitude and the spheres (A) , (B) are the corresponding loci of the circles (A) , (B) as the whole system revolves about the axis of similitude AB .

IV. The *spheres of similitude* (κ) , (L) , (M) of three given spheres (A) , (B) , (C) intersect in the same circle. (Fig. 2.)

This is obvious from (II.): the line PQ being the trace of the plane of section of the spheres two and two, and hence the common section of all three. This circular section is the locus

* Angles contained by the tangents drawn to (A) , &c.

of all the points from which the three spheres subtend equal angles.

V. If a fourth sphere (D) be taken, and its sphere of similitude with one of the three former, its intersection with the circle already determined in (IV.) will give two points from which the four spheres will subtend equal conical angles. This is a solution to the problem modified by the addition above mentioned.

Cor. 1. The three centres of similitude K, L, M of three circles or three spheres are in one line, the line of similitude of (A), (B), (C) : since each line ML, LK, KM bisects PQ at right angles.

Cor. 2. If there be four spheres, the centres of similitude will, three and three, be in one line, and hence all six in one plane.

Cor. 3. The four spheres of similitude, then, are so related that the four planes of section, three and three, all pass through the same line; *scil.* the line passing through the two points determined in (V).

Cor. 4. Except the spheres of similitude all meet one another, the problem is impossible.

The same Analytically.

Take A (supposed to be the centre of the largest sphere) as origin and the line drawn from A to the centres of B, C, D for axes of co-ordinates; denote by x, y, z , the co-ordinates of the point sought; and put

x_0, y_0, z_0 , for the co-ordinates of B, C, D; and

a, b, c, d the radii of the four spheres.

Put also, n = ratio of the distance of xyz from each centre to the radius of that sphere whose centre it is. Then the four distances of the point xyz from the centres will be

$$na = a, nb = b, nc = c, nd = d.$$

But these same distances are, putting the angles xy, xz, yz equal to δ, γ, β respectively,

$$\left. \begin{aligned} x^2 + y^2 + z^2 \\ + 2xy \cos \delta \quad + 2xz \cos \gamma \quad + 2yz \cos \beta \end{aligned} \right\} = a^2 \dots (1)$$

$$\left. \begin{aligned} (x-x_0)^2 + y^2 + z^2 \\ + 2(x-x_0)y \cos \delta + 2(x-x_0)z \cos \gamma + 2yz \cos \beta \end{aligned} \right\} = b^2 \dots (2)$$

$$\left. \begin{aligned} x^2 + (y-y_0)^2 + z^2 \\ + 2x(y-y_0) \cos \delta + 2xz \cos \gamma + 2(y-y_0)z \cos \beta \end{aligned} \right\} = c^2 \dots (3)$$

$$\left. \begin{aligned} x^2 + y^2 + (z-z_0)^2 \\ + 2xy \cos \delta + 2x(z-z_0) \cos \gamma + 2y(z-z_0) \cos \beta \end{aligned} \right\} = d^2 \dots (4)$$

We have these four equations from which to determine x, y, z, n ; and this effected, the question will be solved.

Subtract (2, 3, 4) from (1), and imagine a sphere whose centre is A to be described, and cut by the co-ordinate planes in a spherical triangle BCD.

Then we have

$$x + y \cos \delta + z \cos \gamma = \frac{a^2 - b^2 + x^2}{2x} = b, \dots (5)$$

$$x \cos \delta + y + z \cos \beta = \frac{a^2 - c^2 + y^2}{2y} = c, \dots (6)$$

$$x \cos \gamma + y \cos \beta + z = \frac{a^2 - d^2 + z^2}{2z} = d, \dots (7)$$

Finding the values of x, y, z , and writing in the result, $\cos \beta - \cos \gamma \cos \delta = \sin \gamma \sin \delta \cos B$ &c. we obtain

$$\left. \begin{aligned} x &= \frac{b \sin \beta - c \sin \gamma \cos D - d \sin \delta \cos C}{1 - \cos^2 \beta - \cos^2 \gamma - \cos^2 \delta + 2 \cos \beta \cos \gamma \cos \delta} \times \sin \beta \\ y &= \frac{c \sin \gamma - b \sin \beta \cos D - d \sin \delta \cos B}{1 - \cos^2 \beta - \cos^2 \gamma - \cos^2 \delta + 2 \cos \beta \cos \gamma \cos \delta} \times \sin \gamma \\ z &= \frac{d \sin \delta - b \sin \beta \cos C - c \sin \gamma \cos B}{1 - \cos^2 \beta - \cos^2 \gamma - \cos^2 \delta + 2 \cos \beta \cos \gamma \cos \delta} \times \sin \delta \end{aligned} \right\} \dots (8)$$

Substituting x, y, z of (8) in equation (1) we have,

$$\left\{ \begin{aligned} a^2 &= b^2 \sin^2 \beta + c^2 \sin^2 \gamma + d^2 \sin^2 \delta \\ &- 2 \{ b c \sin \beta \sin \gamma \cos D + b d \sin \beta \sin \delta \cos C + c d \sin \gamma \sin \delta \cos B \} \end{aligned} \right\} \dots (9)$$

Restoring the values of a, b, c, d , in terms of a, b, c, d , and then returning to a, b, c, d , we shall have

$$\begin{aligned} &\left\{ \begin{aligned} &\left(\frac{a^2 - b^2}{x} \right)^2 \sin^2 \beta - 2 \cdot \frac{(a^2 - c^2)(a^2 - d^2)}{y z} \sin \gamma \sin \delta \cos B + \\ &\left(\frac{a^2 - c^2}{y} \right)^2 \sin^2 \gamma - 2 \cdot \frac{(a^2 - b^2)(a^2 - d^2)}{x z} \sin \beta \sin \delta \cos C + \\ &\left(\frac{a^2 - d^2}{z} \right)^2 \sin^2 \delta - 2 \cdot \frac{(a^2 - b^2)(a^2 - c^2)}{x y} \sin \beta \sin \gamma \cos D \end{aligned} \right. \\ &+ 2 \pi^2 \left\{ \begin{aligned} &(a^2 - b^2) \sin^2 \beta - 2 \cdot \left(\frac{(a^2 - c^2)z^2 + (a^2 - d^2)y^2}{y z} \right) \sin \gamma \sin \delta \cos C + \\ &(a^2 - c^2) \sin^2 \gamma - 2 \cdot \left(\frac{(a^2 - b^2)x^2 + (a^2 - d^2)z^2}{x z} \right) \sin \beta \sin \delta \cos C + \\ &(a^2 - d^2) \sin^2 \delta - 2 \cdot \left(\frac{(a^2 - c^2)x^2 + (a^2 - b^2)y^2}{x y} \right) \sin \gamma \sin \delta \cos D \end{aligned} \right. \\ &- 4a^2 \\ &+ \left\{ \begin{aligned} &x^2 \sin^2 \beta + y^2 \sin^2 \gamma + z^2 \sin^2 \delta \\ &- 2 x y \sin \beta \sin \gamma \cos D - 2 x z \sin \beta \sin \delta \cos C - 2 y z \sin \gamma \sin \delta \cos B \end{aligned} \right. \\ &= 0 \dots \dots \dots (10). \end{aligned}$$

Write this $A, n^4 + 2B, n^2 + C, = 0$, and we get

$$n = \pm \sqrt{\frac{-B, \pm \sqrt{(B,^2 - A, C,)}}{A,}} \dots\dots (11).$$

Upon substituting for $B,^2 - A, C,$ its value, it is found to be essentially positive; and therefore if we denote its root by $\pm R,$ we have

$$n = \pm \sqrt{\frac{-B, \pm R,}{A,}}.$$

We shall also find $-B - R$ a negative, and $-B + R$ (when the problem is possible) a positive quantity. Whence, also, n, x, y, z are determined. The latter part of the work is too complicated for insertion here, and as the labour is the only difficulty, I thought it enough to set down the result.

SECOND SOLUTION, by Mr. JOHN RAINES, Thornhill.

Let the three given spheres be denoted by $A, B,$ and $C,$ their radii by $r, r',$ and $r'',$ and the required point by $P.$ Then, on the supposition that the centres of the spheres are not in the same plane, we will imagine two planes ($x, y,$) to pass vertically and horizontally through the centres of A and $B,$ i. e. at right angles to each other, and another plane (z) through the centre of $C,$ parallel to (x). Put $a =$ the distance between the centres A and $B,$ $b =$ the perpendicular distance of the planes (x, z), $\pm c =$ the distance of the centre C from the point opposite the centre of $A,$ measured on z parallel to $y,$ $\pm d =$ its perpendicular distance above or below $y,$ and $x, y,$ and $z =$ the co-ordinates of the point $P,$ referred to the planes $x, y.$ From these assumptions we easily deduce the distances of the point P from the centres of $A, B, C =$

$$\sqrt{\{x^2 + y^2 + z^2\}}, \sqrt{\{(a - x)^2 + y^2 + z^2\}},$$

$$\text{and } \sqrt{\{(x \mp c)^2 + (y \mp d)^2 + (b - z)^2\}}.$$

Now it is evident that the conical angles will be equal when the plane angles made by sections passing through their axes are equal; also that the cosecants of the halves of these plane angles are equal to the above distances divided by the radii $r, r',$ and $r'';$ hence we have two (and only two) equations, viz.

$$\frac{1}{r} \sqrt{\{x^2 + y^2 + z^2\}} = \frac{1}{r'} \sqrt{\{(a - x)^2 + y^2 + z^2\}}$$

$$\text{and } \frac{1}{r} \sqrt{\{x^2 + y^2 + z^2\}} = \frac{1}{r''} \sqrt{\{(x \mp c)^2 + (y \mp d)^2 + (b - z)^2\}},$$

but there are three unknown quantities, and unless either x , y , or z be given, the question is unlimited. When $b = 0$, $z = 0$, and the centres of the spheres are in the same plane. Under this idea, which was probably the intention of the proposer, the question is limited. In this case,

$$\frac{1}{r} \sqrt{x^2 + y^2} = \frac{1}{r'} \sqrt{(a-x)^2 + y^2},$$

$$\text{and } \frac{1}{r} \sqrt{x^2 + y^2} = \frac{1}{r''} \sqrt{(x \mp c)^2 + (y \mp d)^2}.$$

From the first of these we obtain

$$y = \left\{ \frac{r^2(a-x)^2 - r'^2 x^2}{r'^2 - r^2} \right\}^{\frac{1}{2}}, \text{ which substituted in the second gives}$$

$$\{r'^2(a^2 - 2ax) - (r'^2 - r^2) \cdot (x^2 \mp 2cx + c^2 + d^2)\} = 4d^2 r^2 (r'^2 - r^2) \cdot (a^2 - 2ax) + 4d^2 x^2 (2r^2 r'^2 - r^4 - r'^4), \text{ an equation from which } x \text{ may be found. This also furnishes an analytical solution to Quest. IX. (1517), Ladies' Diary, 1830-31.}$$

Solutions were also received from Messrs. GODWARD, MASON, and THOMPSON.

III. QUESTION 513, by DELTA SULIUS.

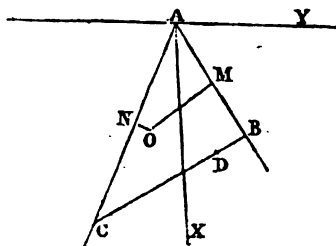
DE, FG are two lines given in position, and E a given point: what is the locus of the centre of the circle circumscribing the triangle DFG, when the base DF passes through E?

FIRST SOLUTION, by — MASON, of Scoulton, Norfolk.

Let GD be the axis of x , GF the axis of y , $\theta = \angle GDF$, x and y the co-ordinates of the centre of the circle, α , β , the co-ordinates of E. Then, by the circle, $GD = 2x + 2y \cos \theta$ and $GF = 2y + 2x \cos \theta$ $\therefore$ $DH = 2x + 2y \cos \theta - \alpha$, if EH be parallel to GF; and by sim. Δ s $DH : HE :: DG : GF$: whence $(x^2 + y^2) \cdot 2 \cos \theta + 2xy \cdot (1 + \cos^2 \theta) - (\alpha + \beta \cos \theta) \cdot y - (\beta + \alpha \cos \theta) x = 0$ is the equation to the required curve, which is the equation to the hyperbola, the co-ordinates of whose centre are $\frac{\alpha - \beta \cos \theta}{2 \sin^2 \theta}$ and $\frac{\beta - \alpha \cos \theta}{2 \sin^2 \theta}$.

SECOND SOLUTION, by Mr. T. S. DAVIES, Bath.

Let the system be referred to rectangular co-ordinates bisecting the angle A. Let $CAB = 2\alpha$, and put $AC = b$, $AB = c$, and the co-ordinates of D = x', y' . Denote also by x_n, y_n the co-ordinates of N, and so of the other points. Then



$$\left\{ \begin{array}{ll} x_n = \frac{1}{2} b \cos \alpha & x_c = b \cos \alpha \\ y_n = -\frac{1}{2} b \sin \alpha & y_c = -b \sin \alpha \\ x_m = \frac{1}{2} c \cos \alpha & x_b = c \cos \alpha \\ y_m = \frac{1}{2} c \sin \alpha & y_b = c \sin \alpha \end{array} \right\} \dots (1).$$

The equations of NO and MO are

$$\left\{ \begin{array}{l} y \sin \alpha - x \cos \alpha = y_n \sin \alpha - x_n \cos \alpha \\ y \sin \alpha + x \cos \alpha = y_m \sin \alpha + x_m \cos \alpha \end{array} \right\} \dots (2).$$

Adding and subtracting equations (2) and inserting for x_n, y_n , &c. their values given by equation (1) we get for the co-ordinates of the centre O of the circumscribing circle

$$y_0 = \frac{(y_n + y_m) \sin \alpha - (x_n - x_m) \cos \alpha}{2 \sin \alpha} = \frac{1}{4} \cdot \frac{c-b}{\sin \alpha} \dots (3)$$

$$x_0 = \frac{(y_m - y_n) \sin \alpha + (x_m + x_n) \cos \alpha}{2 \cos \alpha} = \frac{1}{4} \cdot \frac{c+b}{\sin \alpha} \dots (4).$$

But expressing the line BC in terms of the co-ordinates of its extremities, B, C we have

$$y(x_b - x_c) - x(y_b - y_c) = y_c x_b - y_b x_c$$

and as this line also passes through D, (x', y') we also have

$$y'(x_b - x_c) - x'(y_b - y_c) = y_c x_b - y_b x_c.$$

Now, this becomes, upon the substitutions of the values of x_b, y_b , &c. from (1)

$$y' \cos \alpha (c-b) - x' \sin \alpha (c+b) = -2bc \sin \alpha \dots (5).$$

But from (3), (4), we obtain

$$c-b = 4y_0 \sin \alpha$$

$$c+b = 4x_0 \cos \alpha$$

$$\text{hence } \left\{ \begin{array}{l} c = 2(x_0 \cos \alpha + y_0 \sin \alpha) \\ b = 2(x_0 \cos \alpha - y_0 \sin \alpha) \\ -bc = -4(x_0^2 \cos^2 \alpha - y_0^2 \sin^2 \alpha) \end{array} \right\} \dots (6).$$

Inserting these values of b , c , and dropping the subscripted o from x and y , we find the equation of o to be

$$2y^2 \cos^2 \alpha - 2x^2 \sin^2 \alpha = -yy' + xx' \dots (7).$$

The path pursued, then, by the centre of the circle is the *hyperbola*.

The co-ordinates of the centre are

$$\left. \begin{aligned} x &= -\frac{1}{4} x' \operatorname{cosec}^2 \alpha \\ y &= -\frac{1}{4} y' \sec^2 \alpha \end{aligned} \right\} \dots (8).$$

Transposing the origin to the centre, by means of (8) the equation becomes

$$y^2 \cos^2 \alpha - x^2 \sin^2 \alpha = \frac{1}{8} \{y'^2 \sec^2 \alpha - x'^2 \operatorname{cosec}^2 \alpha\} \dots (9).$$

The principal diameters are found by putting x and y in (9) successively $= 0$. This gives for the intersection of the curve with the *new* axes

$$\left. \begin{aligned} y &= \frac{\sec \alpha \sqrt{(2)}}{4} \{y'^2 \sec^2 \alpha - x'^2 \operatorname{cosec}^2 \alpha\}^{\frac{1}{2}} \\ x &= \frac{\operatorname{cosec} \alpha \sqrt{(-2)}}{4} \{y'^2 \sec^2 \alpha - x'^2 \operatorname{cosec}^2 \alpha\}^{\frac{1}{2}} \end{aligned} \right\} \dots (10).$$

By means of (10) we have for the equation of the asymptotes

$$y = \pm x \cot \alpha \dots \dots (11).$$

Or the asymptotes are *perpendicular* to the lines AC , AB .

Messrs. BAINES, GODWARD, and THOMPSON answered this question.

IV. QUESTION 514, by UMBRA.

A given triangle ABC rests upon two rectangular directrices AO , CO : Find the locus of its vertex B .

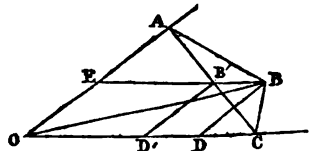
SOLUTION, by Mr. T. S. DAVIES.

For *any* angle of inclination of the lines AO , CO .

The figure being constructed as directed in the enunciation, draw EB , DB parallel to the given directrices, and take these directrices as co-ordinate axes of the system. Put

$\angle OAC = \theta$, $\angle OCA = \phi$, and $\angle AOC = \kappa$:

$EB = OD = x$, $DB = y$, and the sides and angles of the triangle



ABC as usual equal to $a, b, c, \alpha, \beta, \gamma$. Then these three equations are at once obvious

$$\sin(\theta + \alpha) = \frac{x \sin \kappa}{c} \dots \dots (1)$$

$$\sin(\phi + \gamma) = \frac{y \sin \kappa}{a} \dots \dots (2)$$

$$\kappa + \phi + \theta = 180^\circ \dots \dots (3)$$

By eliminating ϕ and θ from these we shall obtain the equation of B in terms of x, y . This may be accomplished as follows:—

$$\text{From (3) we have } \kappa + \theta = 180 - \phi, \dots \dots (4)$$

$$\text{From (4) we get } \begin{cases} \sin \phi = \cos \kappa \sin \theta + \sin \kappa \cos \theta \dots (5) \\ \cos \phi = -\cos \kappa \cos \theta + \sin \kappa \sin \theta \dots (6) \end{cases}$$

Expanding (1) and (2), and in (2) for $\sin \phi, \cos \phi$, inserting their values from (5, 6) we have, after a slight modification of the result,

$$\cos(\kappa - \gamma) \sin \theta + \sin(\kappa - \gamma) \cos \theta = \frac{y \sin \kappa}{a} \dots (7)$$

$$\{\cos \alpha \sin \theta + \sin \alpha \cos \theta = \frac{x \sin \kappa}{c} \dots (8)\}.$$

From (7) and (8) we get

$$\sin \theta = \frac{cy \sin \alpha - ax \sin(\kappa - \gamma)}{ac \sin(\alpha + \gamma - \kappa)} \cdot \sin \kappa \dots (9)$$

$$\cos \theta = \frac{cy \cos \alpha - ax \sin(\kappa - \gamma)}{ac \sin(-\alpha - \gamma + \kappa)} \cdot \sin \kappa \dots (10).$$

Putting the values of $\sin \theta, \cos \theta$, found in (9, 10) in the equation

$$\cos^2 \theta + \sin^2 \theta = 1,$$

we have ultimately for the equation sought,

$$c^2 y^2 - 2acxy \cos(\kappa - \alpha - \gamma) + a^2 x^2 = a^2 c^2 \operatorname{cosec}^2 \kappa \sin^2(\kappa - \gamma - \alpha),$$

which represents an *ellipse* whose centre is O.

Corollaries.

1. If $\alpha + \gamma = \kappa$, then $\sin(\kappa - \gamma - \alpha) = 0$, and $\cos(\kappa - \gamma - \alpha) = 1$ hence

$$\pm cy - ax = 0,$$

or the locus is a straight line through O.

2. Let the point B be turned inwards, and

$\gamma + \alpha + \kappa = 180^\circ$; then also, $\sin(\kappa - \gamma - \alpha) = 0$, and

$$\pm cy - ax = 0,$$

which represents also a straight line.

3. Let $\kappa = 90^\circ$: then the equation becomes that proposed in the question: and

$$c^2y^2 - 2acxy \sin \alpha + \gamma + a^2x^2 = a^2c^2 \cos^2 \gamma + \alpha.$$

4. If $\alpha = 0 = \gamma$, and $a = c$; then the point in question is the middle of AC; and we have, in this case,

$$y^2 - 2xy \cos \kappa + x^2 = a^2,$$

and when κ is a right angle we obtain,

$$y^2 + x^2 = a^2,$$

the equation of a circle, agreeing with a "known genesis" of that curve.

5. When $\alpha = \gamma = 0$, it might seem that $a = c$ was a consequence, since $a : c :: \sin \alpha : \sin \gamma$ in the triangle ABC. The equation upon the former supposition only becomes

$$c^2y^2 - 2acxy \cos \kappa + a^2x^2 = a^2c^2.$$

We can ascertain, however, by a slight modification of the process already employed that this is the equation of the curve when a is not equal to c . For suppose B' to be the point in the line AC: and draw $B'D'$ parallel to OA. Then the co-ordinates of B' are $D'B'$, $B'E$. Then put $AB' = c$ and $B'C = a$.

$$\sin \theta = \frac{x \sin \kappa}{c},$$

$$\sin \phi = \frac{y \sin \kappa}{a},$$

are the three equations corresponding to the new hypothesis.

$$\text{and } \phi + \theta + \kappa = 180.$$

$\sin \phi = \cos \kappa \sin \theta + \sin \kappa \cos \theta$; and inserting the values of $\sin \phi$ and $\sin \theta$, it becomes after squaring

$$\left(\frac{cy - ax \cos \kappa}{ac} \right)^2 = \frac{c^2 - x^2 \sin^2 \kappa}{c^2}, \text{ or by reduction}$$

$$c^2y^2 - 2acxy \cos \kappa + a^2x^2 = a^2c^2.$$

The same as above obtained from the general formula.

Scholium.

The locus in the general case of κ being any angle has been already investigated geometrically by Messrs. Cunliffe and Johnson, in the first vol. of the Repository, p. 266, 7. The method employed by these gentlemen (for the two solutions are essentially the same) is very elegant and simple: viz. that of reducing the general problem to the "case of ease" contained in our Corollary (5), which had been long known.

Ingenious Solutions were also received from Messrs. BAINES, GODWARD, and MASON.

V. QUESTION 515, *by* UMBRA.

Mr. Ferguson constructed a cylindrical dial, whose axis was vertical, and the stile perpendicular to the axis and in the plane of the meridian. Now, when the cylindrical surface is developed, what is the equation to the extremity of the shadow, supposing the sun retained its mean declination during a whole revolution.

FIRST SOLUTION, *by* — MASON, *of* Scoulton.

Let r = the radius of the cylinder, a = the distance of the end of the stile from the axis of the cylinder, x = the length of the circumference of the cylinder from the line parallel to the axis passing through the extremity of the shadow below the horizontal plane passing through the stile. Then x and y are the co-ordinates of the extremity of the shadow upon the developed surface.

Now $\sqrt{\{r^2 + a^2 - 2ar \cos(x \div r)\}}$ = the distance in the horizontal plane of the stile from the vertical line passing through the extremity of the shadow upon the cylindrical surface, $\therefore \sqrt{\{r^2 + a^2 + y^2 - 2ar \cos(x \div r)\}}$ = distance from the end of the stile to the extremity of the shadow; hence $y \div \sqrt{\{r^2 + a^2 + y^2 - 2ar \cos(x \div r)\}}$ = sine of the sun's altitude, and $\sqrt{\{r^2 + a^2 - 2ar \cos(x \div r)\}} \div \sqrt{\{r^2 + a^2 + y^2 - 2ar \cos(x \div r)\}}$ = cosine of the sun's altitude, and if l = the latitude of the place, δ = the sun's declination and A = the altitude of the sun. Then $\frac{\sin \delta - \sin l \sin A}{\cos l \cos A} =$

$$\text{cosine of the sun's azimuth} = \frac{2a^2 - 2ar \cos(x \div r)}{2a \sqrt{\{r^2 + a^2 - 2ar \cos(x \div r)\}}},$$

$$\therefore \sin \delta = \frac{a - r \cos(x \div r)}{\sqrt{\{r^2 + a^2 - 2ar \cos(x \div r)\}}}.$$

$$\cos l \cdot \frac{\sqrt{\{r^2 + a^2 - 2ar \cos(x \div r)\}}}{\sqrt{\{r^2 + a^2 + y^2 - 2ar \cos(x \div r)\}}} =$$

$$\frac{\sin l \cdot y}{\sqrt{\{r^2 + a^2 + y^2 - 2ar \cos (x \div r)\}}} = \frac{\{a - r \cos (x \div r)\} \cos l - y \sin l}{\sqrt{\{r^2 + a^2 + y^2 - 2ar \cos (x \div r)\}}}$$

which is the required equation, the arc x being the abscissa and y the corresponding ordinate.

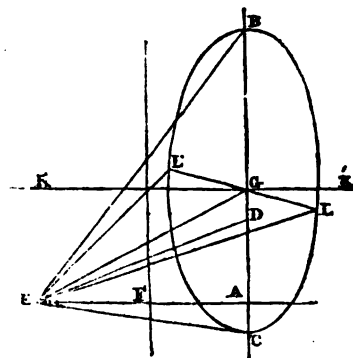
SECOND SOLUTION, by Mr. BAINES, Thornhill.

Put r = the radius of the cylinder, h = the length of the pin; and with respect to the curve traced out upon the surface of the cylinder, put x = the abscissa, or distance from the foot of the pin produced inward, measured on a plane passing through the axis of the cylinder and foot of the pin, y = the ordinate, or distance from the extremity of x , perpendicular to the said plane, and meeting the surface at the end of the pin's shadow, and z = the part of the pin from the surface to the origin of x , then $h + z$ = the distance from the top of the stile from the origin of x , and $\sqrt{\{x^2 + y^2 + (h + z)^2\}}$ = the diagonal of the parallelopipedon formed by x , y and $h + z$, or the distance between the extremities of the shadow and z ; hence, from a little consideration, it is evident that the cosine of the sun's altitude = $(h + z) \div \sqrt{\{x^2 + y^2 + (h + z)^2\}}$, and the cosine of its azimuth from the south = $(h + z) \div \sqrt{\{y^2 + (h + z)^2\}}$; but, by spherics, putting L = the co-latitude, D = the co-declination or polar distance, and A = the co-altitude of the sun, the cosine of the azimuth = $(\cos D - \cos A \cos L) \div (\sin A \sin L)$. Therefore, by substitution and reduction, $\tan L (h + z)^2 = \cos D \sec L \sqrt{\{y^2 + (h + z)^2\}} \cdot [x^2 + y^2 + (h + z)^2] - \sqrt{\{(x^2 + y^2) \cdot (y^2 + (h + z)^2)\}}$ = the equation of the curve upon the surface of the cylinder, when the surface is unopened out. In this state the curve is obviously a curve of double curvature, but if the surface be developed, y becomes = $\sin^{-r} y$, and z (which in the former case = $r - \sqrt{(r^2 - y^2)}$) = 0; hence $\tan L \times h^2 = \cos D \sec L \sqrt{\{(h^2 + (\sin^{-r} y)^2)\}} \cdot \{h^2 + x^2 + (\sin^{-r} y)^2\} - \sqrt{\{x^2 + (\sin^{-r} y)^2\} \cdot \{h^2 + (\sin^{-r} y)^2\}}$ = the equation of the curve.

Cor. When the sun is on the equator, $D=90^\circ$, and $\cos D = 0$;
in this case, $x^2 = \frac{h^2 \cdot \tan^2 L}{h^2 + (\sin^{-1} y)^2} - (\sin^{-1} y)^2$.

THIRD SOLUTION, by Mr. T. S. DAVIES.

Let ABC be a meridian plane drawn through the axis BC of the cylinder; AF the radius of the cylinder, and FE the perpendicular gnomon. Draw ED parallel to the earth's axis, and let BEC be the conical angle of the shadow on the day in question. Then ED is the axis of a right cone whose vertical angle is twice the mean co-declination of the sun on that day. Put



$DEC = DEB = \delta = \text{co-dec.}$

$\angle AEG = \varepsilon = \text{elevation of the pole in the given latitude.}$

$AF = r = \text{radius of cylinder.}$

$AE = a = \text{distance of extremity of the gnomon from the axis.}$

Let, also, BC , and LL' drawn at right angles through O (the middle of BC) lying in the plane of the prime vertical, be taken as the axes of z and y respectively, and the line GK perpendicular to both, as the axis of x . Then denoting for the moment by A' and B' the axes of the elliptic* shadow upon yz , we shall have its equation

$$A'^2 y'^2 + B'^2 z'^2 = A'^2 B'^2 \dots \dots (1)$$

In this case we have the co-ordinates of E , viz. GA , O , AE , or z , o , x ; and whose values we shall seek presently. Hence the equation of the cone $E(G)$ is

$$A'^2 x'^2 y'^2 + B'^2 \{x'z' - z'x'\}^2 = A'^2 B'^2 \{x' - x'\}^2 \dots \dots (2)$$

and the cylinder is denoted by

$$x^2 + y^2 = r^2 \dots \dots (3).$$

* I have said elliptic, for in the case of an hyperbola or parabola, the process is scarcely different. In the hyperbola simply a change of sign, and in the parabola a change of origin by which we get an equation still simpler than (4).

Putting for y^2 in (a) its value derived from (2) we obtain

$$A'^2 x'^2 (r^2 - x^2) + B'^2 \{x, z - z, x\}^2 = A'^2 B'^2 \{x - x'\}^2 \dots (4)$$

This is the projection of the intersection of the cone and cylinder upon the plane of the prime vertical: and it is something remarkable that the projection upon the meridian is only of the second order.

Now, when the cylindrical surface is developed, the value of z remains unaltered: whilst for x is substituted the arc

$r \cos^{-1} \frac{x}{r}$. Then we have for the equation of the curve sought

$$A'^2 x'^2 \left(1 - \cos^{-1} \frac{x}{r}\right) + B'^2 \left\{x, z - z, r \cos^{-1} \frac{x}{r}\right\}^2 = A'^2 B'^2 \left\{r \cos^{-1} \frac{x}{r} - x'\right\}^2$$

It only remains to determine the values of A' , B' , x , z . For this, we have

$$AB = a \tan \overline{\delta + \varepsilon},$$

$$AG = a \tan \overline{\delta - \varepsilon},$$

$$AG = \frac{1}{2}a (\tan \overline{\delta + \varepsilon} - \tan \overline{\delta - \varepsilon}), \text{ and}$$

$$EG = \left\{1 + \frac{1}{4}(\tan \overline{\delta + \varepsilon} - \tan \overline{\delta - \varepsilon})^2\right\}^{\frac{1}{2}} a.$$

$$GL = EG \tan \delta.$$

These give

$$A'^2 = a^2 \cdot \frac{4 \sin^2 \delta \cos^2 \delta}{\cos^2 \delta + \varepsilon \cos^2 \delta - \varepsilon} \text{ and } B'^2 = a^2 \cdot 4 \tan^2 \delta \left\{1 + \frac{\sin^2 \delta \cos^2 \delta}{\cos^2 \delta + \varepsilon \cos^2 \delta - \varepsilon}\right\}$$

$$AG = \frac{1}{2}a = a \cdot \frac{\sin \varepsilon \cos \varepsilon}{\cos \delta + \varepsilon, \cos \delta - \varepsilon}; \text{ and, finally,}$$

$$AE = x' = a.$$

Thus the problem is completely solved.

Cor. Had it been required from the curve developed to assign the azimuth of the point x'', y'' , then putting $x'' = r \cos^{-1} \frac{x}{r} = r\theta$, we have

$$\theta = 2\pi \cdot \frac{\left\{ \frac{B'^2 x' (A'^2 - x'^2)}{A'^2 (A'^2 - x'^2)} + \frac{(A'^2 x'^2 - B'^2 z'^2 - A'^2 B'^2) (A'^2 x'^2 - B'^2 z'^2 + A'^2 B'^2)}{A'^2 x'^2 - B'^2 z'^2 + A'^2 B'^2} \right\}^{\frac{1}{2}}}{\frac{x'}{r}}.$$

Schol. This dial though constructed by Mr. Ferguson was not invented by him. It was known to Ozanam and Bion: and as Bion's work on Instruments having been translated by Stone

many years before, that is doubtless, the source whence Ferguson derived his knowledge of the contrivance. This seems the more likely, as both Bion and Ferguson have made the same error in setting of their vertical ordinates. This method can only be true when the stile is *movable* so as to have always *the same azimuth as the sun has*. The dial of Father Quesnet had for gnomon a circular hoop projecting beyond the surface of the dial, so that the point of the hoop which had the same azimuth as the sun formed a temporary gnomon: but it seems too difficult to imagine by what process this point was, at any moment, determined in practice. See Hutton's *Mentücla*, Vol. III. pp. 308-13. This dial (Quesnet's) has been since considered by M. Mollet, of Lyons, in the III^{me} volume of the *Annales des Mathématiques*, p. 372-7; and in his two little works subsequently published on Gnomonics, 1813 and 1820: but as he treats of a problem altogether different from that now before us, it will be unnecessary to do more than refer to it for the information of those who may be interested in such enquiries.

Mr. GODWARD answered this Question.

VI. QUESTION 516, by Mr. DAVEN.

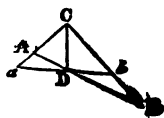
Let one of the solid angles of a regular octahedron be cut off by a plane: then the greatest and least edges thus cut off will be the extremes, and the others the means of an harmonical proportion.

FIRST SOLUTION, by Mr. Wm. GODWARD, Wakefield.

It is manifest that when one of the solid angles of a regular octahedron is cut off by a plane, the opposite edges are the legs of two right angled triangles, and that the line which bisects one right angle, also bisects the other: so that this line and the two hypothenuses have a common point of intersection.

Let therefore $\triangle AOB$, $\triangle acb$ be two right angled triangles having CO the line bisecting the vertical angle, common to both. Put $AC = r$, and $BC = s$; then $AB = \sqrt{(r^2 + s^2)}$, $\sin ACD = \frac{r}{r + s}$, $\sin BAC = \frac{s}{\sqrt{(r^2 + s^2)}}$, and $AC + BC : AC :: AB : AD = r\sqrt{(r^2 + s^2)} \div (r + s)$; whence $\sin ACD : \sin BAC :: AD : CD = rs\sqrt{s} \div (r + s)$.

Similarly, if $ac = r'$ and $bc = s'$, we have $CD = r's'\sqrt{s} \div (r' + s')$.



Hence $\frac{rs}{r+s} = \frac{r's'}{r'+s'}$, or $\frac{r+s}{rs} = \frac{r'+s'}{r's'}$; and $\therefore \frac{1}{r} + \frac{1}{s} = \frac{1}{r'} + \frac{1}{s'}$, an equality indicating that r, s , are the extremes, and r', s' the means of an harmonical proportion.

SECOND SOLUTION, by — MASON, of Scoulton.

Let that diameter of the circumscribing sphere which passes through the angle of the solid be the axis of z , it $\therefore$ makes angles of 45° with the edges. Let the plane yz be coincident with the opposite edges e_1, e_3 and the plane xz with the edges

e_2, e_4 . Let, further, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, be the equation of the cutting plane.

If z_1, z_2, z_3, z_4 be the distances from the plane of xy at which the edges meet the plane, we shall have $x_1 = 0, y_1 = -z_1, x_2 = z_2, y_2 = 0; x_3 = 0, y_3 = z_3; x_4 = -z_4, y_4 = 0$:

Hence $-\frac{z_1}{b} + \frac{z_1}{c} = 1, \therefore z_1 = \frac{bc}{b-c}$ and hence $e_1 = \frac{bc\sqrt{2}}{b-c}$,

also $\frac{z_2}{a} + \frac{z_2}{c} = 1, \therefore z_2 = \frac{ac}{a+c}$ hence $e_2 = \frac{ac\sqrt{2}}{a+c}; \frac{z_3}{b} + \frac{z_3}{c}$

$= 1, \therefore z_3 = \frac{bc}{b+c}$ and $e_3 = \frac{bc\sqrt{2}}{b+c}$; and $-\frac{z_4}{a} + \frac{z_4}{c} = 1,$

$\therefore z_4 = \frac{ac}{a-c}$ and $e_4 = \frac{ac\sqrt{2}}{a-c}$. Now $\frac{1}{e_1} - \frac{1}{e_4} = \frac{1}{\sqrt{2}}$

$\frac{b-a}{ab}$, and $\frac{1}{e_2} - \frac{1}{e_3} = \frac{1}{\sqrt{2}} \cdot \frac{b-a}{ab}$: or $\frac{1}{e_1} - \frac{1}{e_4} = \frac{1}{e_2} -$

$\frac{1}{e_3}$, hence e_1, e_2, e_3 and e_4 are in harmonical proportion,

e_1 and e_4 being the extremes, and e_2, e_3 the means.

THIRD SOLUTION, by Mr. DAVIES.

To establish this theorem with more simplicity, we shall lay down the following Lemma:

Let $\angle BAC$ be any angle, and AD the line bisecting it. Let D be a point in that line, through which any line as BC is drawn. Then denoting AD by a , and $\angle BAC$ by 2α , we shall have

$$\frac{1}{AB} + \frac{1}{AC} = \frac{2 \cos \alpha}{a}.$$

For denote the angle ADC by θ . Then

$$\frac{1}{AC} = \frac{1}{a} \cdot \frac{\sin \overline{2\alpha + \alpha}}{\sin \theta} = \frac{\sin \alpha}{a} \{\cot \alpha + \cot \theta\}$$

$$\frac{1}{AB} = \frac{1}{a} \cdot \frac{\sin \overline{2\alpha - \alpha}}{\sin \theta} = \frac{\sin \alpha}{a} \{\cot \alpha - \cot \theta\}.$$

$$\text{Hence } \frac{1}{AC} + \frac{1}{AB} = \frac{2 \cos \alpha}{a} \dots (a)$$

It may, from this, be shewn, that the property is not confined to the regular octahedron, but that it *appertains to any pyramid raised perpendicularly upon a rectangular base*.

Let D be the centre of the rectangular base $BECK$: then the angles $\angle KAE$, $\angle BAC$ which are equal to one another, are bisected by AD . Let any plane pass through D and cut the four edges of the pyramid in F, H, G, L , as in the figure. Then, lemma,

$$\frac{1}{AF} + \frac{1}{AH} = \left(\frac{2 \cos \alpha}{a} \right) = \frac{1}{AL} + \frac{1}{AG} \dots (b)$$

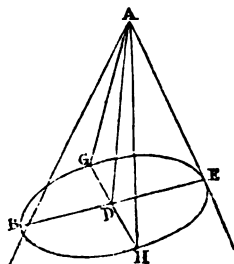
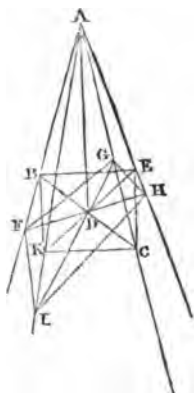
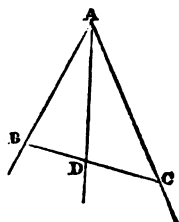
and hence AF, AL, AG, AH are in harmonical proportion, as stated in the theorem.

Cor. 1. If the pyramid be raised perpendicularly on any *parallelogram*, then shall

$$\frac{1}{AF} + \frac{1}{AH} : \frac{1}{AL} + \frac{1}{AG} :: \cos \alpha : \cos \alpha'$$

where α, α' are the semivertical angles of the sections of the pyramid.

Cor. 2. If any plane cut the axis of a given right cone in D , and any second plane drawn through AD cut the sides of the cone in AG, AH .



Then

$$\frac{1}{AG} + \frac{1}{AH} : \frac{1}{AD} = \text{constant ratio.}$$

This formula becomes applicable to the case where an *edge*, as well as an angle, has undergone a decrement by making one of the quantities infinite, the plane in this case being parallel to the edge in question. Thus for such a decrement from the edge of a right pyramid upon a rectangular base we shall have

$$\frac{1}{AB} + \frac{1}{AD} = \frac{1}{AC}.$$

Scholium. The property so far as the regular octahedron is concerned, appears to be due to that able Mineralogist, Mr. Lévy; and was first published in the form

$$“ \frac{1}{AF} + \frac{1}{AH} - \frac{1}{AL} = \frac{1}{AO} ”$$

by Mr. Brooke, in his *Treatise on Crystallography*, p. 317-18. The demonstration given by the latter gentleman, was not founded upon mathematical considerations so much as on the arbitrary, and I had almost said absurd, methods of denoting and determining the decrements of the crystalline edges and angles, which deform, even to the present day, the investigations of those geometrical figures which nature so profusely puts before us in her different mineral productions. In the *Phil. Mag.* June 1826, I gave an investigation of this elegant property (see Vol. LXVII. p. 52), not materially different from the above, but confined to the case where $\alpha = 45^\circ$, or the *regular octahedron*.

Perhaps, whilst I am upon this subject, it might not be irrelevant to express my thorough conviction of the impossibility of employing the co-ordinate methods to investigate the relations between the faces and edges of crystals. The expressions when applied to other than the simplest cases become so fearfully complex as to banish all hope of success by means of them—and especially when the constants are expressed in terms of the axes cut off by each plane, rather than in trigonometrical functions of the planes employed. For a few faces lying near the origin of co-ordinates, or situated in particular relations to them, tolerably neat expressions have already been obtained: but when we apply the method to more complex figures—it may safely be predicted that they will always fail, as they have hitherto done.

An attempt, the first I know of, was made by M. Lamè, in 1818, in his little tract entitled *Examen des differens méthodes employées pour résoudre les problèmes de Géométrie*. He gave a chapter (pp 92-105) on the analytical expressions for the faces,

edges, &c. and applied it to the hexahedron, tetrahedron, rhomboidal dodecahedron, and one or two other figures. Six years after this, (viz. 1824.) a paper in the Philosophical Transactions from the pen of Mr. Whewell made its appearance, containing, however, scarcely more, if any thing at all more, than a repetition of M. Lamé's processes. It is a matter of regret that the labours of the Professor of Mineralogy in the University of Cambridge should be bestowed on fruitless enquiries like these: and fruitless we may expect them,—as well as his attempts to *merely modify* a notation which is *radically defective in principle*—a notation formed before the facts to be included under it, and the detailed purposes to which it is to be applied are at all understood—fruitless, I say, we may fairly consider them, since the chances against the permanency of the system under such circumstances are almost infinity to one. I regret, too, to learn that some of the other ingenious speculations of Mr. Whewell are anticipated with respect to "notation," as we are told in the *British Critic*, Jan. 1831, is the case respecting his *Mathematical principles of Political Economy*, the ground having been already occupied by the Baron Dupin. These trivial accidents, however, can do little in the eyes of Mr. Whewell's admirers towards depriving him of his own selected *nomme du guerre*—*αἰωνιμος*—"worthy of honour."

Other methods have been devised for expressing the figure, and especially the subordinate varieties of figure, which crystals put on during the unknown process of their formation. The method of Häuy is well known, and its defectiveness perfectly understood. The chief contrivances, indeed, that have been employed consist in referring the family of forms to some primary form or nucleus, from which they can be generated by certain specified laws of increment or decrement. This is strikingly the case with the systems of Häuy and Mohs; and the essential difference between these two is the particular nuclei, and consequently the subordinate modifications they undergo, to produce any particular crystalline form. The *Natural-History system* of Mohs, may be seen fully developed in the Edinburgh Philosophical Journal, and in Haidinger's Mineralogy; and some modifications and attempts to improve the notation are printed in the Cambridge Transactions by Mr. Whewell. Several curious properties and mutual relations of the figures which crystals more commonly assume is given in Larkin's Solid Geometry: but the want of systematic classification will doubtless be considered by some a serious defect. Mr. Brooke, too, has beside his list of linear and angular forms produced by decrements from a large collection of alleged pri-

mary forms, devoted one part of his book to the application of spherical trigonometry to the investigation of the angles solid, dihedral and plane, of the faces and edges of crystals. The very complicated calculations to which this process inevitably leads is an objection to its use, and an inducement to search for methods of more facility. Mr. Brooke's utter rejection of all the compendious methods of calculation by subsidiary angles and otherwise, so well known to those ever so little conversant with astronomical reductions, gives a repulsiveness to the whole system, which cannot procure it much favour from the mineralogist who is ignorant of mathematical science—to whom alone his book seems to be addressed. Besides, after all this is done, after all the angles are computed, there is no provision made for the determining the linear magnitudes concerned in the enquiry.

Horizontal and vertical projections have also been proposed, and occasionally employed, but the complexity of the figures render the method ineligible and almost useless for delineative purposes: yet, certainly, so far as perfectly defining the figure is concerned, though it does not furnish the form at a single glance, the Descriptive Geometry gives as complete a statement of the boundaries in question as any method yet devised. Yet I am far from thinking that this is the course that will be ultimately adopted by mineralogists, its inconveniences being of a nature fatal to general use. The Isometrical Perspective of the first President of the Cambridge Philosophical Society, and published in the Transactions of that Body, was considered by Mr. Farish to be well adapted to the delineation of crystalline form. *Had the faces of every crystal been formed by faces parallel to three particular planes*, it is true the linear quantities concerned that is the edges, diagonals, &c. might have been *isometrically projected*: but still the angles of those faces and of those edges would *not* have been so projected. When, however, we find every diversity of angle, solid, dihedral, and plane, occurring in strange varieties of combination, it appears a most extraordinary assertion to state that isometrical perspective is calculated to exhibit by a single figure on one plane, a knowledge of the three dimensions of space which that figure occupies. This, indeed, is the first attempt that I recollect, amongst the many strange fancies that writers on perspective have indulged, to establish the maxim—that the object itself can be inferred from one single projection however it may be made. It is true that in pictorial delineation we can by means of light, shade, colour, and other accessory circumstances give a general notion of the "*third dimension*" which is occupied beyond the plane of the picture by the object delineated: yet it is contrary

to every principle of geometry and common sense to suppose that even these can perfectly enable us to judge of the actual distance of any point in space from its delineation on the picture. Direction alone is furnished by the eye, and this is all the picture tells us of the true place of the object: all other knowledge is collateral and accidental, and hence very inefficient as to numerical quantities. That Mr. Farish's method may serve very well for machinery I have no doubt—but—what is the difference between that method and the old Military Perspective? I am in some degree surprized to see the method brought forward with any pretensions to novelty—and still more to see it stand as the earnest of the labours of that body in the opening of whose transactions it made its appearance.

Mr. BAINES also furnished a Solution to this Question.

VII. QUESTION 517, by Mr. DAVIES.

Two parabolas are given on the same plane, to each of which it is required to draw a tangent, so that they shall be parallel to one another, and the focal radii-vectores of the points of contact shall meet in a given line.

SOLUTION, by Mr. DAVIES.

Let $4a$ and $4a'$ be the parameters of the parabolæ, and κ the angle formed by their axes. Take these axes for co-ordinates of the system, and let b, b' be the distances of the vertices from the origin respectively. Then by the usual methods of transformation, the equations of the parabolæ become

$$y, = \frac{4ab + x,^2 \sin^2 \kappa - 4ax, \cos \kappa}{4a} \dots (1)$$

$$x,, = \frac{4a'b' + y,,^2 \sin^2 \kappa - 4a'y,, \cos \kappa}{4a'} \dots (2)$$

and because the tangents are parallel we have $\frac{dy,,}{dx,,} = \frac{dy,}{dx,}$, or from (1, 2)

$$\frac{2a'}{y,, \sin^2 \kappa - 2a' \cos \kappa} = \frac{x, \sin^2 \kappa - 2a \cos \kappa}{2a} \dots (3).$$

The co-ordinates of the foci of the parabolæ are

$$x' = 0, \text{ and } y' = a + b = c$$

$$x'' = a' + b' = c', \text{ and } y'' = 0.$$

Also the equations of the focal radii vectors are

$$(y - c) x' = (y, - c) x \dots\dots (4)$$

$$y, (c' - x) = (c, - x,) y \dots\dots (5)$$

and the given line in which the focal radii vectors are required to meet may be written

$$Ax + By + c = 0 \dots\dots\dots (6).$$

Eliminating x and y from (4, 5, 6) we obtain

$$Ay, (cc' - cx, - c'y,) + Bx, (cc' - cx,, - c'y,,) + c(cc' - cx,, - c'y, + x,,y, - x,y,,) = 0 \dots\dots (7).$$

Also from equations (2, 3) we have

$$y, = \frac{2a'(x, \cos \kappa + 2a)}{x, \sin^2 \kappa - 2a \cos \kappa} \dots\dots\dots (8)$$

$$x,, = \frac{a(x, \sin^2 \kappa - 4a)(b' \sin^2 \kappa - a' \cos^2 \kappa) + 4a^2(b' \cos^2 \kappa + a')}{(x, \sin^2 \kappa - 2a \cos \kappa)^2} \dots (9)$$

For $y,, y,, x,,$ in (7) put their values given by (1, 8, 9) and we get

$$Mx,^4 + Nx,^3 + Px,^2 + Qx, + R = 0 \dots\dots (10).$$

As the four roots of this equation (10) are all real, or two real, or all imaginary so will there be four, two, or no construction possible. The mode of constructing this equation by means of the conic sections, is too well known to call for further illustration, and we may consider the problem perfectly solved.

Messrs. BAINES, GODWARD, and MASON also answered this Question.

VIII. QUESTION 518, from REYNAUD and DUHAMEL *Problèmes et développemens sur diverses parties Mathématiques.*

"Let $ABCDE$ be any polygon inscribed in a circle, $FGHIK$ a polygon circumscribed whose sides touch in A, B, C, D, E of the inscribed one: if with any radius we describe a new circle concentric with the first, the sum of the squares of the distances of these summits A, B , &c. from any point M of the second circle, multiplied respectively by the tangents KI, KF, FG, GH, HI , will be always the same, that is to say, the sum of the solids, $MA^2 \cdot KI + MB^2 \cdot KF + MC^2 \cdot FG + MD^2 \cdot GH + ME^2 \cdot HI$ is constant."

Then $CL : LE : QL : LM :: CG : GE : GB^2$, by the figure ;

$$CL^2 : LP^2 :: CG^2 : GB^2, \text{ by sim. triangs.}$$

$$\therefore GC : LE : GE : LE :: QL : LM : LP^2, \text{ or,}$$

$$GE : LC - GL : CE : GE : CL :: PL^2 - PM^2 : PL^2$$

$$\therefore GE : CL : GL : CE :: LP^2 : PM^2 \dots (1)$$

In like manner, we shall find

$$KE : BN : KN : BE :: NP^2 : PM^2 \dots (2)$$

But by the parallels GB, LP, CK , we have

$$GE : CL : GL : CE :: KN : BE : KE : BN \dots (3)$$

Hence by comparing the results in (1, 2, 3) we find

$$PN^2 : PM^2 :: PM^2 : PL^2, \text{ or}$$

$$PN : PM :: PM : PL$$

But P is the middle of QM , and by well known properties,

$$QO : OM :: QL : LM$$

Hence, &c.

Q. E. 1^o. D.

Case 1st. Otherwise.

Because BC is a tangent at B , it is conjugate to the point B : and hence (by A , Repos. Vol. V. Part II. p. 150) we have

$$QN \cdot NM = PN \cdot NL$$

Also, because BC is a diameter, and QM ($\parallel BC$) is a conjugate chord, QM is bisected in P , Hence (Lawson's Theorems, th. 1. conv.) the line QM is harmonically divided in N and L .

Q. E. 1^o. D. Aliter.

Case 2. Let the point not be infinitely distant ; that is, let BC, QM, CK meet in one point T .

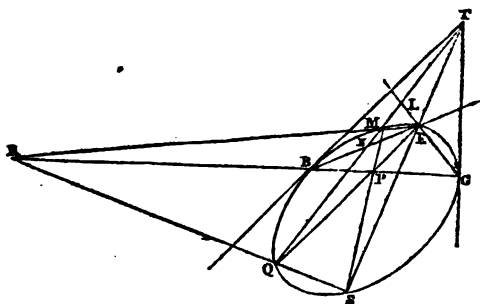
Here the figure is the perspective projection of that considered in the last case: and as QM was there harmonically divided, and the perspective of an harmonical line is itself harmonically divided, the line is also in this case divided as was stated in the proposition.

Q. E. 2^o. D.

Otherwise.

The general proposition, however, might be demonstrated at once, without the aid of projection.

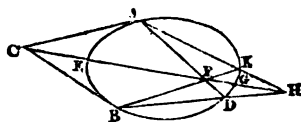
Let T be the intersection of the tangents TB, TC ; and TE be any line drawn from T to cut the curve in M, Q ; and the lines CE, BE (drawn to any point E in the curve from the points of contact) in L, N . Draw TE to cut the curve again in S , and join E, M , and S, Q . These, it is well known meet at some point R in BC . Join Q, E , and M, S ; these also meet in BC , in a point P : and the line BC is harmonically divided in P, R . Then, also, since BME, RNE, QPE , and CEL are drawn to these points of section, these lines are themselves a set of harmonical sectors: and since TQ crosses them, and is cut by them in Q, M, N, L , these four points are also points of harmonical division of QM , as was stated in the enunciation of the theorem,



Q. E. D.

SECOND SOLUTION, by Mr. GODWARD, Wakefield.

Join AH intersecting the curve in K , and BK intersecting AD in F . Then, by § 44, Maclaurin on the properties of Geometrical lines, the three points H, F, C are in the same straight line. Hence CH and the diagonals AD, BK of the trapezium $ABDK$, have a common point of intersection. Wherefore, by Emerson's Conic Sections, B. I. Prop. 71, Cor. 1. B. II. Prop. 64, Cor. 1. B. III. Prop. 38, Cor. 1, $EH : GH :: EF : FG$.



Q. E. D.

Solutions were also given by Messrs. JOHN BAINES and THOMAS THOMPSON,

X. QUESTION 520, by SPHÆRUS.

From any point in one given sphere describe a cone to envelope another given sphere, and find by considerations *purely*

geometrical, the curve surface to which the plane of contact is always tangential.

FIRST SOLUTION, by Mr. T. S. DAVIES, Bath.

I. Let A, B be the centres of the enveloped and directing spheres, and L any point in the latter, which is the vertex of one of the cones. Draw LA cutting (B) again in M and (A) in H. Then the cone whose summit is L has with the sphere (A) a plane of contact which divides AH in I so that

Fig. 1.

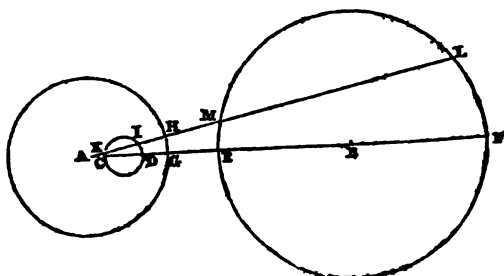
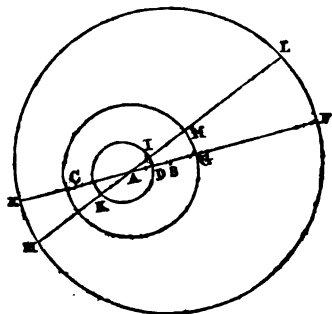


Fig. 2.

$$AI : AH :: AH : AL.$$

It will be sufficient then to seek the locus of I whilst L moves through all the points of the sphere (B).

Let a plane be drawn through ABL. Then it cuts the spheres (A), (B) in circular sections AGH, BLF respectively.



$$\begin{aligned} \text{Then} \quad & AI : AH :: AH : AM \quad \} \\ \text{and} \quad & AL : AF :: AE : AM \quad \} \\ \therefore \quad & AI : AL :: AH^2 : EA \cdot AF. \end{aligned}$$

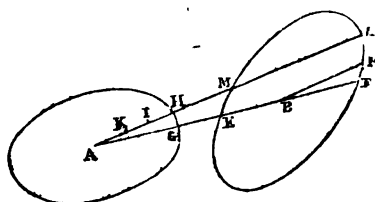
But $AH^2 : EA \cdot AF$ is a given ratio, and therefore also

$AI : AL$ is given; and hence the locus of I is a given circle. Apoll. Loc. Plan. lib. II. pr. 14, Leslie's Geom. Anal. book III. pr. 4, &c.

In like manner all the other positions of L give circles on the diameter CD the assemblage of which constitute a sphere, the locus sought.

II. The similarity of the locus to the enveloped and directing surfaces leads us to enquire further whether the property is confined to the case of the sphere. Let the letters in fig. (3), so far as they are common to those figs. (1), (2) represent corresponding lines and points. Draw BF parallel to AL . Then as before

Fig 3.



$$AK = \frac{AH^2}{AL}.$$

$$\text{But } AL \cdot AM : AG \cdot AF :: BF^2 : FB^2.$$

$$\text{Hence } AK : AM :: \frac{BF^2}{AG \cdot GF} : \frac{AH^2}{BF^2}.$$

Now if $\frac{AH}{BF}$ be constant, then $AK : AM$ is a given ratio, and hence reasoning nearly as is done by Apollonius and Leslie, as referred to above, we shall find the locus of K to be a figure similar and similarly situated with (B). But that $AH : BF$ may be constant, (A) and (B) must be similar and similarly situated figures. Hence we find generally (very obvious steps of the reasoning being supplied by the reader) that when the vertex of a cone moves upon one surface of the second order, and the cone itself envelopes a similar and similarly situated surface of the second order, then the plane of the envelopement will constantly be tangential to a third similar and similarly situated surface of the second order.

III. It might be shewn from the analogy found above, viz.

$$AK : AM :: \frac{BF^2}{GA \cdot AF} : \frac{AH^2}{BF^2}$$

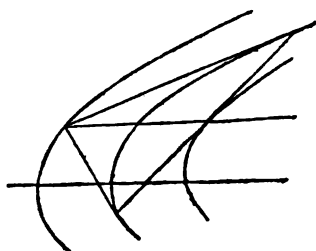
that whether $\frac{AH}{BF}$ be constant, or subjected to such changes as

result from the conditions which those lines are subject to in the figure (3), the surface touched by the planes of envelopement is still of the second order. But it is more readily found by projective considerations: for there can always be found two circles, and a constant position of the eye with respect to their plane and another given plane or picture, such that the perspective of the circles upon the picture shall be any two conic sections whatever. The locus touched by the chords of contact

must then, being itself a circle, be projected into a conic section also. Hence we infer that whatever be the species, magnitudes, and positions of the directing and enveloped surfaces, so long as they are of the second degree, the plane of envelopement will always touch a third surface of the second order.

IV. In these remarks I have not considered the paraboloid particularly: but indeed it scarcely needs it, as this like the other surfaces of the second order are contained in the remarks in III. When similar and similarly situated figures are considered the paraboloids have common axes and are equal in all respects: so that the property will readily offer to the mind of every geometer, its own demonstration the moment he has constructed his figure.

Fig. 4.



V. It may be well to notice that A is the pole of similitude of the two similar surfaces which direct the movement of the vertex and touch the plane of envelopement. Some neat properties of the figure under this aspect may be given did space allow us to do so.

In the preceding remarks I have been rather desirous to point out the principal steps in the reasoning by which the proposition may be established in its most general cases, than to fill up the minutæ details, which though essential to a finished demonstration, are too simple to arrest the progress of any attentive reader: and as the statement of all those details would have taken the room which might be more advantageously employed in giving *different methods* of solution, I thought it better to omit them.

Two analytical solutions of this problem were given in 1806, in the Journal of the Polytechnic School by M. M. Livet and Brianchon, and which are here annexed. What I have to add concerning these solutions, which is very trifling, I shall put in the form of notes.

M. BRIANCHON'S SOLUTION, *Journ. de l'Ec. Polyt. Cah.*
13. pp. 308—10.

GENERAL THEOREM.

A surface of the second degree is circumscribed by a variable

cone, the summit of which moves also upon another surface of the second order, disposed arbitrarily in space: then the plane of contact of the cone and first surface will always touch a surface of the second degree.

Demonstration.

Let $Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy + 2Gx + 2Hy + 2Iz = K \dots (1)$ be the equation of the given surface enveloped by the variable cone.

Let us represent the surface in which the summit of the cone moves by

$$Ma^2 + Nb^2 + Pc^2 = Q \dots (2)$$

a, b, c being the current co-ordinates.

When the conical surface which has its summit in any point (a, b, c) of the directing surface (2) envelopes the surface (1) we know (Monge Feuilles d'Anal. No. 5.) that the curve of contact is found upon the surface,

$$z - c = (x - a) \left(\frac{dz}{dx} \right) + (y - b) \left(\frac{dz}{dy} \right) \dots (3)$$

in which $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are given by equation (1).

The system of equations (1, 3) represent, then, the curve of contact. But in taking the difference of these equations we obtain the following:

$$\left. \begin{aligned} x(Aa + Fb + Ec + G) + y(Fa + Bb + Dc + H) \\ + z(Ea + Db + Cc + I) + Ga + Hb + Ic + K \end{aligned} \right\} = 0,$$

which being linear represents a plane, viz. that upon which this curve of contact is situated.

This last equation may be written in a more simple form:—

$$aa + \beta b + \gamma c + \delta = 0 \dots (4)$$

by making, for brevity,

$$Ax + Fy + Ez + G = \alpha$$

$$Fx + Cy + Dz + H = \beta$$

$$Ex + Dy + Cz + I = \gamma$$

$$Gx + Hy + Iz + K = \delta.$$

Let us now suppose the summit of the cone, though always situated upon the surface (2) to be removed to an infinitesimal distance in any direction from the point (a, b, c) of this surface. Then the plane (4) of the curve of contact will take a new position infinitely near to its former one: and if equation (4) be differentiated relative to a, b, c , considering x, y, z constant,

we shall virtually suppose that x, y, z appertain to the straight line which is the intersection of the consecutive planes of contact. We shall have, then, for x, y, z of this straight line, simply

$$ada + \beta db + \gamma dc = 0.$$

It is now requisite to express that the moving summit of the cone is always found upon a known surface, that is, that the co-ordinates a, b, c of this summit are connected by equation (2); which gives by differentiation

$$mada + nbdb + pc dc = 0.$$

Eliminating dc between these differential equations, we get

$$(ma\gamma - pc\alpha) da + (nb\gamma - pc\beta) db = 0.$$

But the summit (a, b, c) having been displaced in any direction whatever, the ratio $da : db$ is perfectly arbitrary, and may be as great or as little as we chuse. The last equation, then, requires the separate conditions

$$ma\gamma - pc\alpha = 0 \dots\dots\dots(5)$$

$$nb\gamma - pc\beta = 0 \dots\dots\dots(6)$$

then the co-ordinates x, y, z , of which $\alpha, \beta, \gamma, \delta$ are functions, appertains to a *particular point* of the plane (4) which is common to it and any two other consecutive planes whatever. We have, then, for the determination of this particular point, the linear equations (4, 5, 6).

Thus the plane of contact (4) of which the motion is regulated by that of the summit of the cone, touches in this point (x, y, z) a certain surface which it is now required to determine.

For arriving at the equation of this surface, which is the locus of all the points determined as has been just shewn, it is requisite to eliminate the quantities a, b, c , between the four equations

$$ma^2 + nb^2 + pc^2 = Q \dots\dots\dots(2)$$

$$aa + \beta b + \gamma c + \delta = 0 \dots\dots\dots(4)$$

$$ma\gamma - pc\alpha = 0 \dots\dots\dots(5)$$

$$nb\gamma - pc\beta = 0 \dots\dots\dots(6)$$

and the resulting equation in x, y, z is that of the surface sought, since it expresses the relation which exists amongst the co-ordinates of the point (x, y, z) wherever on the surface (2) the summit of the cone be situated.

In performing the calculation we find for the result an equation of the second degree.

$$MNP\delta^2 = Q(NP\alpha^2 + MP\beta^2 + MN\gamma^2) \dots\dots\dots(7)$$

which is the surface to which the plane of contact is always tangential.

W. W. R.

Cor. 1. There exists a remarkable relation between the surface (2) and the surface (7) which is the object of our enquiries: viz. that if the one be a twisted surface the other is so too.

Cor. 2. Converse of the General Proposition: If a surface of the second order (7) be touched by a plane which cuts another surface of the second order (1), then the cone which envelopes the latter at the current plane section, will have its vertex always in a third surface of the second order (2).

Cor. 3. (By Mr. Davies). If $D = E = F = G = H = I = 0$, and $M = A$, $N = B$, and $P = C$: that is if the surfaces (1, 2) are concentric similar and similarly situated, we shall have the equation (7) under the form

$$Ax^2 + By^2 + Cz^2 = \frac{K^2}{B}.$$

That is, this surface is also concentric, similar and similarly situated with them, agreeing with what has been shewn by M. Livet in the same work, p. 296, vel Repos. p. before.

The SOLUTION of M. LIVET, Journ. de l'Ec. Poly. Cah 13. p. 284.

Monge in his lessons to the Polytechnique School has proved that when a conic surface envelopes a surface of the second order, the curve of contact is always plane; that this plane (the position of which depends upon that of the vertex of the cone) possess always by the same straight line when the summit of the circumscribing cone moves in a straight line. Also supposing the summit of the cone moves in a determinate plane, Monge has shewn that the plane of the curve of contact passes always by same point.

We will explain this theory, and add to the propositions enunciated above, several other properties very worthy of observation.

Representing by a, b, c the co-ordinates of the summit of a conic surface, we shall have the equation of partial differences of this surface

$$z - c = (x - a) \frac{dz}{dx} + (y - b) \frac{dz}{dy} \dots\dots (\Delta).$$

Let the equation of the envelope surface be

$$A^2x^2 + B^2y^2 + C^2z^2 = 1 \dots\dots\dots (\text{B}).$$

It is necessary, in order that the equation (A) may adapt itself to the curve of contact of the conical surface and of the surface of the second degree denoted by (B), that the values

of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ be the same as those given by the equation (B)

for these same differential coefficients. We shall have then for one of the equations of the curve of contact

$$z - c = -(x - a) \frac{A^2 x}{C^2 z} - (y - b) \frac{B^2 y}{C^2 z},$$

which reduces itself to

$$A^2 ax + B^2 by + C^2 cz = 1 \dots\dots\dots (1).$$

As this equation is linear in respect to the co-ordinates, x , y , z , it follows that the curve of contact of a conic surface and a surface of the second degree is plane.

The equation (1) may be put under the form

$$z = -\frac{A^2 a}{C^2 c} \cdot x - \frac{B^2 b}{C^2 c} \cdot y + \frac{1}{C^2 c} \dots\dots\dots (C).$$

We see that the coefficients of x and y remain the same whilst the ratios $\frac{a}{c}$ and $\frac{b}{c}$ remain constant; which takes place even when a , b , c are variable, if at the same time they be connected by the equations $a = mc$, $b = nc$, which appertain to a straight line passing through the origin of co-ordinates. It follows from this that if the summit of the circumscribing cone move upon a straight line passing by the origin of co-ordinates, then the curve of contact will also move parallel to itself.

Now considering in the equation (C) the constant term $\frac{1}{C^2 c}$, it represents, obviously, the vertical ordinate of the point of intersection of the plane which includes the curve of contact, and which we shall call the *plane of contact*, with the axis of z . This point will be always the same when the quantities a , b , are indeterminate if c remain constant: which proves, that, if a conical surface circumscribing a surface of the second degree, so move that its summit be always in a horizontal plane always distant from that of x , y , by a quantity c , the plane of contact will always cut the axis of z in the same point, the distance of this point from the origin will be $\frac{1}{C^2 c}$, a quantity which is absolutely independent of A and B , so that it will be the same for

all surfaces of the second degree whose equations do not differ otherwise than by these two parameters.

Returning to equation (1), we see that when a, b, c have fixed and determinate values, the curve of contact will have a fixed and determinate position: but when these quantities are variable, or, which is the same thing, when the summit of the cone changes its position in space, the curve of contact takes also different positions. But the summit of the cone may move in space in three different ways.

1°. By following a curve of double curvature.

2°. By following a curve surface.

3°. The summit of the cone may move in a discontinuous manner.

We shall consider the circumstances which relate to each of these particular movements.

Let $\left\{ \begin{array}{l} F(a, b, c) = 0 \\ f(a, b, c) = 0 \end{array} \right\}$ be equations of the curve of double

curvature upon which the summit of the cone is confined during its motions, and which we shall call the *directing-curve* or *directrix*.

When the summit of the cone corresponds to two consecutive points of the directrix, there will result two different curves of contact; their planes intersect in a straight line of which the position in space is dependant upon that of the summit of the cone: and there will be to each pair of consecutive positions, a similar straight line. The locus of all these lines forms a developable surface, the nature of which depends upon that of the directing curve.

To obtain the equation of this straight line in any position of the cone, it is requisite to differentiate the equation (1) relative to a, b, c , and regarding x, y, z as constant, since the intersections of the two consecutive planes of contact does not vary except when a, b, c vary. The result of this differentiation will be

$$A^2 x da + B^2 y db + C^2 z dc = 0.$$

This equation may be freed from the differentials which it includes by differentiating the equations of the directrix, which will give the values of any two of the quantities da, db, dc . By means of the third we shall arrive at a finite equation between x, y, z, a, b, c .

If we eliminate a, b, c between the equations

$$F(a, b, c) = 0 \dots\dots\dots (1)$$

$$f(a, b, c) = 0 \dots\dots\dots (2)$$

$$A^2ax + B^2by + C^2cz = 1 \dots\dots(3)$$

$$A^2xdx + B^2ydy + C^2zdz = 0 \dots\dots(4)$$

we shall have the equation of the developable surface which is the locus of all the continual intersections of the planes of contact.

Application to the case where the directrix is a straight line.

In this particular case, the equations of the directrix are of the form

$$a = mc + m'$$

$$b = nc + n'$$

from which we have $da = mdc$ and $db = ndc$, and the four formulæ become

$$a = mc + m' \dots\dots\dots(1)$$

$$b = nc + n' \dots\dots\dots(2)$$

$$A^2ax + B^2by + C^2cz = 1 \dots\dots\dots(3)$$

$$A^2mx + B^2ny + C^2z = 0 \dots\dots\dots(4)$$

The equations (1, 2) reduce equation (3) to

$$(A^2mx + B^2ny + C^2z)c + A^2m'x + B^2n'y = 1.$$

Which, by means of equation (4) becomes

$$A^2m'x + B^2n'y = 1 \dots\dots\dots(5).$$

Eliminating successively y and x between the last and equation (4) we shall have

$$A^2(m'n - m'n)x - C^2n'z = n$$

$$B^2(m'n - m'n)y - C^2m'z = m.$$

These and equation (5) referring to a straight line, it follows that the planes of contact pass by the same straight line.

The plane which joins this straight line to the centre of the developable surface being represented by

$$z = mx + ny,$$

the constants M and N are determined by the following equations

$$MB^2n - NA^2m = 0,$$

$$A^2B^2(m'n - mn') - MB^2C^2n' + NA^2C^2m' = 0.$$

They are considerably simplified in the case of the sphere, where $A = B = C$, as they reduce to

$$Mn - Nm = 0$$

$$m'n - mn' - Mn' + Nm' = 0.$$

We deduce from this, $M = -m$ and $N = -n$: which proves that in the sphere, the plane which joins the common

intersection of all the planes of contact is perpendicular to the rectilinear directrix.

Application to the case where the linear directrix is a plane curve of any kind whatever.

In this particular case, the equations of the directrix may be represented under the form

$$C = Ha + Ib + L,$$

$$B = \phi a.$$

The equations (1, 2, 3, 4) referred to above become

$$C = Ha + Ib + L$$

$$B = \phi a$$

$$A^2ax + B^2by + C^2cz = 1$$

$$A^2x da + B^2y db + C^2z dc = 0.$$

Making $db = \phi' \times da$ and eliminating c, b , we shall obtain

$$A^2ax + B^2y \cdot \phi + C^2z (Ha + I\phi + L) = 1$$

$$A^2x + B^2y\phi' + C^2z (H + I\phi') = 0.$$

The nature of the function ϕ remaining unknown it is absolutely impossible to effect the elimination of a between these two equations: but we may at least arrive at the equation of the partial differences of the surface which we seek. For this, it is sufficient to differentiate the first, the former relative to x, y, z , and a , which gives

$$da[A^2x + B^2y\phi' + C^2z(H + I\phi')] + A^2adx + B^2\phi dy$$

$$+ C^2(Ha + I\phi + L) \left(\frac{dz}{dx} \cdot dx + \frac{dz}{dy} \cdot dy \right) = 0.$$

Making, for brevity, $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, and remarking that

the multiplier da is $= 0$ in virtue of the preceding equations, we shall have

$$dx[A^2a + C^2p(Ha + I\phi + L)] + dy[B^2\phi + C^2q(Ha + I\phi + L)] = 0$$

and because of the independence of dx and dy we have also,

$$(A^2 + C^2pH)a + C^2pI\phi + C^2Lp = 0$$

$$C^2qHa + (B^2 + C^2Iq)\phi + C^2Lq = 0.$$

We obtain from these

$$a = - \frac{B^2C^2Lp}{A^2(B^2 + C^2Iq) + B^2C^2pH},$$

$$\phi = - \frac{A^2C^2qL}{A^2(B^2 + C^2Iq) + B^2C^2pH}.$$

Then the equation $A^2ax + B^2y\phi + C^2z(Ha + I\phi + L) = 1$, may be put under the form

$$(A^2x + C^2Hz)a + (B^2y + C^2Iz)\phi + C^2Lz = 1.$$

If we substitute for a and ϕ the values found above, the expression reduces to

$$\frac{A^2B^2(z - px - qy)}{A^2(B^2 + C^2Iq) + R^2C^2\rho H} = \frac{1}{C^2L}.$$

Multiplying out, and arranging the results, we get

$$z - \frac{1}{C^2L} = \left(x + \frac{H}{A^2L}\right)\rho + \left(y + \frac{I}{B^2L}\right)q.$$

Under this form it is obvious that the developable surface engendered by the motion of the plane of contact is a conic surface, the co-ordinates of whose summit are

$$- \frac{H}{A^2L}; - \frac{I}{B^2L}, \text{ and } \frac{1}{C^2L}.$$

As the result to which we have arrived is absolutely independent of the form and position of the directing curve in its plane we may conclude from it that when the summit of a conic surface circumscribing a surface of the second order moves upon a plane surface, the plane of contact passes in all its positions through the same point.

When we consider the motion to be upon a surface we shall demonstrate this proposition in a more simple manner. If the quantities H and I become null, that is if the plane of the directing curve become horizontal the equation

$$z - \frac{1}{C^2L} = \left(x + \frac{H}{A^2L}\right)\rho + \left(y + \frac{I}{B^2L}\right)q \text{ becomes}$$

$$z - px - qy = \frac{1}{C^2L}.$$

which appertains to a conic surface of which the summit is situated upon the axis of z . If the quantities H , I , L become null at the same time, we have

$$z - px - qy = \alpha.$$

This equation appertains to a cylindrical surface of which the generatrix is vertical. In fact, $z - px - qy$ is nothing more than the vertical ordinate from the point of intersection of the tangent plane with the axis of z , since this ordinate is infinite. This proves that the tangent plane is, in all its positions, parallel to the axis of z . Hence, the surface of which the equation is

$$z - px - qy = \alpha.$$

is a vertical cylindrical surface. Finally the quantity L may be nul without H and I being so; in which case, we shall have

$$1 = -\frac{C^2 H}{A^2} p - \frac{C^2 I}{B^2} q.$$

This equation appertains to a cylindrical surface of which the generatrix is parallel to a straight line represented by the equations

$$x = -\frac{C^2 H}{A^2} z + \alpha$$

$$y = -\frac{C^2 I}{B^2} z + \beta.$$

In the case where the surface of the second degree we are considering is a sphere, this straight line is perpendicular to the plane of the directrix, supposed given by the equation

$$c = Ha + Ib + L.$$

When the directrix is a curve surface.

Let its equation be $F(a, b, c) = 0$. Then supposing the summit of the cone to take two positions infinitely near to one another upon this surface, the plane of contact takes also two consecutive positions, and the line of intersection of these planes will have for equations

$$A^2 ax + B^2 by + C^2 cz = 1,$$

$$A^2 x da + B^2 y db + C^2 z dc = 0.$$

Supposing $dc = pda + qdb$, we shall have for the second of the preceding equations

$$A^2 x + C^2 pz + (B^2 y + C^2 qz) \frac{db}{da} = 0.$$

The position of this line depends upon the numerical value of the ratio $\frac{db}{da}$. If we differentiate this equation, only varying $\frac{db}{da}$ we shall have an equation relative to the point of intersection of this straight line with that which corresponds to the variation of $\frac{db}{da}$. This operation leads us to the two following equations:

$$A^2 x + C^2 z \cdot \frac{dc}{da} = 0,$$

$$B^2 y + C^2 z \cdot \frac{dc}{db} = 0,$$

The locus of all the points determined in this manner forms a particular surface of which the nature is intimately related to that of the curve surface which directs the summit of the cone. We shall obtain this surface by eliminating a, b, c between the following equations

$$F(a, b, c) = 0 \dots\dots\dots (1')$$

$$A^2ax + B^2by + C^2z = 1 \dots\dots\dots (2')$$

$$A^2x + C^2z \cdot \frac{dc}{db} = 0 \dots\dots\dots (3')$$

$$B^2y + C^2z \cdot \frac{dc}{db} = 0 \dots\dots\dots (4')$$

Application to the case when the directing surface is a plane.

The equation being now of the form

$$c = Ha + Ib + L,$$

we have $\frac{dc}{da} = H, \quad \frac{dc}{db} = I$

which reduces the four preceding equations to

$$C = Ha + Ib + L$$

$$A^2ax + B^2by + C^2cz = 1$$

$$A^2x + C^2Hz = 0 \dots\dots\dots (A).$$

$$B^2y + C^2Iz = 0 \dots\dots\dots (B).$$

Eliminating c between the two former, we have

$$(A^2x + C^2Hz)a + (B^2y + C^2Iz)b + C^2Lz = 1.$$

This reduces by means of the two last to

$$C^2Lz = 1, \quad \text{or } z = \frac{1}{C^2L}.$$

The equations (A) and (B) give, by substituting the value just found of z , $y = -\frac{I}{B^2L}$,

$$x = -\frac{H}{A^2L}.$$

These results shew us that if the summit of the conic surface move upon a plane, all the planes of contact pass by the same point.

The line which joins the centre of the enveloped surface to this point has for equations

$$x = -\frac{C^2H}{A^2}z, \quad y = -\frac{C^2I}{B^2}z.$$

These reduce to

$$x = -Hx, \quad y = -Iz,$$

in the case where the enveloped surface is a sphere: which proves that when a conic surface circumscribes a sphere so that its summit is always in a plane, the line which joins the centre of this sphere to the point of common intersection of all the planes of contact, is perpendicular to the plane-directrix.

We may here notice some of the circumstances which relate to a discontinued movement of the conical vertex.

It must be remarked that the equations

$$z = \frac{1}{C^2 L}, \quad y = -\frac{1}{B^2 L}, \quad x = -\frac{H}{A^2 L},$$

appertain to a single and unique point, provided the quantities H, I, L are constant; but that will not be the case where these quantities are variable. Let us suppose for example, that the quantities H and I are constant and that L alone is variable, the equation

$$c = Ha + Ib + L,$$

appertains to an infinity of parallel planes, and the values of x, y, z , obtained above come to have reference to a system of points situated upon the same straight line passing by the origin of co-ordinates or by the centre of the enveloped surface. In truth, the elimination of L between the expressions for x, y, z conduct us to

$$y = -\frac{C^2 I}{B^2}, \quad x = -\frac{C^2 H}{A^2} z.$$

If on the contrary the quantity L be constant, and the other two vary, the equation

$$c = Ha + Ib + L,$$

appertains to a system of planes passing by the same point in the axis of z . In this case, the value of z is constant, and those of x and y variable. These values conduct then to a system of points situated upon the same horizontal plane surface.

There remains yet one case more to examine: viz. that where the quantity L is constant at the same time as one of the quantities H or I . To give an application of this case, let us suppose that the summit of the conic surface be subjected to move successively upon an infinity of planes passing by the same straight line given by the equations

$$x = mz + m',$$

$$y = nz + n',$$

we shall have amongst the quantities H, I, L, m , and m' the following relations

$$1 = Hm + In$$

$$0 = Hm' + In' + L.$$

from which we have

$$I = \frac{m' + Ln}{m'n - mn'}$$

$$H = - \frac{n' + Ln}{m'n - mn'}.$$

The formulæ $z = \frac{1}{C^2 L}$, $y = -\frac{I}{B^2 L}$, $x = -\frac{H}{A^2 L}$ become

$$z = \frac{1}{C^2 L}$$

$$y = - \frac{m' + Ln}{B^2 L (m'n - mn')}$$

$$x = \frac{n' + Ln}{A^2 L (m'n - mn')}.$$

The elimination of the arbitrary constant L gives the two equations

$$A^2 (m'n - mn') x - C^2 n' z = n$$

$$B^2 (mn' - m'n) y - C^2 m' z = m.$$

These equations are those of the line by which pass all the planes of contact, in supposing that the summit moves upon the line

$$x = mz + m' \quad y = nz + n'.$$

When the vertex of the conical surface moves upon a surface of the second degree.

Suppose the directrix be denoted by the equation

$$A^2 a^2 + B^2 b^2 + C^2 c^2 = 1.$$

We deduce from this

$$\frac{dc}{da} = - \frac{A^2 a}{C^2 c}, \quad \frac{dc}{db} = - \frac{B^2 b}{C^2 c};$$

and we shall have, to eliminate, a, b, c between the following equations

$$A^2 a^2 + B^2 b^2 + C^2 c^2 = 1 \dots\dots\dots(1)$$

$$A^2 ax + B^2 by + C^2 cz = 1 \dots\dots\dots(2)$$

$$A^2 C^2 cx - A^2 C^2 az = 0 \dots\dots\dots(3)$$

$$B^2 C^2 cy - B^2 C^2 bz = 0 \dots\dots\dots(4).$$

From equations (3, 4) we have for the value of c ,

$$c = \frac{A'^2 B'^2 C'^2 z}{A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2}.$$

There results from this

$$a = \frac{A^2 B'^2 C'^2 x}{A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2},$$

$$b = \frac{A'^2 B^2 C'^2 y}{A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2}.$$

Substitute these values of a, b, c in the equation (1) then we shall have

$$\frac{A^4 A'^2 B'^4 C'^4 x^2 + A'^4 B'^2 B^4 C'^4 y^2 + A'^4 B'^4 C^4 C'^2 z^2}{(A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2)^2} = 1$$

which may be put under the form

$$\frac{A'^2 B'^2 C'^2 (A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2)}{(A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2)} = 1,$$

or which still further reduces to

$$A^4 B'^2 C'^2 x^2 + A'^2 B^4 C'^2 y^2 + A'^2 B'^2 C^4 z^2 = A'^2 B'^2 C'^2 \dots \dots \dots (5).$$

This equation belongs to a surface of the second degree of which the centre is the origin of the co-ordinates.

If the enveloped surface as well as the directing surface is an ellipsoid, then the surface determined by equation (5) will also be an ellipsoid, and its axes will have to those of the two others a very simple relation. Let us represent the equation of the enveloped and directing surfaces by the equations

$$b^2 c^2 x^2 + a^2 c^2 y^2 + a^2 b^2 z^2 = a^2 b^2 c^2$$

$$b'^2 c'^2 x^2 + a'^2 c'^2 y^2 + a'^2 b'^2 z^2 = a'^2 b'^2 c'^2.$$

We shall have

$$A^2 = \frac{1}{a^2}, \quad B^2 = \frac{1}{b^2}, \quad C^2 = \frac{1}{c^2}$$

$$A'^2 = \frac{1}{a'^2}, \quad B'^2 = \frac{1}{b'^2}, \quad C'^2 = \frac{1}{c'^2}.$$

Substitute these in equation (5), and we shall get

$$\frac{x^2}{a^4 b'^2 c'^2} + \frac{y^2}{b^4 a'^2 c'^2} + \frac{z^2}{c^4 a'^2 b'^2} = \frac{1}{a'^2 b'^2 c'^2},$$

or

$$\frac{a'^2}{a^4} x^2 + \frac{b'^2}{b^4} y^2 + \frac{c'^2}{c^4} z^2 = 1.$$

Calling the semi-axes of this ellipsoid a'' , b'' , c'' , we have

$$a'' = \frac{a^2}{a'}, \quad b'' = \frac{b^2}{b'}, \quad c'' = \frac{c^2}{c'}.$$

From the preceding calculations it follows that if the summit of a conical surface circumscribed to a surface of the second degree, so move as to be always in another surface of the second degree concentric with the former, the plane of contact in its positions will be always tangential to a surface of the second degree concentric with the two former. If the two first surfaces, that is the enveloped and the directing surfaces, be ellipsoids the third will also be an ellipsoid, each of the axes of which is a third proportional to the axes of the enveloped and directing surfaces which coincide with it in direction.

From the preceding properties we obtain the two following particular cases:

If the summit of an angle circumscribing a curve of the second degree move in a straight line, then the straight line which joins the points of contact of two sides of the angle with a curve of the second degree pass always by the same point; and if the summit of the circumscribing angle move upon an ellipse concentric with the first curve of the second degree, which we suppose also to be an ellipse, the chord of contact in all its positions will be tangential to a third ellipse, each of whose axes is a third proportional to those of the enveloped and directing ellipses which coincides with it in direction.

Mr. THOMPSON sent a Solution to this Question.

XI. QUESTION 521, by Mr. DAVIES.

Let a, b, c be the three sides of a triangle, and r, r', r'', r''' , the radii of the four circles which touch them, then

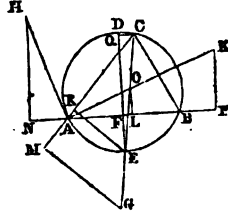
$$ab + ac + bc = rr' + rr'' + rr''' + r'r'' + r'r''' + r''r'''.$$

That is the sum of all the rectangles made by the binary combination of the sides is equal to the sum of all the rectangles made by the binary combinations of the four radii of the circles of contact.

FIRST SOLUTION, by Mr. GODWARD.

Let ACB be any triangle, O, G, H, K the centres, and OL,

GM, HN, KP the radii of the four circles of contact. About the triangle ABC describe a circle and draw the diameter DE bisecting AB in F. Join AH, AK, and draw ER, CQ, perpendicular to AC, DE. By a known property $CM = BN = AP = \frac{1}{2}(AB + AC + BC)$, hence $FN = FP = \frac{1}{2}(AC + BC) = CR$. Also since AH, AK, bisect the angles NAC, BAC, the angle HAK is a right angle, and the triangles HAN, KAP are therefore similar: hence HN



: AN :: AP : KP, or $HN : KP = AN . AP = (NF - AF) \times (NF + AF) = NF^2 - AF^2 = CR^2 - AF^2 = FD . (EQ - EF = FD . FQ)$. But (Prop. 46, M. G. Student) $OL . GM = EF . FQ$; $\therefore OL . GM + HN . KP = FQ . (EF + FD) = FQ . ED = AC . CB$. Similarly $OL . HN + GM . KP = AB . BC$, and $OL . KP + GM . HN = AB . AC$. Wherefore $OL . KP + OL . HN + OL . GM + HN . KP + GM . KP + GM . HN = AC . CB + AB . BC + AB . AC$, which is the theorem enunciated in the question.

Cor. 1. $HN . KP = FD . FQ = (\text{Prop. 14.}) CF^2 - FL^2$.

Cor. 2. $OL . GM + HN . KP = AC . BC$.

Cor. 3. $2(HN . KP - OL . GM) = 2(FD . FQ - EF . FQ) = 2(FD . EQ - FD . EF - AL . LB) = 2CK^2 + 2FL^2 - 4AF^2 = AC^2 + BC^2 - AB^2$.

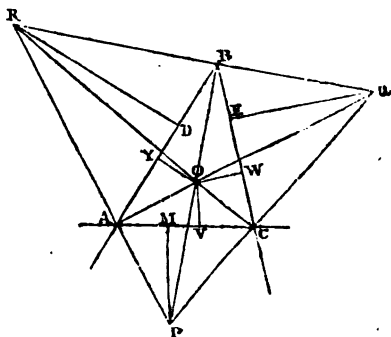
Cor. 4. By Cor. 3, $HN . KP - OL . GM = \frac{1}{2}(AC^2 + BC^2 - AB^2)$, $GM . KP - OL . HN = \frac{1}{2}(AB^2 + BC^2 - AC^2)$ and $HN . MG - OL . KP = \frac{1}{2}(AB^2 + AC^2 - BC^2)$. Whence, by addition, $HN . KP + GM . KP + HN . MG - OL . GM - OL . HN - OL . KP = \frac{1}{2}(AC^2 + BC^2 + AB^2)$.

Cor. 5. Adding the theorem deduced in Cor. 4, to that in the question, we have $HN . KP + GM . KP + HN . GM = \frac{1}{2}(AC^2 + BC^2 + AB^2) + \frac{1}{2}(AC . BC + AB . BC + AB . AC) =$ the square of half the perimeter of the triangle, a property which has been before noticed by Mr. Lowry, at page 5, Vol. V. of the Repository.

Cor. 6. By Cors. 4 and 5, we obtain $OL(GM + HN + KP) =$ the square of half the perimeter of the triangle $-\frac{1}{2}(AC^2 + BC^2 + AB^2)$.

SECOND SOLUTION, by Mr. DAVIES, Bath.

Let the sides and angles of the triangle be denoted by a, b, c , and $2\alpha, 2\beta, 2\gamma$, respectively, and let the centres of the circles of contact be O, P, Q, R . Then if the lines be drawn as in the figure, we shall readily find



$$\left. \begin{aligned} AR &= \frac{c \cos \beta}{\sin \alpha + \beta} = \frac{c \cdot \cos \beta}{\cos \gamma}; \text{ and } RD = \frac{c \cos \beta \cos \alpha}{\cos \gamma} = R_{III} \\ PM &= \frac{b \cos \alpha \cos \gamma}{\cos \beta} = R_{II} \\ QE &= \frac{a \cos \beta \cos \gamma}{\cos \alpha} = R, \end{aligned} \right\} \dots (1)$$

$$\left. \begin{aligned} OW = OV = OY = \\ \frac{a \sin \beta \sin \gamma}{\cos \alpha} = \frac{b \sin \alpha \sin \gamma}{\cos \beta} = \frac{c \sin \alpha \sin \beta}{\cos \gamma} \end{aligned} \right\} = R$$

$$\text{Now } RR_1 = \frac{a \cos \beta \cos \gamma}{\cos \alpha} \cdot \frac{a \sin \beta \sin \gamma}{\cos \alpha} \dots \dots (2)$$

$$\text{but } a = \frac{b \sin \alpha \cos \alpha}{\cos \beta \sin \beta} = \frac{c \sin \alpha \cos \alpha}{\cos \gamma \sin \gamma} \dots \dots (3).$$

Putting for a in (2) each of the values in (3) we shall get

$$\left. \begin{aligned} RR_1 &= bc \sin^2 \alpha \\ \text{Similarly, } RR_{II} &= ac \sin^2 \beta \\ RR_{III} &= ab \sin^2 \gamma \end{aligned} \right\} \dots \dots \dots (4).$$

$$\left. \begin{aligned} \text{Also } R_{II} R_{III} &= \frac{b \cos \alpha \cos \gamma}{\cos \beta} \cdot \frac{c \cos \beta \cos \alpha}{\cos \gamma} = bc \cos^2 \alpha \\ R, R_{III} &= ac \cos^2 \beta \\ R, R_{II} &= ab \cos^2 \gamma \end{aligned} \right\} \dots (5).$$

Hence (4, 5)

$$RR_1 + RR_{II} + RR_{III} + R, R_{II} + R, R_{III} + R_{II} R_{III} = ab + bc + ca.$$

Schol. Many properties already known, but otherwise discovered, may be proved by means of these equations, and

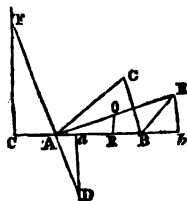
several new ones may be also deduced from them. It is not, however, necessary to enter further into the enquiry on this occasion, especially as I shall shortly by combining them with trigonometrical expressions for other parts of a plane triangle, compose a little dissertation for a separate part of the Repository.

THIRD SOLUTION, by — MASON, of Scoulton.

Let a, b, c be the sides of the Δ , $s =$ half the perimeter, then the radius r of the inscribed $\odot = \sqrt{\{(s-a)(s-b)(s-c) \div s\}}$, $r_1 = \sqrt{\{s(s-b)(s-c) \div (s-a)\}}$, the radius of that circle which touches a , and b , and c produced, $r_2 = \sqrt{\{s(s-a)(s-c) \div (s-b)\}}$, the radius of that touching b , and a and c produced, and $r_3 = \sqrt{\{s(s-a)(s-b) \div (s-c)\}}$ = the radius of the other. Now by multiplication $r \cdot r_1 + r \cdot r_2 + r \cdot r_3 + r_1 \cdot r_2 + r_1 \cdot r_3 + r_2 \cdot r_3 = (p-b)(p-c) + (p-a)(p-c) + (p-a)(p-b) + p(p-b) + p(p-c) + p(p-a) = (p-a)(2p-c) + (p-b)(2p-a) + (p-c)(2p-b) = ab + ac + bc$.

FOURTH SOLUTION, by Mr. THOS. THOMPSON, Newcastle-upon-Tyne.

Let D, E, F be the centres of the outward circles, O that of the interior one; a, b, c , R the points of contact with AB . Then, as FD bisects the outward angle A , and AOE the interior one, the $\angle OAR = \angle ADa = \angle AFC$, $\therefore$ by similar right angled Δs , $AD + AC : AD + CF :: OR + Eb : Ab + AR$, but it is well known that $AD + AC = BC$ and $Ab + AR = AC + AB$, $\therefore BC : AD + CF :: OR + Eb : AC + AB$: whence $AD \cdot OR + AD \cdot Eb + CF \cdot Eb = BC \cdot AC + BC \cdot AB \dots\dots (1)$. Again, by sim. Δs , $BR : OR :: Eb : bb$ and $AB : AD :: EC : CB$, therefore $OR \cdot Eb + AD \cdot EC = BR \cdot Eb + AB \cdot BC$. Let P represent the semi-perimeter, then $BR = P - AC$, $Bb = P - AB$, $ab = P - BC$ and $ac = P$, $\therefore OR \cdot Eb + AD \cdot EC = BP^2 - P(AC + AB + BC) + AC \cdot AB \dots\dots (2)$, which being added to (1) we have the sum, giving the required proof.



Mr. BAINES sent an answer to this Question.

XII. QUESTION 532, by M. CHAMBANON DE MAUGRIS,
(*tom. IV. Mem. Etrang. p. 646-655, Ann. 1763.*)

AO, OR are two planes, any how inclined to the horizon, intersecting in an angle AOR; OR is an inflexible rod, devoid of gravity, loaded in m with a given body M fixed to it there; and the whole apparatus thus formed, acted on by gravity glides along the planes AO, OR. Determine the velocity of M , including the effect of friction.

The following notice, by the Secretary of the Académie, (M. Grandjean de Fouchy) explains the nature of the difficulty involved in this problem. It is taken from the summary prefixed to the volume which contains the *mémoire*, p. xix.

"M. Chabanon de Maugris has given the solution of a problem in Dynamics, which consists in finding the velocity of an inflexible rod without weight, gliding between two planes inclined to the horizon in any angles whatever, and moved by the action of a weight fixed to this rod, taking into account the effect of friction. The difficulty of this problem is the perturbation, so to speak, that the motion of the body (already composed of its gravity and of its acquired motion in its own direction) undergoes at each instant, from the friction of the planes between which it moves. From this there results at each instant, a destruction of forces and a continual change of direction, which it is necessary to examine, in order to determine the course of the body in motion. For effecting this, M. Chabanon employs the principle of D'Alembert: he has also slightly used the principle of the conservation of living forces, but this latter principle, which is very easily applicable when the plane is infinitely polished and without friction, is not applicable to the general question which we have proposed. He obtains an equation integrable or constructible by quadratures: and not content to give the solution of the problem in the case proposed, he has annexed new conditions to his solution, which we may regard as being very ingenious and very complete."

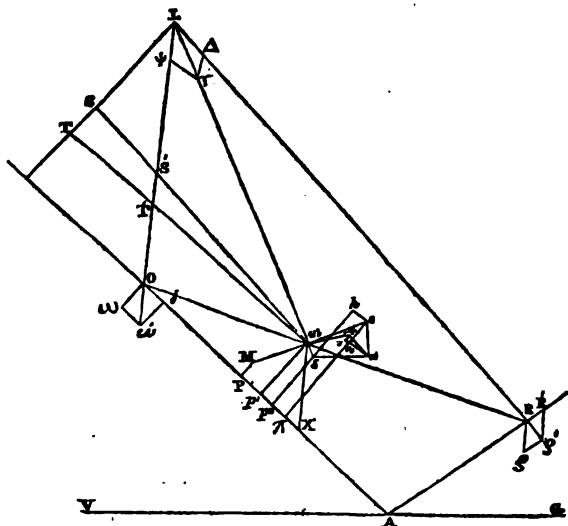
SOLUTION.

I consider in general the resistance which friction opposes to the motion of a body on a plane as a force applied to that body; and I assume that this force is to the pressure in a given ratio, $g:k$. Thus a body being placed upon a horizontal plane, which it presses by its weight p , I suppose that we apply to it

a force in the direction of the plane, a force $= \frac{g}{k} \cdot p$, for neu-

trализing the resistance created by the friction, and putting the body in a state ready to move in the direction of the force applied.

Let mm be the line which the body describes in a given mi-



nute interval of time, $me = mm$, and in the same direction as the line which it would describe in the following instant (equal to the first) if it were abandoned to itself and no accelerating force affected its motion. Let ea be the space which its gravity tends to make it pass through in the second instant. Finally instead of arriving at a as it otherwise would do, let it in consequence of its attachment to the rod, be found at μ .

We can decompose the motion ma , which the body would have, into two others, of which one μm is that which it actually retains, and of which the other μa is destroyed; from which it follows that the line $m\mu$ prolonged ought to meet the lines aL , RL in L , these lines being drawn from the extremities of the bar under this condition,—that each of the motions $L\psi$, $L\Delta$, into which the motion Ly or $m\mu$ is decomposed, is destroyed, 1° by the resistance perpendicular to each of the planes, 2° by the friction.

This condition determines the directions of the lines OL , RL . In fact we have only to draw from the points O , and R , the perpendiculars ow , Rq , and to take upon AO and AR , parts oo' , RR' having to the perpendiculars ow , Rq the ratio of g to h ; then the diagonals ow' , Rq' of the parallelograms formed upon the sides oo' , ow and RR' , Rq will shew the directions which the

lines OL , RL ought to take in order that the motions $L\psi$, $L\Delta$ should be destroyed.

If we draw perpendicular to oA the ordinates PM , $p'm$, $p''\mu$, &c. of the curve described by the body, and Πe , zL parallel to them; if also, we draw parallel to oA the line $mT'T$ and the line ani , we shall have the similar triangles $ai\mu$, mTL : then we shall have the analogy

$$ai : \mu i :: mT : TL.$$

The problem is then reduced to expressing analytically each term of this proportion.

Put $OM = a$, $MR = b$; the abscissa $AP = x$, the ordinate $MP = y$; the sine of the right angle $= 1$; sine of angle nea or oAV formed by the plane and the horizon $= n$; its cosine $= m$; p = the weight of the body; a = its velocity at M ; t = the time employed by the body to arrive at M . Then

$$an = npdt^2; en = mpdt^2; ni = ddx; \text{ and } \mu h = -ddy^*.$$

$$\text{and hence, } ai = ddx + npdt^2, \mu i = ddy + mpdt^2.$$

Having the expressions for ai and μi it only remains to seek those of mT and TL .

Let mx be parallel to oL , ms 's parallel to RL : then

$$1 : g : h :: oL : LZ : oZ.$$

The value of oL being found we shall have the values of LZ and oZ . From LZ subtracting zT or y , we shall have LT ; and adding oZ to $op' = \sqrt{a^2 - y^2}$ the sum will be $p'z$ or mT . It is required, then, to determine oL , which we may do by means of the similar triangles oms' , oRL , from which we shall have the expression of os' , $os' = \{OT' + s'T'\}$, or $mx + s'T'$. The triangles OLZ and xmp' are also similar. Then

$$mx = \frac{y}{h}; \text{ and } p'x = \frac{yg}{h}: \text{ also}$$

$$ox = mT' = \sqrt{a^2 - y^2} + \frac{gy}{h}.$$

All the angles of the triangle $s'mT'$ are known, and therefore $mT' : s'T'$ is a given ratio ($:: q : r$ for example); $s'T'$ being

* I treat ddx as positive and ddy as negative, because the first differentials of x and y ought to decrease: but these are $-dx$ and dy , since when t increases y increases also, and x diminishes.

† It must be remarked that the values of ai and μi are the same if the point π fall beyond i with respect to a , and if the point k be below μ relative to the plane ao , for then ai would become $am - ni$; but ni would be $= -ddx$, since t increases, and $-dx$ increases; and μi would be $ik + h\mu$, or $\mu h = +ddy$.

therefore z is between Λ and o , which again changes the signs of the general equation.

It is easy to convince ourselves whether the extremity, o , of the rod approaches towards the plane ΛR or recedes from it. We need only seek upon the curve $\Lambda K M F O$ (an arc of which the body M pursues,) the point K most elevated above the horizon: then accordingly as by the initial position of the rod, the body M is found between K and F or between K and o it will tend towards the plane ΛR or the plane ΛO . With respect to the point K we find it readily by transforming the equation of the curve from the co-ordinates Ap , pM to the co-ordinates ΛN , NM (ΛN being horizontal and NM vertical), and then seek v , the new axis of the abscissas, to which corresponds a *maximum* ordinate. This *maximum* will be vK .

Making the calculation we find (supposing for brevity that ΛOR is a right angle)

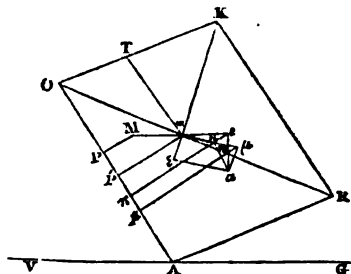
$$AV = \frac{mn(b^2 - a^2) \sqrt{a^2 n^2 + b^2 m^2}}{\sqrt{a^2 b^2 + m^2 n^2 (b^2 - a^2)^2}} = \frac{mn b^2 - m n a^2}{\sqrt{b^2 n^2 + a^2 m^2}},$$

from which we see that the point K will be differently situated according to the different positions of the planes with respect to the horizon; and according to the different values of a and b .

If we have m or $n = 0$, that is to say if one of the planes be vertical, v falls upon the point Λ , as is evident. The points Λ and v will also coalesce if $a = b$: and in fact on this supposition the curve is a circle of which the horizontal axis of abscissas is the diameter.

Application.

Suppose the planes perfectly polished. In this case $g = 0$, $h = 1$, and the equation of the preceding problem is



$(npdt^2 + ddx)\{(ar + br)\sqrt{a^2 - y^2} + bqy\} = (mpdt^2 + ddy)aq\sqrt{a^2 - y^2}$
which becomes on the supposition that ΛOR is right,

$$(npdt^2 + ddx) \cdot \frac{by}{a} = (mpdt^2 + ddy) \sqrt{a^2 - y^2}$$

If we put in this equation instead of y and ddy their values drawn from the equation of the curve we have

$$(ddx + npdt^2) \frac{b}{a} \sqrt{b^2 - x^2} = \{mpdt^2 + \frac{a}{b} dd\sqrt{b^2 - x^2}\} x.$$

Multiply by $\frac{dx}{\sqrt{b^2 - x^2}}$ and integrate, and we shall find

$$\frac{b dx^2 + 2bnpx}{2a} = -mpdt^2 \sqrt{b^2 - x^2} - \frac{a}{2b} \cdot \frac{x^2 dx^2}{b^2 - x^2} + \kappa dt^2, \text{ or}$$

$$\frac{dx^2}{dt^2} = \frac{\{-2bnpx - 2abmp \sqrt{b^2 - x^2} + 2ab\kappa\} (b^2 - x^2)}{b^2(b^2 - x^2) + a^2x^2}, \text{ or}$$

$$v^2 = \frac{\{-2bnpx - 2abmp \sqrt{b^2 - x^2} + 2ab\kappa\} \{b^2(b^2 - x^2) + a^2x^2\}}{\{b^2(b^2 - x^2) + a^2x^2\}b^2}$$

(by putting for dt its value $\frac{mm}{v}$) and consequently we obtain

$$v^2 = -2npx - \frac{2amp}{b} \sqrt{b^2 - x^2} + \frac{2a\kappa}{b};$$

the constant κ being determined by the initial position of the rod.

REMARK II.

If there be two bodies attached to the rod, the resultant of the destroyed motion mz , $m'z'$ or $a\mu$, $a'\mu'$ will still pass through the point L : then if we decompose each of these motions into two others, ai and μi , $a'i'$ $\mu'i'$ one of which is parallel to the plane AO and the other perpendicular, we shall have

$$M \cdot ai \cdot TL + M' \cdot a'i' \cdot tL = M \cdot \mu i \cdot mT + M' \cdot \mu'i' \cdot m't \dots (x).$$

But denoting as above AP by x , PM by y , AQ by x' , and QM' by y' , we shall have ai , μi , TL , mT expressed in terms of x , dx , ddx , and dt ; and in like manner, $a'i'$, $\mu'i'$, iL , $m't$ expressed in terms of x' , ddx' , and dt . But we easily find the ratio of x to x' : and then the equation x changes itself into another composed of x , dx , ddx , and dt^2 . This equation will be (always, supposing for the sake of abridging the calculus the planes perfectly polished and at right angles, and putting $om = a$, $mm' =$

$$b, m'R = c), M \{(npdt^2 + ddx) \sqrt{b + c)^2 - x^2}\} +$$

$$M \{npdt^2 + \frac{c}{b + c} ddx\} \times \frac{c}{b + c} \sqrt{b + c)^2 - x^2}$$

$$= M \{mpdt^2 + \frac{a}{b + c} dd\sqrt{b + c)^2 - x^2}\} \frac{ax}{b + c}$$

horizontally. Then

$$p : \phi :: dr : mn = mt + eq = -ndx - mdy.$$

Hence $u du = -pndx - gmdy$, or $u^2 = c - 2np x - 2mpy$.

Similarly we find $v^2 = c' - 2np'x - 2mp'y$: and hence

$$\begin{aligned} m v^2 + m' v'^2 &= K - 2mnp x - 2mpm \cdot \frac{a}{b+c} \sqrt{(b+c)^2 - x^2} \\ &\quad - 2m'np' \frac{c}{b+c} \cdot x - 2m'p'm \frac{a+b}{b+c} \sqrt{(b+c)^2 - x^2} \end{aligned}$$

an equation precisely like that already found.

REMARK IV.

The principle of the conservation of living forces which as we have seen applies very happily to the solution of the problem in the case where the plane is perfectly polished, does not seem to apply so well when we take friction into the account. In truth, for determining in this case the velocity with which the body freely moves in its path in virtue of an accelerating force, it is necessary that we should be acquainted with this accelerating force, and consequently it requires us to determine the effect of friction upon each body in diminishing the effect of its gravitation—which is a second problem to resolve.

Cor. If in the equation given by remarks II. and III. we make $m' = 0$, then all the terms affected by m' disappearing, m will multiply all the remaining terms of the equation: from which we conclude that the motion of a body simply attached to a rod without weight does not depend at all upon its mass, whilst if there be several bodies the motion of each of them depends upon all the masses. By this supposition ($m' = 0$) we again fall upon the equation which gives the motion of a single body.

REMARK V.

If we have more than two bodies attached to the rod, it is obvious that the difficulty of the problem is not in any way increased, provided the bodies can be determined in number, magnitude, and position: the single case where the rod is loaded with an infinity of small bodies, that is where itself becomes heavy or subject to gravity may give birth to some difficulty: and we shall therefore examine this case.

We suppose that each point of the bar is charged with a small body of which the mass is any function whatever (Λ) of

its distance a from the extremity of the bar. The equation of the problem, which ought to be generally

$M \cdot ai \cdot L + M' \cdot a'i' \cdot tL + \dots = M\mu i \cdot mT + M' \cdot \mu'i' \cdot mt + \dots$
if we preserve the same denominations as above, and put $OR = e$, will become

$$\int \Delta da \cdot (npdt^2 + ddx) \cdot \frac{e-a}{y} = \int \Delta da (mpdt^2 + ddy) \sqrt{a^2 - y^2}$$

an equation which is integrable, or at least, which is reduced to quadratures, if we substitute $z = \sqrt{a^2 - y^2}$ for x , or $AO = AP'$

(calling $AO = z$) and if we put in place of y its value $\frac{a}{e} \sqrt{e^2 - z^2}$.

By these substitutions we transform the equation found above into this

$$\int \left\{ \Delta da (npdt^2 + \frac{e-a}{e} ddz) \frac{e-a}{e} \sqrt{e^2 - z^2} \right\} = \int \left\{ \Delta da (mpdt^2 + \frac{a}{e} dd \sqrt{e^2 - z^2}) \frac{az}{e} \right\}.$$

of which we take in the first place the integral for any position of the bar in treating only a as variable, and then for all positions of the bar, by making z the variable.

I shall suppose the former integration already made, and the equation written thus,

$$npdt^2 \sqrt{e^2 - z^2} \cdot \int \Delta da \cdot \frac{e-a}{e} + ddz \sqrt{e^2 - z^2} \int \Delta da \left(\frac{e-a}{e} \right)^2 = mpdt^2 \cdot z \int \Delta da \cdot \frac{a}{e} + add \sqrt{e^2 - z^2} \cdot \int \Delta da \cdot \frac{a^2}{e^2}.$$

Then multiply by $\frac{dz}{\sqrt{e^2 - z^2}}$ and integrate, which gives,

$$2npzdt^2 \int \Delta da \frac{e-a}{e} + dz^2 \cdot \int \Delta da \left(\frac{e-a}{e} \right)^2 = - 2mpdt^2 \sqrt{e^2 - z^2} \times \int \Delta da \cdot \frac{a}{e} - \frac{z^2 dz^2}{e^2 - z^2} \int \Delta da \frac{a^2}{e^2} + Kdt^2, \text{ or}$$

$$\frac{dz^2}{dt^2} = \frac{\left\{ K - 2npz \int \Delta da \cdot \frac{e-a}{e} - 2mp \sqrt{e^2 - z^2} \cdot \int \Delta da \cdot \frac{a}{e} \right\} (e^2 - z^2)}{(e^2 - z^2) \int \Delta da \left(\frac{e-a}{e} \right)^2 + z^2 \int \Delta da \cdot \frac{a^2}{e^2}},$$

and putting for dt^2 its value $\frac{\left(\frac{e-a}{e}\right)^2}{v^2} dz + \frac{a^2}{e^2} \times \frac{z^2 dz^2}{v^2}$, or

$\frac{\{(e-a)^2(e^2-z^2) + a^2z^2\} dz^2}{v^2(e^2-z^2)e^2}$, we have

$$v^2 = \frac{\{(e-a)^2(e^2-z^2) + a^2z^2\} \left\{ K - 2npz \int \Lambda da \frac{e-a}{e} - 2mp \sqrt{e^2-z^2} \cdot \int \Lambda da \cdot \frac{a}{e} \right\}}{(e^2-z^2) \int \Lambda da (e-a)^2 + z^2 \int \Lambda da \cdot a^2}$$

If we have the rod equally heavy in all its points, we have only to make $\Lambda = \text{const.}$ Let this const. = 1, then we have

$$v^2 = \{(e-a)^2(e^2-z^2) + a^2z^2\} \times \frac{K - 2npz \cdot \frac{2ea-a^2}{2e} - 2mp \sqrt{e^2-z^2} \cdot \frac{e^2}{2e}}{(e^2-z^2) \frac{3e^2a-3ae^2+a^3}{3} + z^2 \cdot \frac{a^3}{3}};$$

and as we take the integral through all the extent of the bar, it requires us to put e for a in terms free from the sign $\int$: thus we have

$$v^2 = \{(e-a)^2(e^2-z^2) + a^2z^2\} \cdot \frac{3K - 8enpz - 3emp \sqrt{e^2-z^2}}{e^3}.$$

Let $e = 1$, the angle $\Lambda RO = s$, the angle ean (whose sine is m , and cosine n) = c , we shall have $z = \sin s$, and

$$v^2 = \{a^2 + (1-2a)\cos^2 s\} \times \{3K - 3p \cdot \sin(c+s)\};$$

and the velocity of the centre of gravity is expressed by

$$\frac{1}{2} \sqrt{\{3K - 3p \sin(c+s)\}}.$$

If the plane ΛO is vertical the extremity O of the bar continually approaches the plane ΛR , as we have already seen in Remark I.; and the velocity of the centre of gravity will be expressed by

$$\frac{1}{2} \sqrt{3K - 3p \sin s},$$

and when the extremity O of the bar arrives at the plane ΛR the velocity of the centre of gravity will be

$$\frac{1}{2} \sqrt{3K}.$$

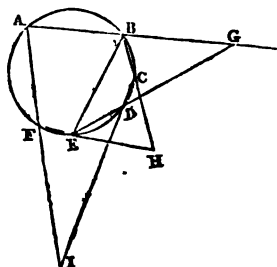
XIII. QUESTION 523, by Mr. DAVIES.

Let the opposite sides of a hexagon inscribed in a circle be produced to meet: then of the three angles thus formed the

middle one will be equal to both the others. Geometrical and analytical proofs are required.

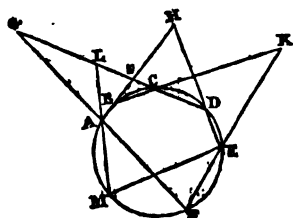
FIRST SOLUTION, by — MASON, of Scoulton.

Let the interior angles of the annexed figure be represented by the letters placed at the angular points respectively. Join EB. By Euclid $\angle G + \angle GEB = \angle EBA$, or $G + 180^\circ - C = 180^\circ - F$, since $\angle EBA + F = 180^\circ$ and $\angle AED + C = 180^\circ$. $\therefore G = C - F$. In like manner it appears that $I = E - B$, and $H = D - A$. $\therefore G + I = C - F + E - B = 180^\circ - \angle BED - 180^\circ + \angle EBA + \angle FEB + \angle BED - \angle EBA - \angle EBC = (180^\circ - \angle EBC) - (180^\circ - \angle FEB) = D - A = H$. Q. E. D.



SECOND SOLUTION, by Mr. GODWARD.

Geometrically. Let G, H, K be the concourses of the opposite sides of the hexagon, and N the concourse of AH, DG. Through A draw LAM making the $\angle LAG = \angle BKE$, and join EM. Then the $\angle BKE = \angle LAG = \angle MAF = \angle MEF$; hence ME is parallel to BK or BC, and $\therefore$ the $\angle EDC = \angle MAB$, or the $\angle NDH = \angle NAL$; but the $\angle DNH = \angle ANL$; hence the $\angle DHN = \angle ALN = \angle AGC + \angle GAL = \angle AGC + \angle BKE$.



This question was answered by Messrs. BAINES and DAVIES.

XIV. QUESTION 524, by Mr. T. S. DAVIES, Bath.

Let A, B, C, D, E, F be the six summits of a complete quadrilateral,* and G, H, K the intersections of its diagonals; from

* The term *quadrilateral* has been usually employed by English geometers to designate a four sided figure, all whose angles were *salient* (Fig. 1.): and in this they followed, in common with the earlier continental

one of the intersections as G, draw GL perpendicular to the third diagonal, meeting it in L: then

$$1^{\text{st}}. \angle DLB = \angle E \mp F,$$

$$2^{\circ}. \angle CLA = \angle E \pm F,$$

$$3^{\circ}. \text{The angles } DLB, CLA \text{ are bisected by } GL.$$

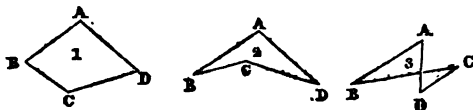
Analytical and geometrical demonstrations are required.

SOLUTION, by Mr. T. S. DAVIES, the Proposer.

In writing out this question, I had inadvertently attributed to one single point L properties which, commonly, belong only

writers, the great fathers of the ancient geometry. As, however, the French military writers had frequent occasion to mention the *angle of the brisure* and the *angle of the curtain* in their descriptions of the star-forts and bastioned works, they, for brevity, called it the *re-entrant angle* (*re-entrant*); and the term was gradually adopted by writers on pure mathematics, when describing an angle of a four sided figure whose point lay within the triangle ABD (fig. 2). A third kind

has received the name of *intersecting quadrilateral* (vid. Phil. Mag. V. 68. p. 117) as ABCD (fig. 3). In this case each of the points A, C is without the tri-



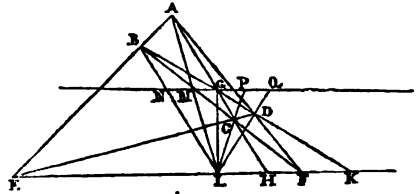
angle upon BD, and yet both on the same side of it. Carnot guided by more general views respecting the rectilinear transversals of a polygonal figure, called any figure composed of four lines each intersecting the other three, by the name of *quadrilateral*, employed the terms *simple* and *complete* for distinguishing them. The complete one is that which has two more intersections, made by AB with CD and AC with AD. This is his *complete quadrilateral*, and each of the figures 1, 2, 3 in their present state he called a *simple quadrilateral*. Combining this term with that expressive of its particular character, (*salient*, *re-entrant*, and *intersectant*) we shall be able to define every possible variety of figure produced by four straight lines on a plane.

It will also be seen that though the complete quadrilateral is always of the same general features, yet it may be conceived as having for its elemental quadrilateral, either of the simple ones. The advantages of this classification will be best understood by those geometers who have attempted to investigate the properties of figures formed by assemblages of numerous lines. Setting aside the influence of the method as enabling us to discriminate and distribute lines more readily into known figures having certain analogies, it is of great use for the compactness which it confers upon our enunciation of the properties themselves. I trust therefore it will be employed in our future elementary works, and in the statements of propositions of this class in our periodical mathematical publications.

to two different ones, L and L' . These points coincide when the simple salient quadrilateral is inscribable in a circle.

It was shewn in the Repository, Vol. I. p. 170, that if circles be described about the component triangles EAD , EBC , FBA , FCD they will all intersect in the point L' . It is to this point that the first and second statements apply: and their demonstration is very simple.

Join AL' , BL' , CL' , DL' , EL' , FL' . Then, because L' is a point in all the circles we have



$$CL'D = CFD = BFA = BL'A: \text{ and } \\ AL'D = AED = BEC = BL'C.$$

Hence $BFA + AED = CL'D + CL'B$
that is $\angle F + \angle E = DL'B \dots\dots\dots Q. E. 1^o. D.$

Again $BFA - AED = CL'D - CL'B$
 $= CL'D - AL'D$

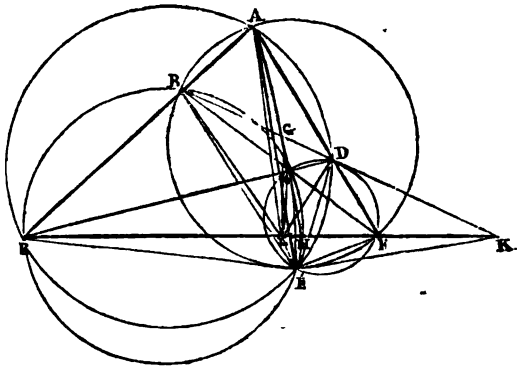
that is, $\angle F - \angle E = CL'A \dots\dots\dots Q. E. 2^o. D.$

By interchanging AC with BD , the demonstration adapted to the other sign will result.

Through G let GL be drawn perpendicular to EF : and join AL , BL , CL , DL . We have, first, to prove that GL bisects the angles BLD , CLA : and, secondly, that this is not true when we substitute L' for L .

Draw through G a parallel to AB cutting AL , BL , CL , DL in P , N , M , Q .

Then because AC is harmonically divided in G and H , and MP is parallel to one of the harmonical radiants from L , it is bisected in G by the radiant LG . And because MG is parallel to EF and GL perpendicular to it, LG



bisects MP at right angles. Hence MLP , or CLA is bisected by GL .

In the same we have BD harmonically divided in G , K , and similar reasoning shews that BLD is also bisected by GL .

Q. E. 3^o. D.

To shew that CL' does not bisect the angles $AL'C$, $BL'D$, we may, for the moment, admit that it does, and thence derive the conclusion *ex absurdo*.

Then because $BL'D$ is bisected by GL' , therefore $BG : GD :: BL' : L'D$; but $BG : GD :: BK : KD$; therefore also $BL' : L'D :: BK : KD$, and hence the exterior angle at L' is bisected (Euc. VI. A. Sims.) by the line $L'K$; and $GL'K$ is a right angle.

In the same way we shall find that $GL'H$ is a right angle. Hence therefore, either the point L' is at the foot of the perpendicular from G , (that is it coincides with L), or else the line GL' does not bisect the angles $AL'C$, $BL'D$: and it only remains to ascertain *when* this coincidence takes place.

Now $CL'F = CDA$, and $CL'E = CBA$. Hence $CL'F + CL'E = CDA + CBA$: and when L' is in the line EF , the two angles $CL'E$ and $CL'F$ are equal to two right angles: and hence also $ABC + ADC =$ two right angles, or the simple salient quadrilateral is inscribed in a circle.

SCHOLIUM.

Many other curious properties belong to this figure (the complete quadrilateral) the investigation of which are interesting and without difficulty: and, but for other purposes which I have in view, I would enter upon them here. In order to give an opportunity of resuming the inquiry, I shall propose one in the set of questions which accompany the present number: and now merely throw together a few notes respecting the properties just established.

The first and second properties are, I believe, for the first time given here: the third was proposed by me in the last published number of the Leeds Correspondent, (No. 19, or No. 3 of Vol. V. July 1821); and the proof I gave to Mr. Whitley, the learned editor of that excellent little work, was, as far as I recollect, very like the one now published. I was led to notice the property whilst studying a particular case of it which had been proposed by Mr. Baines in the same work, (Vol. III. p. 49.), and very elegantly solved by two gentlemen; and as these deserve to be more extensively known than the confined circulation of that work can effect, I have affixed them here. The same property, however, had been noticed (in 1751) by a

correspondent signing "R", (probably Edward Rollinson) in "*The Mathematician*," (p. 47), and answered by the editor, Mr. John Turner, and the proposer "R," (p. 107). These also I shall insert. The general property as stated in the Leeds Correspondent, was taken up by Mr. Rutherford and Mr. Beverley in "*The Scientific Receptacle*," printed at Holbeach, only one number (and that imperfect) of which has fallen in my way. It appeared about the end of 1824: and I have not learnt whether more than this number was published or not. The solution of Mr. Beverley which was for the case where the quadrilateral was inscriptible, was not printed. Mr. Rutherford's is very well executed: and for the sake of collecting into one view all that has been done, I shall insert that solution here. I shall take them in Chronological order.

1. *Mathematician*, p. 47.

If from the Extremities of the Base of any plane Triangle, two right Lines be drawn intersecting each other in the Perpendicular, and terminating in the opposite Sides: right Lines drawn from them to where the Perpendicular meets the Base, will make Angles with the Base equal to each other: *Quare* the Demonstration.

Answered by Mr. JOHN TURNER, p. 107.

Draw *CH* parallel to *AB* intersecting *DE*, *DF* produced in *H* and *I*: Then, since, by similar Triangles,

$$BE : BD :: CE : CH = \frac{BD \times CE}{BE}$$

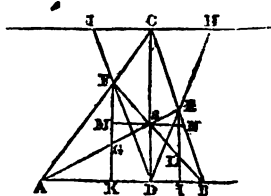
$$\text{and } AF : AD :: CF : CI = \frac{AD \times CF}{AF},$$

we shall have

$$CH : CI :: \frac{BD \times CE}{BE} : \frac{AD \times CF}{AF}$$

$$:: BD \times CE \times AF : AD \times CF \times BE.$$

But these last two Terms are equal, by a known Property of the Triangle;* therefore *CH* and *CI* are also equal, and consequently the Angle *CDH* = the Angle *CDI*.



* This property of the triangle quoted here appears to have been first noticed in England about 1737, Mr. Samuel Ashby having pro-

The same answered by the PROPOSER, R. (ibid.)

To the Base AB from the Points F and E, let fall the Perpendiculars FK and EI, and parallel to the same through s the common Intersection of the Lines AE, BF, and CD draw MN : Then, by similar triangles,

$$\begin{aligned} AC : AF :: CS : GF \\ :: CD : FK ; \end{aligned}$$

therefore, by Permutation, $CS : CD :: GF : FK$;
after the very same Manner, it will appear that

$$CS : CD :: EL : EI ;$$

whence by Equality, $GF : FK :: EL : EI$,
or again by Permutation, $GF : EL :: FK : EI$.

But $GF : EI :: FS : LS$

$$:: MS : NS .$$

Whence again, by Equality, $FK : EI :: MS : NS$, or as $KD : DI$; therefore the two Triangles DFK , DEI having one Angle K equal to one Angle I, and the Sides about the other Angles DFK , DES proportional, are equiangular, (Euc. 7. 6.) ; and consequently the Angle FDK equal to the Angle IDK .

Q. E. D.

As the lettering of the diagram in the Leeds Correspondent was something different from that before us, we must repeat the question as there proposed.

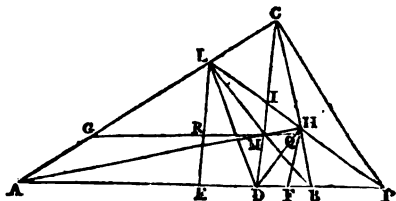
Let ABCD be any quadrilateral having its opposite sides AC, AD produced to meet in H, and AB, CD in G, and let E be the point in which the diagonals intersect ; if GH be joined, and EF demitted perpendicularly upon it, the angles BFC, AFD being found will be each bisected by EF.

Answered by AMICUS, (Mr. WHITLEY), Leeds Cor. III. p. 49.

Let AH and BL, any two straight lines drawn from the angles A, B of the plane $\triangle ACB$ intersect in CD the perpendicular to AB, and meet the opposite sides BC, AC in H and L : LD and HD joined, make equal angles with the base AB.

posed it as the Prize of that year in the Ladies' Diary, (vid. *Leybourn's Ed.* p. 246, vol. I.). The demonstration commonly given of it was first published in Simpson's Geom. Book IV. pr. 32 : but it appears to have been known to John Bernouilli thirty Years earlier. *Op. tom. IV. p. 33.1*

Draw LE and HF perpendicular to AB , and through L , H draw LP intersecting CD in I , and AB produced in P . Then by question 145, No. 13, *Math. Companion*, or question 14, *Ladies' Diary*, 1818, AP is harmonically divided in D and B , and consequently by the property of harmonicals, LP is also divided harmonically in the points I and H ; that is $LI : HI :: LP : HP ::$ (by parallel lines) $ED : DF :: EL : FH$, $\therefore$ the right angled triangles, DEL , DFH having the sides about the right angles proportionals, are equiangular, and consequently the angles ADL , BDH are equal. Q. E. D.



Cor. 1. CD bisects the angle LDH .

Cor. 2. Hence if AH be perpendicular to BC , it will bisect the angle DHL .

Cor. 3. If AH and CD be perpendicular to BC and AB , the point of intersection M will be the centre of the circle inscribed in the triangle DHL .

Again, by Mr. J. WHITEHEAD, (understood to be by the late Mr. J. WOOLFENDEN, of Hollingwood), Leeds Cor. p. 50.

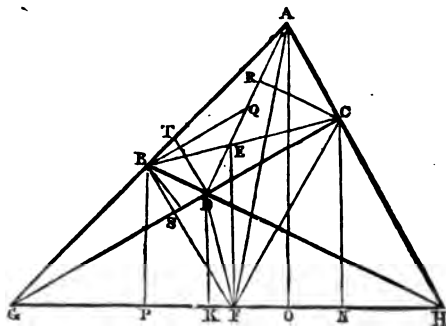
Let the figure be drawn as for above, and draw GMN perpendicular to AB : then, by sim. triangles we shall have the following proportions, viz.

$$\begin{aligned} MQ &= DF : AF :: MN : AB \\ &: MN : BD :: CM : CD \\ AF : FH &:: AD : DM \\ AD : GM &:: CD : CM \\ BD : DM &:: BE : LE \\ GM : AB &:: DE : BE \end{aligned}$$

OR $DF : FH :: DE : LE$, by mult.

Hence, as the triangles DEL , DFH have the sides about the right $\angle$ s proportional they are similar, and consequently, $\angle ADL = \angle BDH$. Q. E. D.

Demit CN, DK, AO, BP , perpendicular to GH and draw CR, BQ respectively parallel to HB, GC : Then, by sim. triangles.



AH : HC :: AO : CN, but

AH : HC :: AD : DR, whence

AO : CN :: AD : DR.

In a similar manner we have

AO : BP :: AD : DQ ; therefore,

CN : BP :: DR : DQ

:: CR : DB

$$:: CE : EB$$

:: NF : FP:

Whence the $\triangle s$ CNF, BFP are similar: therefore $\angle CNF = \angle BFP$, and consequently $\angle CFB$ is bisected by the straight line EF. Q. E. 1^o. D.

Again, draw DT and BS each $\parallel$ AC; then by sim. Δ s

AO : BP :: AG : BG :: AC : BS, and

$$\mathbf{DK : BP :: DH : BH :: CD : CS,}$$

:: DT : BS , therefore we have

AO : DK :: AC : DT :: AE : ED* :: OF : FK.

Whence the $\angle AFO = \angle DFR$, and therefore QFD is bisected by the straight line EF . Q. E. 2°. D.

* This is an evident mistake for CN : DK, and a mistake in the copying for press, not of the investigation itself.

SECOND SOLUTION, *Analytically.*

Fig. 1.

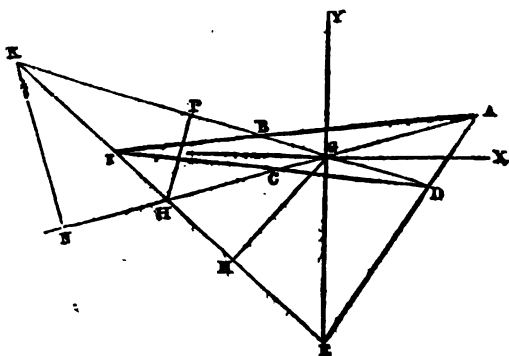


Fig. 2.

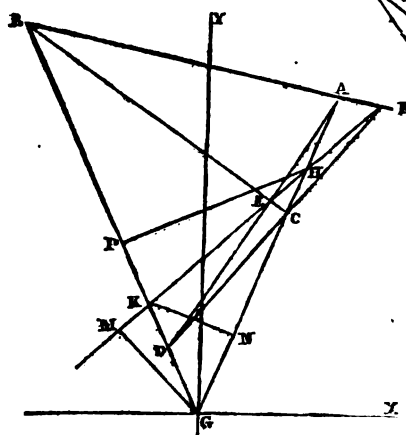
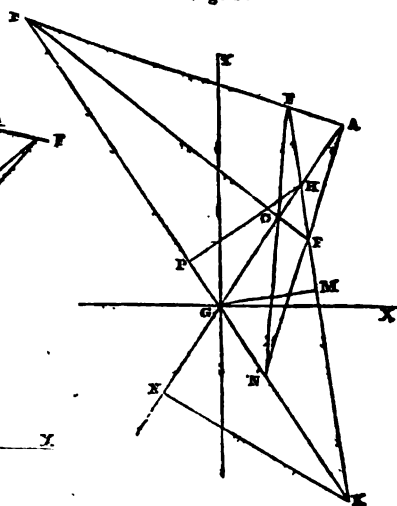


Fig. 3.



Refer the system of lines to rectangular co-ordinates bisecting the angles at G, and let A lie in the *plus-plus* region. Call the angle AGX, ϵ ; and denote the several distances of the angles A, B, C, D, from G, by a, b, c, d , respectively,

$$\begin{array}{lcl} \text{Then } x_a = a \cos \epsilon & y_a = a \sin \epsilon & \\ x_b = -b \cos \epsilon & y_b = b \sin \epsilon & \\ x_c = -c \cos \epsilon & y_c = -c \sin \epsilon & \\ x_d = d \cos \epsilon & y_d = -d \sin \epsilon & \end{array} \left. \vphantom{\begin{array}{lcl} \text{Then } x_a = a \cos \epsilon \\ x_b = -b \cos \epsilon \\ x_c = -c \cos \epsilon \\ x_d = d \cos \epsilon \end{array}} \right\} \dots \dots (1).$$

Expressing the lines AB, CD, AD, BC, by means of the values of the points A, B, C, D, in (1) we shall have, respectively adapted to each of them in order, the following equations.

$$AB \dots x(a-b) \sin \varepsilon - y(a+b) \cos \varepsilon = -2ab \sin \varepsilon \cos \varepsilon \dots (2)$$

$$CD \dots x(c-d) \sin \varepsilon - y(c+d) \cos \varepsilon = 2cd \sin \varepsilon \cos \varepsilon \dots (3)$$

$$AD \dots x(a+d) \sin \varepsilon - y(a-d) \cos \varepsilon = 2ad \sin \varepsilon \cos \varepsilon \dots (4)$$

$$BC \dots x(b+c) \sin \varepsilon + y(b-c) \cos \varepsilon = -2bc \sin \varepsilon \cos \varepsilon \dots (5)$$

To find the co-ordinates of the points E, F, we employ (2, 3) and (4, 5) respectively: and by the usual processes obtain

$$y_f = \frac{ab(c-d) + cd(a-b)}{bc-ad} \cdot \sin \varepsilon \dots (6)$$

$$x_f = \frac{ab(c+d) + cd(a+b)}{bc-ad} \cdot \cos \varepsilon \dots (7)$$

$$y_e = -\frac{ad(b+c) + bc(a+d)}{ab-cd} \cdot \sin \varepsilon \dots (8)$$

$$x_e = \frac{ad(b-c) - bc(a-d)}{ab-cd} \cdot \cos \varepsilon \dots (9)$$

The equation of EF is $x(y_e - y_f) - y(x_e - x_f) = y_e x_f - x_e y_f$, or inserting the constants from (6, 7, 8, 9, it becomes

$$\begin{aligned} & x \cdot \left\{ \frac{ad(b+c) + bc(a+d)}{ab-cd} + \frac{ab(c-d) + cd(a-b)}{bc-ad} \right\} \cdot \sin \varepsilon \\ & + y \cdot \left\{ \frac{ad(b-c) - bc(a-d)}{ab-cd} - \frac{ab(c+d) + cd(a+b)}{bc-ad} \right\} \cdot \cos \varepsilon \\ & = \left\{ \frac{ad(b+c) + bc(a+d)}{ab-cd} \cdot \frac{ab(c+d) + cd(a+b)}{bc-ad} + \right. \\ & \quad \left. \frac{ab(c-d) + cd(a-b)}{bc-ad} \cdot \frac{ad(b-c) - bc(a-d)}{ab-cd} \right\} \cdot \sin \varepsilon \cos \varepsilon. \end{aligned} \dots (10)$$

$$\text{But } ad(b+c) + bc(a+d) = ac(b+d) + bd(a+c) \} \dots (11)$$

$$\text{and } ab(c-d) + cd(a-b) = ac(b+d) - bd(a+c) \}$$

Inserting (11) in (10) the equation becomes

$$\begin{aligned} & x \cdot \left\{ \frac{ac(b+d) + bd(a+c)}{ab-cd} + \frac{ac(b+d) - bd(a+c)}{bc-ad} \right\} \cdot \sin \varepsilon \\ & \quad \left\{ \frac{bd(a+c) - ac(b+d)}{ab-cd} - \frac{ac(b+d) + bd(a+c)}{bc-ad} \right\} \cdot \cos \varepsilon \\ & = \left\{ \frac{\{ac(b+d) + bd(a+c)\}^2 - \{ac(b+d) - bd(a+c)\}^2}{(ab-cd)(bc-ad)} \right\} \sin \varepsilon \cos \varepsilon \end{aligned} \dots (12)$$

Cancelling the denominators, and otherwise reducing we arrive by easy steps at the following equation of EF, viz.

$$x \{ac(b-d) - bd(a-c)\} \sin \varepsilon - y \{bd(a-c) + ac(b-d)\} \cos \varepsilon = 4abcd \sin \varepsilon \cos \varepsilon \dots\dots(13)$$

Again, the equations of the lines AC, BD are, respectively,

$$y \cos \varepsilon - x \sin \varepsilon = 0 \dots\dots\dots(14)$$

$$y \cos \varepsilon + x \sin \varepsilon = 0 \dots\dots\dots(15)$$

From which and equation (13) we obtain the co-ordinates of H and K.

$$\left. \begin{aligned} x_h &= -\frac{2ac}{a-c} \cdot \cos \varepsilon, & y_h &= -\frac{2ac}{a-c} \cdot \sin \varepsilon \\ x_k &= \frac{2bd}{b-d} \cos \varepsilon, & y_k &= -\frac{2bd}{b-d} \cdot \sin \varepsilon \end{aligned} \right\} \dots\dots(16).$$

We now proceed to determine the co-ordinates of the points M, N, P.

For M;—let us put A, B, C instead of the coefficients of the equation (13), or in other words write it thus:

$x \cdot A \sin \varepsilon + y \cdot B \cos \varepsilon = -c$: then the perpendicular upon it from the origin will be denoted by

$$x \cdot B \cos \varepsilon - y \cdot A \sin \varepsilon = 0 \dots\dots\dots(17).$$

From these we obtain

$$\left. \begin{aligned} y_m &= -\frac{BC \cos \varepsilon}{A^2 \sin^2 \varepsilon + B^2 \cos^2 \varepsilon} \\ x_m &= -\frac{AC \sin \varepsilon}{A^2 \sin^2 \varepsilon + B^2 \cos^2 \varepsilon} \end{aligned} \right\} \dots\dots\dots(18).$$

For N:—the co-ordinates of K found in (16) being inserted in the usual expression for the perpendicular from that point upon AM, (eq. 14) gives

$$\begin{aligned} \{x - \frac{2bd}{b-d} \cos \varepsilon\} \cos \varepsilon + \{y + \frac{2bd}{b-d} \sin \varepsilon\} \sin \varepsilon &= 0, \text{ or} \\ x \cos \varepsilon + y \sin \varepsilon &= \frac{2bd}{b-d} \cos 2\varepsilon \dots\dots\dots(19) \end{aligned}$$

is the equation of KN. Combining (14, 19) we have at once

$$\left. \begin{aligned} x_n &= \frac{2bd}{b-d} \cos \varepsilon \cos 2\varepsilon \\ y_n &= \frac{2bd}{b-d} \sin \varepsilon \cos 2\varepsilon \end{aligned} \right\} \dots\dots\dots(20).$$

For P;—The co-ordinates of H in (16) being put in the

equation of the perpendicular upon GK (eq. 15), we get for the equation of HP,

$$\{x + \frac{2ac}{a-c} \cos \epsilon\} \cos \epsilon - \{y + \frac{2ac}{a-c} \sin \epsilon\} \sin \epsilon = 0, \text{ or}$$

$$x \cos \epsilon - y \sin \epsilon = -\frac{2ac}{a-c} \cos 2\epsilon \dots \dots (21).$$

From (15, 21) we obtain by elimination,

$$\left. \begin{aligned} y_r &= \frac{2ac}{a-c} \cos 2\epsilon \sin \epsilon \\ x_p &= -\frac{2ac}{a-c} \cos 2\epsilon \cos \epsilon \end{aligned} \right\} \dots \dots \dots (22).$$

To find the *lengths* of the lines MA, MC, MB, MD, we have

$$MA^2 = (x_a - x_m)^2 + (y_a - y_m)^2$$

$$MC^2 = (x_c - x_m)^2 + (y_c - y_m)^2$$

$$MB^2 = (x_b - x_m)^2 + (y_b - y_m)^2$$

$$MD^2 = (x_d - x_m)^2 + (y_d - y_m)^2$$

The method of computation of all these is alike, and it will therefore be sufficient to put down that of one.

$$\begin{aligned} AM^2 &= (x_a - x_m)^2 + (y_a - y_m)^2 \\ &= \left\{ a \cos \epsilon + \frac{A \sin \epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \right\}^2 + \left\{ a \sin \epsilon + \frac{BC \cos \epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \right\}^2 \\ &= \left\{ a^2 \cos^2 \epsilon + \frac{2a \cdot AC \cdot \sin \epsilon \cos \epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} + \frac{A^2 C^2 \sin^2 \epsilon}{(A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon)^2} \right\} \\ &\quad + \left\{ a^2 \sin^2 \epsilon + \frac{2a BC \sin \epsilon \cos \epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} + \frac{B^2 C^2 \cos^2 \epsilon}{(A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon)^2} \right\} \\ &= a^2 + \frac{2a(A+B)C \cdot \sin \epsilon \cos \epsilon + C^2}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \\ &= a^2 + \frac{2a\{2bd(a-c) + 2bcd\} \times 4abcd \sin^2 \epsilon \cos^2 \epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \\ &= a^2 \left\{ 1 + \frac{4 \cdot ab^2cd^2 \sin^2 2\epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \right\} \dots \dots \dots (23). \end{aligned}$$

$$\text{Similarly, } MC^2 = c^2 \left\{ 1 + \frac{4ab^2cd^2 \sin^2 2\epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \right\} \dots \dots (24)$$

$$MB^2 = b^2 \left\{ 1 + \frac{4a^2hc^2d \sin^2 2\epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \right\} \dots \dots (25)$$

$$MD^2 = d^2 \left\{ 1 + \frac{4a^2bc^2d \sin^2 2\epsilon}{A^2 \sin^2 \epsilon + B^2 \cos^2 \epsilon} \right\} \dots \dots (26).$$

For the lengths of the lines BN, DN, EN, FN we shall have in like manner;—

$$BN^2 = (x_b - x_n)^2 + (y_b - y_n)^2$$

$$DN^2 = (x_d - x_n)^2 + (y_d - y_n)^2$$

$$EN^2 = (x_e - x_n)^2 + (y_e - y_n)^2$$

$$FN^2 = (x_f - x_n)^2 + (y_f - y_n)^2.$$

$$\begin{aligned} \text{Now } BN^2 &= (x_b - x_n)^2 + (y_b - y_n)^2 \\ &= \left\{ -b \cos \epsilon - \frac{2bd}{b-d} \cos \epsilon \cos 2\epsilon \right\}^2 + \left\{ b \sin \epsilon - \frac{2bd}{b-d} \cos 2\epsilon \sin \epsilon \right\}^2 \\ &= \frac{b^2}{(b-d)^2} \{ b^2 + 2bd \cos 4\epsilon + d^2 \} \dots\dots\dots (27), \end{aligned}$$

$$\text{Similarly, } DN^2 = \frac{d^2}{(b-d)^2} \{ b^2 + 2bd \cos 4\epsilon + d^2 \} \dots\dots\dots (28).$$

In the same way we may obtain the values of all the other lines of the figure: but as the processes are necessarily longer in proportion as the values of the co-ordinates of the extremities are more complexly expressed, the details of the operations may be dispensed with.

We see now from (23, 24) that $\frac{MA^2}{MC^2} = \frac{a^2}{b^2}$, or the angle

AMC is bisected by GM, and in like manner from (25, 26,) that BMD is likewise bisected by GM. The other results are obtained in a precisely similar way. The property then is also established analytically.

The symmetry and simplicity of these results are quite as great as could be expected when the number of processes necessarily gone through to arrive at them, are considered. It is, however, in reference to other properties of the trapezium that they are chiefly valuable—properties which do not apparently admit of any geometrical solution at all. For this purpose it was that I entered on the enquiry: and the values and formulæ laid down already, I have found of great use in the investigation of other properties of the complete quadrilateral. The length to which I have already gone in the present solution, forbids my entering more at large on the subject here: but it is likely that I shall resume it on an early occasion,—and if so, will put down the values also of the other lines in the figure.

$$\begin{aligned}
& \therefore \frac{(PH - \Delta \cdot PH)^{\frac{1}{2}}}{(SP - TQ)^{\frac{1}{2}}} :: 1 - \frac{n}{2} \cdot \frac{TQ}{SP} : 1 - \frac{1}{2} \cdot \frac{\Delta PH}{PH}; \text{ and } PQ : QR \\
& :: SY - \Delta \cdot SY : SY :: 1 - \frac{\Delta SY}{SY} : 1; \text{ compound this propor-} \\
& \text{tion with the former, we have } QR : QR :: 1 - \frac{n}{2} \cdot \frac{TQ}{SP} : 1 - \\
& \frac{1}{2} \cdot \frac{\Delta PH}{PH} - \frac{\Delta SY}{SY} \therefore QR : Rr :: 1 - \frac{1}{2} \cdot \frac{\Delta PH}{PH} - \frac{\Delta SY}{SY} \\
& : \frac{\Delta SY}{SY} + \frac{\Delta PH}{2PH} - \frac{n}{2} \cdot \frac{TQ}{SP} \therefore PQ : Rr :: 1 - \frac{n}{2} \cdot \frac{TQ}{SY} : \\
& \frac{\Delta SY}{SY} + \frac{\Delta PH}{2PH} - \frac{n}{2} \cdot \frac{TQ}{SP}. \text{ But as above } F : R :: \frac{PQ^2}{PH} : Rr \\
& \therefore \frac{PQ}{PH} \{PQ\} : Rr :: \frac{PQ}{PH} \left\{ 1 - \frac{n}{2} \cdot \frac{TQ}{SP} \right\} : \frac{\Delta SY}{SY} + \frac{\Delta PH}{2PH} - \\
& \frac{n}{2} \cdot \frac{TQ}{SP} \therefore \frac{PQ}{PH} \cdot \frac{\Delta SY}{SY} + \frac{\Delta PH}{2PH} - \frac{n}{2} \cdot \frac{TQ}{SP} \text{ ultimately } :: \frac{2ds}{c} \\
& : \frac{dp}{p} + \frac{dc}{2c} - \frac{n}{2} \cdot \frac{dr}{r}, \text{ if } c = \text{chord of curvature at } P = 2PH \\
& \text{and } p = \text{perpendicular } SY \text{ upon the tangent at } P \text{ and } s = \text{arc} \\
& \text{described. Hence } R = F \cdot \frac{c}{2ds} \left\{ \frac{dp}{p} + \frac{dc}{2c} - \frac{n}{2} \cdot \frac{dr}{r} \right\} = \frac{h}{r^2 2ds} \\
& \left\{ \frac{dp}{p} + \frac{dc}{2c} - \frac{n}{2} \cdot \frac{dr}{r} \right\} \text{ which is also } = kD \cdot v^m \therefore D = \frac{h}{kv^m} \\
& \cdot \frac{1}{r^2} \cdot \frac{c}{2ds} \left\{ \frac{dp}{p} + \frac{dc}{2c} - \frac{n}{2} \cdot \frac{dr}{r} \right\} \text{ which are the equations} \\
& \text{required,}
\end{aligned}$$

SECOND SOLUTION, by Messrs. BAINES and GODWARD.

Put r = the spiral's radius vector, p = the perpendicular upon the tangents, q = $\frac{1}{2}$ th of the chord of curvature at the same point, s = the arc of the curve described, R = the resistance, D = the density of the medium, and $F = m \div r^n$ = the centripetal force. Now, by Whewell's Dynamics, Art. 64 and 65, $q = \frac{pdr}{2dp}$, $ds = \frac{r^2 dv}{p}$, and $F = - \frac{dpq + pdr}{ds} = - \frac{m}{ds}$

$\left(d \frac{q}{r^n} + \frac{dr}{r^n}\right)$. Again, by Art. 65, $\text{veloc.} = \frac{ds}{dt} = (2Pq)^{\frac{1}{2}} = \left(\frac{2mq}{r^n}\right)^{\frac{1}{2}}$; hence $R = D \times (\text{vel.})^m = D \times \frac{2mq}{r^n}^{\frac{m}{2}}$, and $D R \div \left(\frac{2mq}{r^n}\right)^{\frac{m}{2}} = -\frac{m}{ds} d \frac{q}{r^n} + \frac{dr}{r^n} \div \left(\frac{2mq}{r^n}\right)^{\frac{m}{2}}$, so that when the relation between p and r is given, the preceding formulæ will enable us to determine R and D .

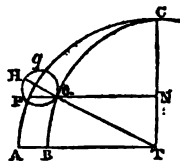
Mr. DAVIES answered this Question also.

XVI. QUESTION 526, by R. N.

Art. 340, page 335, Vol. II. Playfair's Outlines of Natural Philosophy, it is stated, that a circle whose diameter is equal to the whole height of the tide being placed perpendicular to the horizon the lowest point being at the level of low water, the tide will ascend and descend through equal arcs in equal times. Required the proof. The earth is supposed a fluid sphere and disturbed only by the action of the sun and moon.

SOLUTION, by — MASON, of Scoulton.

Let AB be the whole height of the tide in the given latitude, t the number of hours since low water as at C , Q the place of the observer, $QH = AB = \text{diameter of the circle}$, cPA a quadrant of the spheroid. Draw TQH , and PQN perpendicular to CT or parallel to AT ; then the $\angle QTN = (15 \cdot t)^\circ$ and the height of the tide $pQ = \text{vers } Qq = \frac{1}{2} HQ$, vers of degrees in $Qq = \frac{1}{2} AB$ vers degrees in Qq . But, by the ellipse, $AB : PQ :: AT : PN :: BT : QN :: QT : QN$ and $PQ : pQ :: QT : QN$ nearly; therefore, compounding, $AB : pQ :: QT^2 : QN^2 :: 1 : \sin^2 (15 \cdot t)^\circ \therefore pQ = AB \sin^2 (15 \cdot t)^\circ = \frac{1}{2} AB$ vers $(15 \cdot t)^\circ$. Hence the number of degrees in $Qq = (15 \cdot t)^\circ$, or the arc Qq is proportional to the time since low water.



Not materially different was the solution of Mr. DAVIES.

XVII. QUESTION 527, by R. N.

A fine inextensible and flexible string passes freely through a small hole in an horizontal smooth table; two weights P and Q

are attached to the string, Q hanging down, and P being on the horizontal table or plane; a given velocity is communicated to P in a given direction; required the equation of the path it describes on the plane.

SOLUTION, by — MASON, of Scoulton.

Let T be the tension of the string at the time (t) , x and y the co-ordinates of P , the axis of x being that line which is perpendicular to the direction of P 's motion at the beginning, θ = the angular distance of P from the axis of x , and r = the distance of P from the orifice.

$$\therefore \frac{d^2x}{dt^2} = -\frac{T}{P} \cos \theta, \frac{d^2y}{dt^2} = -\frac{T}{P} \sin \theta \text{ and } \frac{d^2r}{dt^2} = -g + \frac{T}{Q},$$

are the equations of the motions P and Q ,

$$\text{hence } \sin \theta \frac{d^2x}{dt^2} - \cos \theta \frac{d^2y}{dt^2} = 0 \dots\dots\dots(1)$$

$$\text{and } P \cos \theta \frac{d^2x}{dt^2} + P \sin \theta \frac{d^2y}{dt^2} + Q \frac{d^2r}{dt^2} + Qg = 0 \dots\dots\dots(2).$$

But $x = r \cos \theta$ and $y = r \sin \theta$; we thus reduce equation (1)

$$\text{to } 2 \cdot \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \cdot \frac{d^2\theta}{dt^2} = 0, \therefore \frac{c^2}{r^2} = \frac{d\theta}{dt}: \text{ let } r = a \text{ at first}$$

$$\text{and } \frac{d\theta}{dt} = w; \therefore \frac{a^2 w}{r} = \frac{d\theta}{dt} \dots\dots\dots(3), \text{ and equation (2) is}$$

$$(P + Q) \frac{d^2r}{dt^2} - Pr \left(\frac{d\theta}{dt} \right)^2 + Qg = 0, \text{ or, from (3), } (P + Q)$$

$$\frac{d^2r}{dt^2} - \frac{P \cdot a^4 \cdot w^2}{r^3} + Qg = 0, \therefore (P + Q) \left(\frac{dr}{dt} \right)^2 = -\frac{P \cdot a^4 \cdot w^2}{r^2}$$

$$- 2Qgr + c. \text{ But } \frac{dr}{dt} = 0 \text{ when } r = a \therefore (P + Q) \left(\frac{dr}{dt} \right)^2 =$$

$$P \cdot a^4 \cdot w^2 \frac{r^2 - a^2}{a^2 \cdot r^2} - 2Qg(r - a) \text{ also } \frac{d\theta}{dt} = \frac{a^2 w}{r^2} \therefore (P + Q)$$

$$\left(\frac{dr}{r d\theta} \right)^2 \text{ or } (P + Q) \frac{r^2 - p^2}{p^2} = P \frac{r^2 - a^2}{a^2} - \frac{2Qgr^2(r - a)}{a^4 w^2},$$

$$\therefore \frac{1}{p^2} = \frac{Q}{P + Q} \frac{1}{r^2} + \frac{P}{P + Q} \frac{1}{a^2} - \frac{2Qg}{(P + Q)w^2} \frac{r - a}{a^4} \text{ the}$$

equation to the path described by P in terms of the radius vector and the perpendicular on the tangent. The equation be-

tween r and θ cannot be found, the differential equation not being integrable.

The same might have been determined by the principle of *vis viva*, and the equable description of areas, the force tending always to a fixed point.

Let v and v' be the velocities of P and Q , $\therefore c = 2g(r - a)$
 $Q = Pv^2 + Qv'^2$; if aw = velocity of P when $r = a$, then
 $c = Pa^2w^2$, $\therefore Pa^2w^2 = 2gQ(r - a) = Pv^2 + Qv'^2 \frac{r^2 - p^2}{p^2} =$
 $(P + Q \frac{r^2 - p^2}{p^2}) (\frac{a^4w^2}{p^2}) = (P + Q) \frac{a^4w^2}{p^2} - Q \frac{a^4w^2}{r^2}$, $\therefore \frac{1}{p^4} =$
 $\frac{Q}{Q + P} \cdot \frac{1}{r^2} - \frac{2Qg}{(P + Q)a^2} \cdot \frac{r - a}{a^4}$; the same as before.

Mr. DAVIES also answered this Question.

XVIII. QUESTION 528, by DELTA SULIUS.

Two sides of a triangle inscribed in a conic section pass constantly by the foci: to what curve is the third side always a tangent?

SOLUTION, by Mr. T. S. DAVIES.

Let x, y , be the vertex, x_{II}, y_{II} , x_{III}, y_{III} , the extremities of the base of one of the triangles in question: and let the whole system be referred to the centre and principal axes. Then the equation of the ellipse is

$$a^2y^2 + b^2x^2 = a^2b^2 \dots\dots\dots(1).$$

And because the triangle is inscribed in the ellipse, we have also the following equations

$$a^2y_{II}^2 + b^2x_{II}^2 = a^2b^2 \dots\dots\dots(2)$$

$$a^2y_{III}^2 + b^2x_{III}^2 = a^2b^2 \dots\dots\dots(3)$$

$$a^2y_{II}^2 + b^2x_{III}^2 = a^2b^2 \dots\dots\dots(4).$$

The equations of the sides of the inscribed triangle will be

$$y(x, -x_{II}) - x(y, -y_{II}) = y_{II}x_{III} - y_{III}x_{II} \dots\dots\dots(5)$$

$$y(x, -x_{III}) - x(y, -y_{III}) = y_{III}x_{II} - y_{II}x_{III} \dots\dots\dots(6).$$

But (5, 6) pass by the focus, and hence, putting $a^2 - b^2 = a^2e^2$, they give

$$\begin{cases} ae(y, -y_{II}) = y_{II}x_{III} - y_{III}x_{II} \\ -ae(y, -y_{III}) = y_{III}x_{II} - y_{II}x_{III} \end{cases} \text{ or}$$

$$y_{II} = \frac{x_{II} + ae}{x_I + ae} \cdot y_I, \dots\dots\dots(7).$$

$$y_{III} = \frac{x_{III} - ae}{x_I - ae} \cdot y_I, \dots\dots\dots(8).$$

Subtract equations (3, 4) from equation (2), and the result will be

$$b^2(x_I^2 - x_{II}^2) + a^2(y_I^2 - y_{II}^2) = 0 \dots\dots\dots(9)$$

$$b^2(x_I^2 - x_{III}^2) + a^2(y_I^2 - y_{III}^2) = 0 \dots\dots\dots(10).$$

Insert the values of y_{II} , y_{III} , from (7, 8) in (9, 10): the result is

$$x_{II} = -\frac{(1 + e^2)x_I + 2ae}{(1 + e^2)a + 2ex_I} \cdot a, \dots\dots\dots(11)$$

$$x_{III} = -\frac{(1 + e^2)x_I - 2ae}{(1 + e^2)a - 2ex_I} \cdot a, \dots\dots\dots(12).$$

Substitute these values of x_{II} , x_{III} , in equations (7, 8), and there will result

$$y_{II} = -\frac{(1 - e^2)ay_I}{(1 + e^2)a + 2ex_I}, \dots\dots\dots(13)$$

$$y_{III} = -\frac{(1 - e^2)ay_I}{(1 + e^2)a - 2ex_I}, \dots\dots\dots(14).$$

Now the equation of the variable base, in terms of the co-ordinates of its extremities, is

$$y(x_{II} - x_{III}) - x(y_{II} - y_{III}) = y_{III}x_{II} - y_{II}x_{III} \dots\dots\dots(15),$$

or putting the values of x_{II} , x_{III} , y_{II} , y_{III} from (11, 12, 13, 14) we find

$$y(1 + e^2)(a^2 - x_I^2) + x(1 - e^2)xy_I = a^2(1 - e^2)y_I, \dots\dots\dots(16).$$

Eliminating y , between (2) and (16) we find

$$a^2y^2(1 + e^2)^2(a^2 - x_I^2) = b^2(1 - e^2)^2(a^2 - xx_I)^2 \dots\dots\dots(17).$$

Differentiating (17) and equating the result to 0, on account of the locus being the intersection of the lines (16) in their consecutive positions, we have

$$x_I = \frac{a^2b^2x(1 - e^2)^2}{(1 + e^2)^2a^2y^2 + (1 - e^2)^2b^2x^2} \dots\dots\dots(18).$$

And finally eliminating x , between (17, 18) we are led to the equation sought: viz.

$$a^2y^2(1 + e^2)^2 + b^2x^2(1 - e^2)^2 = a^2b^2(1 - e^2)^2 \dots\dots\dots(19)$$

which last is a solution of the problem in question, being the equa-

tion of a concentric ellipse, whose semi-axes are a and b . $\frac{1 - e^2}{1 + e^2}$.

and are coincident with those of the given ellipse.

The same method applies to the hyperbola and with corresponding results ; but the parabola has no corresponding property, on account of having no second focus.

This is only a very confined case of a mere general theorem : viz. If a polygon of any number of sides be inscribed in a given conic section so that each side except one may pass through a given point, that side will touch a given conic section. This may be proved by projection as in the next question.

There is another case of this very curious question which has not, I think, been noticed. When some of the sides pass through given points, and others are parallel to given straight lines, or make given angles with given straight lines, the last side will then also touch a given conic section. In this case we may consider the lines which are parallel to given lines or which make given angles with given lines, as passing through points at an infinite distance from the centre of the given conic section, and situated upon given diameters of those given conic sections or upon parallels to them.

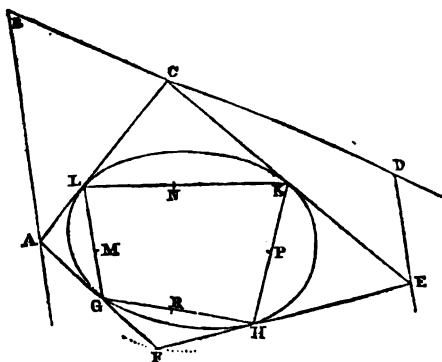
XIX. QUESTION 529, by POLYGONUS.

All the angles but one of a polygon circumscribing a given conic section are upon given lines : what is the locus of the remaining one ?

SOLUTION, by Mr. T. S. DAVIES, Bath.

Project the conic section and the given polygon upon any plane where the section shall be circular : then—

Let AB, BD, DE , &c. be the given lines, and $ACEF$. . . one of the polygons circumscribing the section. Denote by $ab, bd, de, \dots$ and $acef, \dots$ the corresponding lines on the projection. Now because the angle A moves upon AB , the chord LG passes through a given point M . In like manner all the other sides but one of the in-



scribed polygon pass through given points. Let these points $m, n, \dots$ be projected on the new plane into $m, n, \dots$; then the polygon inscribed in the circle will have all its sides but one passing through $m, n, \dots$

Now it has been shewn in a remarkably elegant manner by Mr. Lowry, in the Mathematical Companion, 1814, p. 492, that in this case the last side gh , will touch a given conic section. Let this conic section be again projected on the original plane ABC : then this will be the locus of consecutive intersections of the sides $GH, G'H', \dots$

Lastly, it has been shewn in the solutions to Question 523, of this series, that when a line GH touches one conic section in R and cuts another in G, H : then the tangents at G, H will always meet in a given conic section, This is the locus required.

Note. I have solved this question also by an analytical process; but it is long and complicated; and is similar in its principles to the method employed in my solution of the last question, and the prize of last number, (Part II. p. 57). This renders it unnecessary to say more on the subject here: especially, as I do not think the analytical solution I speak of could be materially contracted.

XX. PRIZE QUESTION 530.

By Mr. W. S. B. WOOLHOUSE.

AC and BC are two *heavy metal* and equally uniform bars of given lengths, and freely moveable about C in the plane ACB by means of a joint at C ; BC rolls over a small pulley P , while AC moves freely round a pin which is fixed at A in the vertical right line PA , the distance PA being given; Being placed in a given position they are allowed to descend by the force of gravity. It is required to investigate the velocities when they arrive at any other position: and also to give a method of computing their position at the instant BC quits the pulley?

FIRST SOLUTION, by — MASON, of Scoulton, Norfolk.

Let AC, CB be any positions of the two rods, AP vertical,

$$7(2a)^2 \frac{c^2 \cos^2 \theta}{a^2 + c^2 - 2ac \sin \theta} \left(\frac{d\theta}{dt} \right)^2 \text{ since } \tan \phi = \frac{c - a \sin \theta}{a \cos \theta}$$

$$\text{and } y = 2a \frac{\cos \theta}{\cos \phi} = 2\sqrt{a^2 + c^2 - 2ac \sin \theta} \text{ and hence } \left(\frac{d\phi}{dt} \right)^2$$

$$= \frac{a^2 (c \sin \theta - a)^2}{(a^2 + c^2 - 2ac \sin \theta)^2} \left(\frac{d\theta}{dt} \right)^2. \text{ Put these values in equation (1) and we have (velocity)}^2 \text{ of the point c} = (2a)^2 \left(\frac{d\theta}{dt} \right)^2 =$$

$$6g \left\{ pa + 2q \cdot a (\sin \alpha - \sin \theta) + qb (\sin \beta - \frac{c - a \sin \theta}{\sqrt{a^2 + c^2 - 2ac \sin \theta}}) \right\} \\ + 3q + q \frac{(c \sin \theta - a)^2}{(a^2 + c^2 - 2ac \sin \theta)^2} \left\{ b^2 - 8b \sqrt{a^2 + c^2 - 2ac \sin \theta} \right\} \quad (A).$$

If we divide this equation by $(2a)^2$ we have $\left(\frac{d\theta}{dt} \right)^2$ or the square of the angular velocity of the rod AC; and since $\frac{d\phi}{dt} = \frac{a(c \sin \theta - a)}{a^2 + c^2 - 2ac \sin \theta} \left(\frac{d\theta}{dt} \right)$, we know the angular velocity of BC.

To find when AC quits the pulley.

Let x', y' be the co-ordinates of the centre of gravity of BC, s the pressure of the rod upon the pulley,

u = the horizontal pressure at c, which is necessarily equal upon the two rods and opposite; let q be the vertical pressures at c, which are also equal on the two rods and opposite,

$$\therefore \frac{s \sin \phi - u}{q} = \frac{d^2 x'}{dt^2}, \quad \frac{s \cos \phi + q}{q} - g = \frac{d^2 y'}{dt^2}$$

$$\text{and } \frac{r \sin \theta + q \cos \theta + \frac{p}{2} g \cos \theta}{\frac{1}{3} p} = -2a \frac{d^2 \theta}{dt^2}.$$

Multiply the first and second of these equations by $\sin \theta$ and $\cos \theta$ respectively and subtract, therefore

$$\frac{s}{q} \cos (\theta + \phi) + \frac{p}{q} \frac{r \sin \theta + q \cos \theta}{p} - g \cos \theta = \cos \theta \frac{d^2 y'}{dt^2} - \sin \theta \frac{d^2 x'}{dt^2}, \text{ we must substitute in this equation the value of } \frac{r \sin \theta + q \cos \theta}{p} \text{ deduced from the third equation.}$$

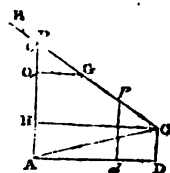
Now $y' = 2a \sin \theta + b \sin \phi$ and $x' = 2a \cos \theta - b \cos \phi$,
we hence find $\cos \theta \frac{d^2 y'}{dt^2} - \sin \theta \frac{d^2 x'}{dt^2}$ and thence

$$\left. \begin{aligned} \frac{1}{g} \cos(\theta + \phi) &= \frac{p}{g} \frac{2a}{3} \frac{d^2 \theta}{dt^2} + \frac{p}{g} \frac{b}{2} \cos \theta + g \cos \theta \\ + 2a \frac{d^2 \theta}{dt^2} - b \sin(\theta + \phi) \left(\frac{d\phi}{dt} \right)^2 + b \cos(\theta + \phi) \frac{d^2 \theta}{dt^2} \end{aligned} \right\} \dots (B).$$

We have already found in equation (A), $\frac{d\theta}{dt}$, we $\therefore$ find by
differentiating $\frac{d^2 \theta}{dt^2}$ and also from $\frac{d\phi}{dt}$ we find $\left(\frac{d\phi}{dt} \right)^2$, and put
them in equation (B) we thus find s the pressure of BC upon
the pulley in terms of θ ; if we make $s = 0$, the value of θ
resulting gives the position of AC at the instant BC quits
the pulley, and θ being known ϕ is also known from the
equation $\tan \phi = \frac{c - a \sin \theta}{a \cos \theta}$, and the problem will be com-
pletely solved.

SECOND SOLUTION, by Mr. W. S. B. WOOLHOUSE.

Draw AD perpendicular to the vertical right line CD ;
 CH perp. to PA ; let p be any particle in BC ,
and assume $AC = a$; $BC = b$; $\angle CAD = \phi$;
 $\angle BCH = \psi$; $\angle ACB = \phi + \psi = \theta$; ga feet
 $= g$, and $cp = k$. Then the values of the
co-ordinates of p referred to AD are $Ad =$
 $a \cos \phi - k \cos \psi$ and $pd = a \sin \phi + k \sin \psi$.



$\therefore d . Ad = -ad\phi \sin \phi + kd\psi \sin \psi$, $d . pd =$
 $= ad\phi \cos \phi + kd\psi \cos \psi$.

$$\begin{aligned} d . Ad^2 + d . pd^2 &= a^2 d\phi^2 + k^2 d\psi^2 + 2akd\phi d\psi (\cos \phi \cos \psi - \sin \phi \sin \psi) \\ &= a^2 d\phi^2 + k^2 d\psi^2 + 2akd\phi d\psi \cos(\phi + \psi) \\ &= a^2 d\phi^2 + k^2 d\psi^2 + 2akd\phi d\psi \cos \theta. \end{aligned}$$

$$\therefore \text{vel.}^2 p = \frac{d . Ad^2}{dt^2} + \frac{d . pd^2}{dt^2} = a^2 \frac{d\phi^2}{dt^2} + k^2 \frac{d\psi^2}{dt^2} + 2ak . \frac{d\phi d\psi}{dt^2} \cos \theta$$

and consequently the sum of the vis viva of all the particles
of which BC is composed $= \Sigma p . \text{vel.}^2 p = \Sigma dk . \text{vel.}^2 p =$

$$a^2 b \frac{d\phi^2}{dt^2} + \frac{b^3}{3} . \frac{d\psi^2}{dt^2} + a . b^2 \frac{d\phi d\psi}{dt^2} \cos \theta.$$

Also the vires viva of the particles composing AC = $\frac{a^3}{3} \cdot \frac{d\phi^2}{dt^2}$.

Therefore the sum of the vis vivæ of both bars =

$$\left(\frac{a^3}{3} + a^2b\right) \cdot \frac{d\phi^2}{dt^2} + \frac{b^3}{3} \cdot \frac{d\psi^2}{dt^2} + ab^2 \frac{d\phi d\psi}{dt^2} \cos \theta;$$

and the descent of their common centre of gravity being

$$\begin{aligned} & \left(\frac{a^2}{2} + ab\right) \sin \phi, + \frac{b^2}{2} \sin \psi, \quad \left(\frac{a^2}{2} + ab\right) \sin \phi + \frac{b^2}{2} \sin \psi \\ = & \frac{\left(\frac{a^2}{2} + ab\right) \sin \phi, + \frac{b^2}{2} \sin \psi}{a + b} - \frac{\left(\frac{a^2}{2} + ab\right) \sin \phi + \frac{b^2}{2} \sin \psi}{a + b} \\ = & \frac{\left(\frac{a^2}{2} + ab\right) (\sin \phi, - \sin \phi) + \frac{b^2}{2} (\sin \psi, - \sin \psi)}{a + b}, \end{aligned}$$

we have by the property of the vis viva,

$$\begin{aligned} & \left(\frac{a^3}{3} + a^2b\right) \cdot \frac{d\phi^2}{dt^2} + \frac{b^3}{3} \cdot \frac{d\psi^2}{dt^2} + ab^2 \cdot \frac{d\phi d\psi}{dt^2} \cos \theta = \\ & g \left\{ (a^2 + 2ab) (\sin \phi, - \sin \phi) + b^2 (\sin \psi, - \sin \psi) \right\}. \end{aligned}$$

But PA = CH. $\tan \psi = a \cos \phi \tan \psi$ and $\therefore a \cos \phi \tan \psi + a \sin \phi = PH + HA = PA$ and

$$\cos \phi \tan \psi + \sin \phi = \frac{PA}{a},$$

the differential of which multiplied by $\cos \psi$ gives

$$d\psi \frac{\cos \phi}{\cos \psi} + d\phi (\cos \phi \cos \psi - \sin \phi \sin \psi) = d\psi \frac{\cos \phi}{\cos \psi} + d\phi \cos \theta = 0,$$

$\therefore d\psi = -d\phi \cdot \frac{\cos \psi}{\cos \phi} \cos \theta$, and by substitution

$$\begin{aligned} & \left\{ \frac{a^3}{3} + a^2b - ab^2 \frac{\cos \psi}{\cos \phi} \cos^2 \theta + \frac{b^3}{3} \cdot \frac{\cos^2 \psi}{\cos^2 \phi} \cos^2 \theta \right\} \cdot \frac{d\phi^2}{dt^2} \\ & = g \left\{ (a^2 + 2ab) (\sin \phi, - \sin \phi) + b^2 (\sin \psi, - \sin \psi) \right\}. \end{aligned}$$

Consequently

$$-\frac{d\phi}{dt} = \sqrt{\frac{3g \left\{ (a^2 + 2ab) (\sin \phi, - \sin \phi) + b^2 (\sin \psi, - \sin \psi) \right\}}{a^3 + 3a^2b - 3ab^2 \frac{\cos \psi}{\cos \phi} \cos^2 \theta + b^3 \frac{\cos^2 \psi}{\cos^2 \phi} \cos^2 \theta}}$$

$$\text{And } \frac{d\psi}{dt} = -\frac{d\phi}{dt} \cdot \frac{\cos \psi}{\cos \phi} \cos \theta =$$

$$\cos \theta \sqrt{\frac{3g \left\{ (a^2 + 2ab) (\sin \phi, - \sin \phi) + b^2 (\sin \psi, - \sin \psi) \right\}}{a^3 \frac{\cos^2 \phi}{\cos^2 \psi} + 3a^2b \frac{\cos^2 \phi}{\cos^2 \psi} - 3ab^2 \frac{\cos \phi}{\cos \psi} \cos^2 \theta + b^3 \cos^2 \theta}}.$$

Which two equations express the angular velocities of AC and BC, the former of which multiplied by a will give the velocity of the joint c.

To determine the position of the bars when BC quits the pulley let G be its centre of gravity, and assume $PG = x$ and the co-ordinates of G referred to PA will be $PQ = x \sin \psi$ and $QG = x \cos \psi$. Let the pressure acting at C on BC in direction DC = p , and that in direction CH = f . Then since at this instant there is no reaction from the pulley at P, we have for the motion of BC,

$$g - \frac{p}{b} = \frac{d^2(x \sin \psi)}{dt^2}; \quad \frac{f}{b} = - \frac{d^2(x \cos \psi)}{dt^2}$$

$$\text{and } \frac{f}{b} \sin \psi - \frac{p}{b} \cos \psi = \frac{b}{6} \cdot \frac{d^2 \psi}{dt^2}.$$

Substituting the values of $\frac{p}{b}$ and $\frac{f}{b}$ from the 1st and 2nd in the 3rd, we derive,

$$-\sin \psi \frac{d^2(x \cos \psi)}{dt^2} + \cos \psi \frac{d^2(x \sin \psi)}{dt^2} - g \cos \psi = \frac{b}{6} \cdot \frac{d^2 \psi}{dt^2}.$$

Or by expansion,

$$\frac{a dx d\psi}{dt^2} - g \cos \psi = \frac{b}{6} \cdot \frac{d^2 \psi}{dt^2}, \text{ or } \frac{dx d\psi}{dt^2} - \frac{b}{12} \cdot \frac{d^2 \psi}{dt^2} = \frac{g}{2} \cos \psi.$$

But $PG = a \cos \phi \sec \psi$, and since GC is constant, we have $dx = d.p.c = ad.(\cos \phi \sec \psi)$, which expanded and the value of $d\psi$, before deduced, substituted and the whole reduced gives $dx = -ad\phi \sin(\phi + \psi) = -ad\phi \sin \theta$,

$$\therefore -a \frac{d\phi}{dt} \cdot \frac{d\psi}{dt} \sin \theta - \frac{b}{12} \cdot \frac{d^2 \psi}{dt^2} = \frac{g}{2} \cos \psi.$$

Now the differential of the foregoing expression for $\frac{d\psi}{dt}$ will obviously give $\frac{d^2 \psi}{dt^2} = p d\phi + q d\psi$, (p and q being determinate functions of ϕ and ψ) and $\therefore \frac{d^2 \psi}{dt^2} = p \frac{d\phi}{dt} + q \frac{d\psi}{dt}$.

Hence

$$-a \frac{d\phi}{dt} \cdot \frac{d\psi}{dt} \sin \theta - \frac{b}{12} \left\{ p \frac{d\phi}{dt} + q \frac{d\psi}{dt} \right\} = \frac{g}{2} \cos \psi;$$

and substituting the values of $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt}$, we derive an

equation by which the values of ϕ and ψ may be computed, but the operation would be too complex for insertion.

Mr. DAVIES also solved this Question.

NOTICES.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF LONDON.

The Volume for 1831, contains the following Mathematical Papers.

1. Observations on the Second Comet of 1822, made at Rio de Janeiro. By Captain William Robertson, R. N.
2. On the performance of Fluid Refracting Telescopes, and on the applicability of the principle of construction to very large instruments. By Peter Barlow, Esq.
3. Researches in Physical Astronomy: viz. on the Precession of the Equinoxes, and on the Theory of the Planets. By J. W. Lubbock, Esq.
4. On the Transient Magnetic state of which various substances are susceptible. By William Snow Harris, Esq.
5. On the Nature of Negative and Imaginary Quantities. By Davies Gilbert, Esq.
6. On the probable Electric origin of all the Phenomena of Terrestrial Magnetism, with an Illustrative Experiment. By Peter Barlow, Esq.
7. On the Equilibrium of Fluids, and on the Figure of a Homogeneous Planet in a Fluid State. By James Ivory, Esq.
8. An account of Operations carried on for ascertaining the Difference of Level between the River Thames at London Bridge and the Sea; and also for determining the Height above the Level of the Sea, of intermediate points passed over between Sheerness and London Bridge. By John Augustus Lloyd, Esq.
9. On the variable intensity of Terrestrial Magnetism, and the Influence of Aurora Borealis upon it. By Robert Were Fox, Esq.
10. Description of a Graphic Register of Tides and Winds. By H. R. Palmer, Esq. Civil Engineer.
11. On the Errors in the Course of Vessels, occasioned by Local Attraction, with some remarks on the loss of his Majesty's Ship Thetis. By Peter Barlow, Esq.
12. Researches in Physical Astronomy: viz. on the Theory of the Moon; on the Precession of the Equinoxes, supposing the Earth to revolve in a Resisting Medium; and on the Planetary Inequalities, the equations extending as far as the Cubic Orders of the Eccentricities. By J. W. Lubbock, Esq.
13. On a peculiar class of Acoustic Figures: and on certain forms assumed by Groups of Particles upon Vibrating Elastic Surfaces. By Michael Faraday, Esq.
14. Table for facilitating the Computations relative to Suspension Bridges. By Davies Gilbert, Esq.
15. An account of the Construction and Verification of a Copy of the Imperial Standard Yard, made for the Royal Society. By Capt. Henry Kater.
16. On the Theory of Elliptic Transcendents. By James

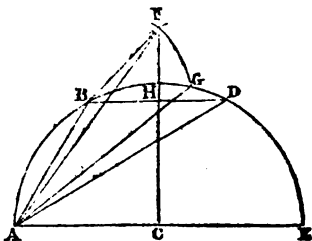
Ivory, Esq. 17. On the Tides in the Port of London, By J. W. Lubbock, Esq. 18. On the Extensive Atmosphere of Marr. By Sir James South. 19. On the Friction and Resistance of Fluids. By George Rennie, Esq. 20. An account of a new Register Pyrometer, for Measuring the Expansion of Solids. By J. F. Daniell, Esq. 21. On the length of the Seconds' Pendulum at the Royal Observatory at Greenwich. By Capt. Edward Sabine. 22. On the Influence of Screens in arresting the progress of Magnetic Action; and on the power of Masses of Iron to control the attractive force of a Magnet. By W. S. Harris, Esq.

NUGÆ.

* * We intend under the present head to present our readers with such miscellaneous notes on mathematical subjects as, either from their brevity or their individual want of importance, could not constitute separate articles. ED.

Constructions and Expressions for the Circumference of the Circle.

1. Let AE be the diameter and C the centre of the circle, and let the semicircle be described upon AE . Set off AB , ED each equal to radius AC . With centres A and E and distances respectively equal to AD , EB describe arcs intersecting in F . Then with centre B and radius BF describe a circle cutting the circumference (on the side D) in G . The chord AG is nearly equal to the quadrant of the circle.



$$\text{For } AF = AD = 2 \sin 60^\circ = \sqrt{3}$$

$$\text{then } CF = \sqrt{AF^2 - AC^2} = \sqrt{3 - 1} = \sqrt{2}.$$

But H being the intersection of CF and BD , we shall have

$$FH = CF - CH = \sqrt{2} - \frac{1}{2}\sqrt{3} = \frac{2\sqrt{2} - \sqrt{3}}{2}.$$

The triangle BHF gives $BH = BF = \sqrt{BH^2 + HF^2} = \sqrt{3 - \sqrt{6}}$.

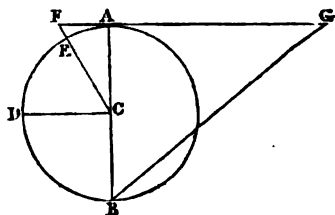
Hence we have $\sin \frac{1}{2}BG = \frac{1}{2}\sqrt{3 - \sqrt{6}}$, $\sin \frac{1}{2}AB = \frac{1}{2}$

$$\cos \frac{1}{2}BG = \frac{1}{2}\sqrt{1 + \sqrt{6}}, \cos \frac{1}{2}AB = \frac{1}{2}\sqrt{3}.$$

$$\begin{aligned} \text{Then } 2AG &= 4 \sin \frac{1}{2}ABG = 4 \sin \left(\frac{1}{2}AB + \frac{1}{2}BG \right) \\ &= 4 \sin \frac{1}{2}AB \cos \frac{1}{2}BG + 4 \sin \frac{1}{2}BG \cos \frac{1}{2}AB, \\ &= \{ \sqrt{1 + \sqrt{6}} + \sqrt{9 - 3\sqrt{6}} \} \\ &= 3.14239917 \\ &= \text{semicircumf. to radius unity} \} \text{ very} \\ &= 2 \text{ quadrantal arcs} \} \text{ nearly.} \end{aligned}$$

This, which is a tolerably close approximation for practical purposes, is effected by *the compasses only*. It is due to Mascheroni.

II. Let AB be the diameter of a circle, and c its centre. Let there be drawn an indefinite tangent at the point A, and a radius CD parallel to this tangent. If we set off the radius DC towards A, terminating in E, and draw CE to meet the tangent in F; and take upon this tangent from F on the side of A, FG = 3CD, the straight line BG is nearly equal the semicircle



We have by construction

$$AF = \tan 30^\circ = \frac{1}{2}\sqrt{3}.$$

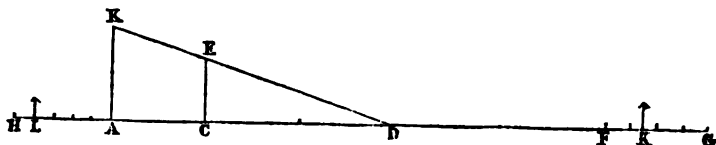
$$\text{Hence } AG = FG - AF = 3 - \frac{1}{2}\sqrt{3} = \frac{1}{2}(9 - \sqrt{3})$$

$$\begin{aligned} BG &= \sqrt{AB^2 + AG^2} = \sqrt{4 + \frac{1}{4}(9 - \sqrt{3})^2} \\ &= \frac{1}{2}\sqrt{6(20 - 3\sqrt{3})} \\ &= 3.1415384 \end{aligned}$$

which is nearly the length of a semicircle.

This method is by an anonymous German author.

III. From any point A in an indefinite straight line draw a



For the similar triangles BAC , cmn give

$$mn : nc :: ma : ab, \text{ whence } mn = ac = \frac{ac \cdot ma}{ab} = \frac{ab \cdot D}{15ab} = \frac{D}{15}.$$

Then $BC = BA - AC = \frac{15D}{15} - \frac{D}{15} = \frac{14D}{15}$. Also, the right-angled triangles $D''cB$ gives

$$D''c = D \sqrt{\left(\frac{14}{15}\right)^2 + 9} = \frac{D}{15} \sqrt{2025 + 196} = \frac{D}{15} \sqrt{2221} =$$

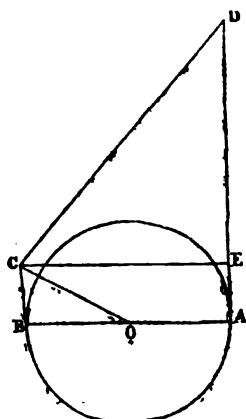
3.1418.

This is the method of M. le Professor Quetelet of Brussels.

It is taken from (vol. I. pp. 253, 4.) his *Correspondance*—where also (Vol. II. pp. 257, 8.) we find the following by Professor De Gelder, of Leyden.

V. Upon the circumference whose centre is O and radius $OB=1$, take the arc $BC=30^\circ$ (which is found by the ruler and compasses); draw the tangent BC , and by the other extremity A of the diameter BO , draw the indefinite tangent AD , upon which set off AD equal to three times the radius OB . Through C draw CE parallel to BA , and join CD which will represent very nearly the semicircle to radius OB .

The right angled triangle DCE gives



$$DC = \sqrt{DE^2 + CE^2} = \sqrt{\{(DA - CB)^2 + CE^2\}}$$

$$\text{But } BC = \tan 30^\circ = \frac{1}{\sqrt{3}} \text{ and } CE = 2. \text{ Hence}$$

$$DC = \sqrt{\left\{ \left(3 - \frac{1}{\sqrt{3}} \right)^2 + 4 \right\}} = 3.14153^*$$

* M. Garnier at the end of this article gives the following useful approximate expressions for functions of the circle and of the sphere.

$$\text{area of the circle} = D \cdot \frac{7}{4} D.$$

$$\text{surface of the sphere} = D (3 + \frac{1}{4}) D$$

$$\text{volume of the sphere} = D \cdot \frac{3}{4} D \cdot \frac{3}{4} D.$$

These suppose the circumference = 3 + times the diameter.

ARTICLE III.

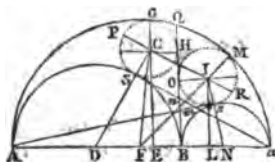
Solutions to Questions proposed in No. XXII.

I. QUESTION 531, by N. Y.

If on the segments AB , Ba , of the diameter of a semicircle, the semicircles ASB , aSB be constructed, and the ordinate BQ drawn; it is required, first, to inscribe a circle in each of the spaces $AQSBA$, $aQBSa$; secondly, to shew that these circles are equal; and thirdly, to prove that the least circle circumscribing the two equal circles, is equal to the space included between the arcs of the three semicircles.

SOLUTION, by Mr. JOHN BAINES, *Thornhill, near Wakefield.*

Suppose the circles GHS , MOs to be those inscribed in the spaces $AQBSA$, $aQBSa$, whose centres are c and i . Demit the radii CH , IO perpendicular to BQ : then, D , N and F being the centres of the semicircles ASB , BsA and AQa , join DC , NI , FO and FI , and produce the two latter to the circumference at G and M ; also demit the perpendiculars CE , IL and draw the tangents as , As .



Put $AD = DB = DS = a$, $BN = aN = NS = b$, and radius $CS = CG = CH = BE = x$, then $AF = a + b$, $BF = a - b$, $CD = a + x$, $DE = a - x$, and $EF = (a - b) - x$. Now $CD^2 - DE^2 = CE^2 = HE^2 = 4ax$, and $CE^2 + FE^2 = FC^2 = (a - b)^2 + 2x(a + b) + x^2$; but $FC = FG - CG = AF - CH = a + b - x$, $\therefore$ equating the square of this expression with the above value of FC^2 , we obtain $x = cs = ab \div (a + b)$. In the very same manner it may be easily proved that the radius is $is = ab \div (a + b)$, and hence the circles GHS , MOs are equal.

Again, by substitution, we have $BH = \frac{2a\sqrt{b}}{\sqrt{(a+b)}}$, and $BO = \frac{2b\sqrt{a}}{\sqrt{(a+b)}}$; hence $HO = \frac{2a\sqrt{b} - 2b\sqrt{a}}{\sqrt{(a+b)}}$. Now, if through

the centres C and I , the right line PR be drawn to meet the circumferences at P and R , it is evident that PR will be the diameter of a circle circumscribing the two circles GHS , MOs ;

$$\text{but } HO^2 + (2CH)^2 = CI^2 = 4ab - \frac{2ab\sqrt{ab}}{a+b} + \frac{4a^2b^2}{(a+b)^2},$$

and consequently $CI = 2\sqrt{ab} - \frac{2ab}{a+b}$; hence, adding the

radii CP , IR , we have $PR = 2\sqrt{ab}$, and a circle whose diameter is $2\sqrt{ab}$ = the space $ASBSaQA = BQ^2$; for the square of its diameter $= 2(a+b)^2 - 2a^2 - 2b^2 = 4ab = PR^2$.

The lines within the above diagram possess so many remarkably neat properties that it will, perhaps, not be deemed intrusive to notice some of them.

Cor. 1. Since $AB \cdot Ba = BQ^2 = 4ab$, $BQ = PK$.

$$\text{Cor. 2. } BH^2 + BO^2 = \frac{4a^2b + 4ab^2}{a+b} = 4ab = BQ^2 = PR^2.$$

$$\text{Cor. 3. } EF = \frac{ab}{a+b} - (a-b), \text{ and } CF = a+b - \frac{ab}{a+b},$$

hence by addition, $EF + CF = 2b = Ba$.

Similarly it is shewn that $FL + FI = 2a = AB$.

$$\text{Cor. 4. } \cos \angle CDE = \sin \angle sAD = DE \div CD = \frac{a}{a+2b},$$

and $\sin \angle sAD : DS :: \text{rad.} : a+2b = DB + Ba = Da$, $\therefore$ a tangent drawn through the point of contact s passes through a . In like manner a tangent from the point of contact s passes through A .

Cor. 5. $sa = 2\sqrt{b(a+b)}$, and $sA = 2\sqrt{a(a+b)}$, $\therefore$ by similar triangles $sa : sD :: Ba : Bn = a\sqrt{\left(\frac{b}{a+b}\right)}$, hence $Bn = \frac{1}{2}BH$, and for the same reason $Bm = \frac{1}{2}BO$.

These two last Corollaries furnish an easy method of describing the circles in the required spaces. Thus draw the tangent as and on BQ take $nH = Bn$, then a perpendicular to BQ from the point H will meet DS in C , the centre of the circles GHS .

$$\text{Cor. 6. } DE = a - \frac{ab}{a+b} = \frac{a^2}{a+b}, \text{ hence } DE \cdot DN = BD^2,$$

and consequently $LN \cdot DN = BN^2$; by addition $DN (DE + LN)$

$= BD^2 + BN^2$. Also it is obvious, $2CH \cdot AB = BH^2$, and $OL \cdot BA = BO^2$. Moreover $BH \cdot BO = 2CH \cdot RP$, or $Bn \cdot Bm = \frac{1}{2}CH \cdot RP$.

Cor. 7. $CD : IN :: a + \frac{ab}{a+b} : b + \frac{ab}{a+b} :: a(a + 2b) : b(b + 2a) :: AD \cdot Da : aN \cdot NA$. Also $CD \cdot IN = PR^2 + CH^2$.

Cor. 8 $\sin \angle$ which CI makes with OH $= \frac{\sqrt{ab}}{a+b-\sqrt{ab}}$.
 $\sin \angle aNB = \frac{2\sqrt{a+b}}{a+2b}$, $\sin \angle aMB = \frac{2\sqrt{a(a+b)}}{2a+b}$, $\sin \frac{1}{2} \angle CPI = \frac{(a+b)\sqrt{ab}-2ab}{a^2+ab+b^2}$, and $\frac{\sqrt{a+b} \times (\sqrt{a^3} + \sqrt{b^3})}{a^2 + 2\frac{1}{2}ab + b^2}$
 $= \sin \angle$ formed by DC, NI.

Solutions were also received from Messrs. GODWARD, LAWS, and THOMPSON.

II. QUESTION 532, by R. N.

If two sums of money s_1 and s_2 become due respectively at the end of T_1 and T_2 years; the equated time for paying both the sums is found by some authors, on the supposition that the amount of the present values of s_1 and s_2 at the expiration of the equated time must $= s_1 + s_2$, when simple interest is allowed. The rate of interest being (r) : It is required to explain the cause of the difference of the results obtained thus and by supposing the interest of s_1 , for the time it is held after its becoming due, equal to the discount of s_2 for payment before it becomes due.

SOLUTION, by Mr. JOHN BAINES.

Put r = the interest of £1 for one year, and x = the equated time between the two payments, then according to the first

hypothesis, $s_1 \cdot \frac{1+rx}{1+rT_1} + s_2 \cdot \frac{1+rx}{1+rT_2} = s_1 + s_2$, which

by a very slight reduction becomes $\frac{rs_1(x-T_1)}{1+rT_1} = \frac{rs_2(T_2-x)}{1+rT_2}$,

From this result we easily perceive that this assumption is

equivalent to that of computing the interest upon $\frac{s_1}{1+rT_1}$ in-

stead of upon s_1 for $(x - T_1)$ years, or upon a sum less than what it ought to be by $\frac{rs_1 T_1}{1 + rT_1}$; also the latter side of this

equation is evidently the interest, for $(T_2 - x)$ years, upon the present worth of s_2 due T_2 years hence, instead of the discount of s_2 for $(T_2 - x)$ years, or, which is equivalent, the interest for $(T_2 - x)$ years upon the present worth of s_2 due $(T_2 - x)$ years hence. It is therefore, sufficiently obvious, that the first hypothesis differs from the second in two particulars, viz. in the first supposition, the first payment upon which the interest is

computed is too little by $\frac{rs_1 T_1}{1 + rT_1}$, and the present worth of the second is considered as due x years before its proper time. Both these errors are in defect, but they are neither equal nor similar, and hence the two suppositions give different results. That both the suppositions give the same results, when compound interest is used, may be easily proved: for, according to

to the first $s_1(1+r)^{\frac{x}{T_1}} + s_2(1+r)^{\frac{x}{T_2}} = s_1 + s_2$, and according to the second, $s_1\{1 + r\}^{x-T_1} - 1 = s_2\{1 - (1+r)^{x-T_1}\}$ or $s_1(1+r)^{x-T_1} + s_2(1+r)^{x-T_2} = s_1 + s_2$. Multiplying and dividing the first side of this equation by $(1+r)^{T_1+x}$, we

$$\text{have, after a little reduction, } \frac{s_1(1+r)^{T_1+x} + s_2(1+r)^{T_2+x}}{(1+r)^{T_1+T_2}} \\ = \frac{s_1(1+r)^x}{(1+r)^{T_1}} + \frac{s_2(1+r)^x}{(1+r)^{T_2}}, = s_1(1+r)^{\frac{x}{T_1}} + s_2(1+r)^{\frac{x}{T_2}} =$$

$s_1 + s_2$, the same as the first.

These anomalous results are peculiarly remarkable, and seem to argue that the principle of simple interest, with regard to this subject, is not equitable, for both the suppositions seem to be founded on justice. Malcolm's rule which is the same as the second supposition, is without doubt as correct as can be when simple interest is used; indeed before the appearance of this question I always considered it strictly true.

*Messrs. DAVIES, GODWARD, LAWS, and THOMPSON
answered this Question.*

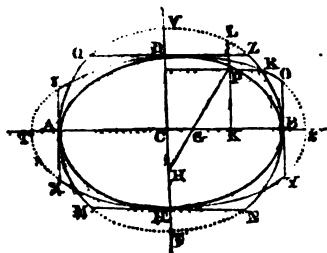
III. QUESTION 633, by Mr. THOMAS STEPHENS DAVIES, Bath.

If the rectangular ordinates of a conic section be prolonged till they be equal to the normals of the same points, what is the equation of the curve traced out by their intersection?

SOLUTION, by Mr. T. S. DAVIES, Bath.

In the ellipse and hyperbola, put a, b for the semiaxes CB, CD,

and let CK, KP the current co-ordinates of P a point in the given conic section be denoted by x', y' respectively. Then the normals intercepted by the two axes are



$$CP^2 = KL^2 = y'^2 = \frac{a^4 y'^2 + b^4 x'^2}{a^4} \dots (1)$$

$$HP^2 = NM^2 = x'^2 = \frac{a^4 y'^2 + b^4 x'^2}{b^4} \dots (2)$$

But by the equation of the given ellipse and hyperbola we have,

$$y'^2 = \frac{b^2 (a^2 \mp x'^2)}{a^2} \text{ and } x'^2 = \frac{a^2 (b^2 \mp y'^2)}{b^2};$$

these inserted in (1) and (2) respectively give, after dropping all the accents,

$$a^4 y^2 \pm b^2 (a^2 \mp b^2) x^2 = a^4 b^2 \dots (3)$$

$$\pm a^2 (a^2 \mp b^2) y^2 + b^4 x^2 = a^2 b^4 \dots (4)$$

which are the equations of the loci of L and M respectively.

These are conic sections, (ellipses or hyperbolas) concentric with the given ones, and having coincident axes. The axes of

(3) are b and $\frac{a}{\sqrt{a^2 - b^2}}$; and those of (4) are a and $\frac{b}{\sqrt{a^2 - b^2}}$:

and the resulting loci are TSDE, AVBU. The points L and M do not, however, trace out the whole curve denoted by (3), (4),

they being limited by the value $x = \pm a$ in (3) and $y = \pm b$ in (4). [The dotted parts of the two figures are those which are superfluous in fulfilling the conditions of the proposed problem.] *This, then, is a case of a more general problem:— what is that problem?*

In case of the parabola, let $4a = \text{latus rectum}$: then $4a^2 + y'^2 = \text{normal}^2$; and as $y'^2 = av' \therefore \text{normal}^2 = y'^2 = 4a^2 + ax$, dropping accents, which is also a parabola, whose axes coincide with the given one, and whose vertex has $x = -4a$. We here, however, only have part of the curve applicable to the question, as in the ellipse and hyperbola.

Schol. D'Alembert has assigned the portions of these curves which are actually generated by the law of the problem, as the representation of the areas of the surfaces of revolution which are generated by the given conic sections about their major and minor axes respectively. *Encycl. Méth., Art. Conoïde.* Here follows the passage:—

“ Nous donnerons à cette occasion une méthode particulière pour mesurer la surface d'un *conoïde*; cette méthode est assez simple; nous croyons nouvelle, et elle peut être utile en quelques cas.

“ D'un point quelconque de la courbe qui engendre le *conoïde*, soit menée une ordonnée perpendiculaire à l'axe de rotation, et une perpendiculaire à la courbe qui aboutisse à l'axe: soit prolongée l'ordonnée hors de la courbe, jusqu'à ce que le prolongement soit égal à l'excès de la perpendiculaire sur l'ordonnée; et imaginant que, l'on fasse la même chose à chaque point de la courbe, soit supposée une nouvelle courbe qui passe par les extrémités des ordonnées ainsi prolongées: je dis que la surface courbe du *conoïde* sera à l'aire de cette nouvelle courbe, comme la circonférence du cercle est au rayon. Cette proposition est fondée sur ces deux-ci: 1st. L'élément de la surface du *conoïde* est le produit du petit côté de la courbe par la circonférence du cercle dont l'ordonnée est la rayon: 2d. La perpendiculaire est à l'ordonnée, comme l'élément de la courbe est à l'élément de l'abscisse; deux propositions dont la démonstration est très-facile.

“ Par le moyen de cette proposition, on peut trouver aisément la surface courbe du *conoïde* qu'une section conique quelconque engendre en tournant autour de son axe. Car on trouvera que la courbe formée par les ordonnées prolongées est toujours une section conique; et par conséquent la mesure de la surface courbe se réduira à la quadrature de quelque section conique, c'est-à-dire à la quadrature de la

parabole, qui est connue depuis long-temps, ou à la quadrature du cercle, d'Ellipse ou à celle de l'hyperbole."

This question was also answered by Messrs. BAINES, GODWARD, LAWS, and THOMPSON.

IV. QUESTION 534, by Mr. W. S. B. WOOLHOUSE.

Four given equal spheres being placed in close contact with each other, it is required to find the volume of the space enclosed between them and the four triangular planes through each three centres.

FIRST SOLUTION, by Mr. WM. GODWARD, London.

The volume (V) of the space enclosed between the spheres and planes is equal to the volume (A) of a tetrahedron, whose side is equal to the diameter (d) of one of the equal spheres minus four times the volume (B) of a spherical pyramid, each side of whose base contains 60° .

In an equilateral spherical triangle whose side is 60° , we have each angle $= 70^\circ \cdot 52877$, and the spherical excess $= 70^\circ \cdot 52877 \times 3 - 180^\circ = 31^\circ \cdot 58633$. Hence the area of the spherical triangle $= \frac{\pi d^2 \times 31^\circ \cdot 58633}{720^\circ} = \cdot 04387 \pi d^2$, which

multiplied by $\frac{1}{6} d$, gives (B) $= \cdot 02297 d^3$. Also $\cdot 02297 d^3$ (A) $= \frac{1}{12} d^3 \sqrt{2} = \cdot 11785 d^3$, and (V) $= (A) - 4(B) = \cdot 02597 d^3$, as required.

SECOND SOLUTION, by Mr. WOOLHOUSE, London.

The four planes through the centres of the spheres form a regular tetrahedron whose edges each $= 2r$ and vo-

lume $= (2r)^3 \times \frac{1}{12} \sqrt{2} = \frac{2}{3} r^3 \sqrt{2}$. By taking away from

this the four equal spheric pyramids cut off by the four spheric surfaces, the remainder is the required volume. Now the sides of the equilateral spheric triangular bases being

each $= \frac{1}{3} \pi$, we have by the formula, Bonnycastle's Trig.

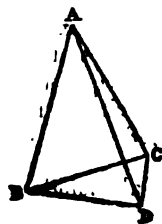
p. 366, each angle $= \cos^{-1} \frac{2}{3}$. Therefore the spherical excess $\varepsilon = 3 \cos^{-1} \frac{2}{3} - \pi$; the surface of each triangle $= r^2 \varepsilon$; the value of each pyramid $= \frac{1}{3} r^3 \varepsilon$; and hence the required volume $= \frac{2}{3} r^3 \sqrt{2} - \frac{4}{3} r^3 \varepsilon = \frac{r^3}{3} (2\sqrt{2} - 12 \cos^{-1} \frac{2}{3} + 4\pi)$
 $= r^3 \times .2076 = \text{sphere} \times .0496 = \frac{\text{sphere}}{20.14}.$

THIRD SOLUTION, by Mr. T. S. DAVIES, Bath.

Generally, when any four spheres, radii r_1, r_2, r_3, r_4 , are taken.

Let A, B, C, D, be the centres of the four spheres, whose radii are respectively r_1, r_2, r_3, r_4 . Then the edges of the tetrahedron are respectively

$$\left. \begin{aligned} AB &= r_1 + r_2 \\ AC &= r_1 + r_3 \\ AD &= r_1 + r_4 \\ BC &= r_2 + r_3 \\ BD &= r_2 + r_4 \\ CD &= r_3 + r_4 \end{aligned} \right\} \dots (1)$$



Denote the angle CAD, DAB, BAC by $\alpha_2, \alpha_3, \alpha_4$ respectively. Then

$$\cos BAC = \frac{(r_1 + r_2)^2 + (r_1 + r_3)^2 - (r_2 + r_3)^2}{2(r_1 + r_2)(r_1 + r_3)}, \text{ or}$$

* My object in proposing this question was to form a plan of ascertaining the space which is unoccupied in the interstices between any number of equal spheres, when packed closely together. When the spheres are numerous, there will evidently be about as many of these spaces as spheres, and hence we may observe that nearly

$\frac{1}{20}$ th of the whole space is unoccupied. Thus, for example, a bag of

shot will be found to weigh nearly $\frac{1}{20}$ th less than the same volume

of solid lead; and if a vessel contains a number of equal spheres, it will also hold in addition a quantity of water, or other fluid, which

is nearly $\frac{1}{20}$ th of the joint volume of the spheres, or $\frac{1}{21}$ part of the

capacity of the vessel.

$$\left. \begin{aligned} \cos A_4 &= \frac{r_1^2 + r_1 r_2 + r_1 r_3 - r_2 r_3}{r_1^2 + r_1 r_2 + r_1 r_3 + r_2 r_3} \\ \text{Similarly, } \cos A_3 &= \frac{r_1^2 + r_1 r_2 + r_1 r_4 - r_2 r_4}{r_1^2 + r_1 r_2 + r_1 r_4 + r_2 r_4} \\ \text{and } \cos A_2 &= \frac{r_1^2 + r_1 r_3 + r_1 r_4 - r_2 r_4}{r_1^2 + r_1 r_3 + r_1 r_4 + r_2 r_4} \end{aligned} \right\} \dots (2)$$

But each solid angle of the tetrahedron cuts from its concentric sphere, a spherical triangle, which is the base of the spherical pyramid which is common to that sphere and the tetrahedron. The sides of the triangle are the measures of the angles A_4, A_3, A_2 respectively. But as these sides are only given by means of their cosines, to find the spherical excess, we must have recourse to a theorem first given by De Gua in the *Mém. de l'Acad. des Scien.* 1783, and which so far as I know has not found its way into any other work, except *Young's Trigonometry*. It is this:—

$$\text{ver. sin E} = \frac{1 - \cos^2 A_2 - \cos^2 A_3 - \cos^2 A_4 + 2 \cos A_2 \cos A_3 \cos A_4}{(1 + \cos A_2)(1 + \cos A_3)(1 + \cos A_4)} \dots (3)$$

Leaving now this part of the enquiry for a moment, we will attend to the *volume of the tetrahedron*. The edges which meet in A are $r_1 + r_2, r_1 + r_3, r_1 + r_4$; and the volume of the solid is (Brewster's Legendre, p. 251.)

$$P = \frac{(r_1 + r_2)(r_1 + r_3)(r_1 + r_4)}{6} \times$$

$$\{1 - \cos^2 A_2 - \cos^2 A_3 - \cos^2 A_4 + 2 \cos A_2 \cos A_3 \cos A_4\}^{\frac{1}{2}} \dots (4)$$

Or putting for $\cos A_2, \cos A_3, \cos A_4$ their values as found in (2), this becomes after due reduction

$$P = \frac{r_1 r_2 r_3 r_4}{8} \times$$

$$\left\{ \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right)^2 - 2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \right) \right\}^{\frac{1}{2}} \dots (5)$$

This may be put into other forms, more elegant at least, if not more brief; as, for example,

$$P = \frac{r_1 r_2 r_3 r_4}{8} \times$$

$$\left\{ \begin{aligned} & -\frac{1}{r_1} \left(-\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) + \frac{1}{r_2} \left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right) \\ & + \frac{1}{r_3} \left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r_3} + \frac{1}{r_4} \right) + \frac{1}{r_4} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4} \right) \end{aligned} \right\} \dots (6)$$

X 2

Or, again, if we put $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{s}$, then (6) will be changed into the following:—

$$P = \frac{r_1 r_2 r_3 r_4}{8} \times \left\{ \frac{2}{r_1} \left(\frac{1}{s} - \frac{1}{r_1} \right) + \frac{2}{r_2} \left(\frac{1}{s} - \frac{1}{r_2} \right) + \frac{2}{r_3} \left(\frac{1}{s} - \frac{1}{r_3} \right) + \frac{2}{r_4} \left(\frac{1}{s} - \frac{1}{r_4} \right) \right\}^{\frac{1}{2}} \dots (7)$$

To return to the excess:—its numerator is found by (4) to be

$$= \frac{36 P^2}{(r_1 + r_2)^2 (r_1 + r_3)^2 (r_1 + r_4)^2}$$

which inserted in (3) and the values of the cosine of that denominator furnished by (2) also inserted in it, we obtain after some reduction

$$\text{vers } E = \frac{36 P^2}{8 r_1^3 (r_1 + r_2 + r_3) (r_1 + r_3 + r_4) (r_1 + r_2 + r_4)} \dots (8)$$

or putting for P its value from (7), and performing the like operation with respect to the other E s, we have the four following expressions for the surfaces of the respective spheres enclosed within the faces of the tetrahedron :

$$\left. \begin{aligned} \text{vers } E_1 &= \frac{r_1^2 r_2^2 r_3^2 r_4^2}{r_1^3} \times \frac{\left\{ \frac{1}{r_1} \left(\frac{1}{s} - \frac{1}{r_1} \right) + \frac{1}{r_2} \left(\frac{1}{s} - \frac{1}{r_2} \right) + \left(\frac{1}{s} - \frac{1}{r_3} \right) + \left(\frac{1}{s} - \frac{1}{r_4} \right) \right\}^{\frac{1}{2}}}{(r_1 + r_2 + r_3) (r_1 + r_3 + r_4) (r_1 + r_2 + r_4)} \\ \text{vers } E_2 &= \frac{r_1^2 r_2^2 r_3^2 r_4^2}{r_2^3} \times \frac{\left\{ \frac{1}{r_1} \left(\frac{1}{s} - \frac{1}{r_1} \right) + \frac{1}{r_2} \left(\frac{1}{s} - \frac{1}{r_2} \right) + \frac{1}{r_3} \left(\frac{1}{s} - \frac{1}{r_3} \right) + \frac{1}{r_4} \left(\frac{1}{s} - \frac{1}{r_4} \right) \right\}^{\frac{1}{2}}}{(r_2 + r_3 + r_4) (r_2 + r_1 + r_3) (r_2 + r_1 + r_4)} \\ \text{vers } E_3 &= \frac{r_1^2 r_2^2 r_3^2 r_4^2}{r_3^3} \times \frac{\left\{ \frac{1}{r_1} \left(\frac{1}{s} - \frac{1}{r_1} \right) + \frac{1}{r_2} \left(\frac{1}{s} - \frac{1}{r_2} \right) + \frac{1}{r_3} \left(\frac{1}{s} - \frac{1}{r_3} \right) + \frac{1}{r_4} \left(\frac{1}{s} - \frac{1}{r_4} \right) \right\}^{\frac{1}{2}}}{(r_3 + r_1 + r_2) (r_3 + r_1 + r_4) (r_3 + r_2 + r_4)} \\ \text{vers } E_4 &= \frac{r_1^2 r_2^2 r_3^2 r_4^2}{r_4^3} \times \frac{\left\{ \frac{1}{r_1} \left(\frac{1}{s} - \frac{1}{r_1} \right) + \frac{1}{r_2} \left(\frac{1}{s} - \frac{1}{r_2} \right) + \frac{1}{r_3} \left(\frac{1}{s} - \frac{1}{r_3} \right) + \frac{1}{r_4} \left(\frac{1}{s} - \frac{1}{r_4} \right) \right\}^{\frac{1}{2}}}{(r_4 + r_3 + r_2) (r_4 + r_2 + r_1) (r_4 + r_3 + r_1)} \end{aligned} \right\} \dots (9)$$

Hence $\frac{1}{3} (r_1^3 E_1 + r_2^3 E_2 + r_3^3 E_3 + r_4^3 E_4) =$ space enclosed between the spheres and tetrahedron;

and $P = \frac{1}{3} (r_1^3 E_1 + r_2^3 E_2 + r_3^3 E_3 + r_4^3 E_4) =$ space between the spherical surfaces.

These several expressions cannot fail to interest the geometer by their symmetry; and by taking the four values equal we shall obtain from them a solution to the proposed problem.

In that case vers $E = \frac{4}{27}$, or $\cos E = \frac{23}{27}$, and the enclosed space is found to be = sphere $\times .0496$.

A Solution was also given by Mr. JOHN BAINES.

V. QUESTION 535, by Dr. BROOK TAYLOR.

Let ov , da be parallels cut by a transversal in v , v respectively. From a point o in one of them, (ov), draw four lines meeting the transversal in D , C , B , A and the other parallel in d , c , b , a . Then

$$\frac{CD}{cd} : \frac{AB}{ab} :: CV \cdot VD : AV \cdot VB.$$

FIRST SOLUTION, by Mr. THOMAS THOMPSON, Newcastle upon Tyne.

Draw cn parallel to ov meeting OD in N , then by similar triangles

$$CD : CN :: VD : VO$$

$$CN : cd :: (CO : co) :: CV : vb;$$

Therefore $CD : cd :: VD \cdot CV : VO \cdot vb$,

$$\text{and } \frac{CD}{cd} = \frac{CV \cdot VD}{VO \cdot vb}.$$

In like manner if ap be drawn parallel to ov meeting OB in P , we have

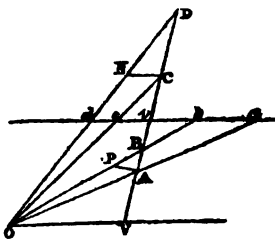
$$AB : AP :: VB : VO,$$

$$AP : ab :: (OA : oa) :: AV : vb;$$

Therefore $AB : ab :: AV \cdot VB :: VO \cdot vb$,

$$\text{and } \frac{AB}{ab} = \frac{AV \cdot VB}{VO \cdot vb}.$$

$$\text{Therefore } \frac{CD}{cd} : \frac{AB}{ab} :: CV \cdot VD : AV \cdot VB.$$



the study of them. It is denominated by them, very appropriately, "the Geometry of the Rule."

Messrs. BAINES, and GODWARD, answered this question.

VI. QUESTION 596, by Mr. GRIFFITH JONES,
Liverpool.

A pendulum composed of three cylindrical rods of unequal lengths SA, SB, SC, lying in the same plane ($\angle ASB = \angle BSC$) is supposed to vibrate about the point S in the plane of the figure. Required the distances of the centres of gravity and oscillation from the point of suspension?

SOLUTION, by Mr. WM. GODWARD, London.

Let $SA = a$, $SB = b$, $SC = c$, $\angle ASB = \angle BSC = \theta$, and the distances of the centres of gravity and oscillation from the point of suspension $= d$ and D respectively. Also let ϕ be the angle which SA makes with the horizon; then $\pi - \theta - \phi$ and $\pi - 2\theta - \phi$ are the angles which SB and SC make with it. Now, by a property of the lever,

$$a^2 \cos \phi + b^2 \cos (\theta + \phi) + c^2 \cos (2\theta + \phi) = 0;$$

and by a property of the centre of gravity,

$$a^2 \sin \phi + b^2 \sin (\theta + \phi) + c^2 \sin (2\theta + \phi) = 2d(a + b + c).$$

Eliminating ϕ from these two equations which may be effected by taking the sum of their squares, and we have

$$a^4 + b^4 + c^4 + 2b^2(a^2 + c^2) \cos \theta + 2a^2 c^2 \cos 2\theta = 4d^2(a + b + c)^2,$$

$$\text{Hence } d = \frac{\sqrt{\{a^4 + b^4 + c^4 + 2b^2(a^2 + c^2) \cos \theta + 2a^2 c^2 \cos 2\theta\}}}{2(a + b + c)},$$

$$\begin{aligned} \text{and } D &= \frac{a \cdot \frac{1}{2}a + b \cdot \frac{1}{2}b + c \cdot \frac{1}{2}c}{d(a + b + c)} = \frac{a^3 + b^3 + c^3}{2d(a + b + c)} \\ &= \frac{2}{3} \cdot \frac{a^3 + b^3 + c^3}{\sqrt{\{a^4 + b^4 + c^4 + 2b^2(a^2 + c^2) \cos \theta + 2a^2 c^2 \cos 2\theta\}}} \text{ the} \\ &\text{distances required.} \end{aligned}$$

Messrs. BAINES, DAVIES, and LAWS answered this question.

VII. QUESTION 537, by Mr. SAMUEL JONES,
Liverpool.

Sum the series,

$$\frac{1 \cdot 7}{2^3 \cdot 4^2 \cdot 6} + \frac{3 \cdot 11}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5 \cdot 15}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \frac{3^2 \cdot 5^2 \cdot 7 \cdot 19}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdot 12} + \&c.$$

ad infinitum in finite terms.

SOLUTION, by Mr. WILLIAM GODWARD.

By expansion

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{5}{2}x^2 + \frac{5 \cdot 3}{2 \cdot 4}x^4 - \\ 3 \cdot 5 \left(\frac{x^6}{2 \cdot 4 \cdot 6} + \frac{x^8}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{3x^{10}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} + \&c. \right) \\ \therefore \frac{1}{15} - \frac{x^2}{6} + \frac{x^4}{8} - \frac{1}{15}(1-x^2)^{\frac{1}{2}} = \\ \frac{x^6}{2 \cdot 4 \cdot 6} + \frac{x^8}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{3x^{10}}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} + \&c. \dots (1)$$

Multiply this by $\frac{dx}{x^2(1-x^2)^{\frac{1}{2}}}$ and take the integrals between

$x = 0$ and $x = 1$, and

$$-\frac{5}{48} \cdot \frac{\pi}{2} + \frac{8}{45} = \frac{3}{2^3 \cdot 4^2 \cdot 6} \cdot \frac{\pi}{2} + \frac{3 \cdot 5}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8} \cdot \frac{\pi}{2} + \\ \frac{3^2 \cdot 5 \cdot 7}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} \cdot \frac{\pi}{2} + \frac{3^2 \cdot 5^2 \cdot 7 \cdot 9}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdot 12} \cdot \frac{\pi}{2} + \&c. \dots (2)$$

Multiply (1) by $\frac{dx}{x^4(1-x^2)^{\frac{1}{2}}}$, and integrate as before, then

$$\frac{1}{8} \cdot \frac{\pi}{2} - \frac{8}{45} = \frac{1}{2^2 \cdot 4 \cdot 6} \cdot \frac{\pi}{2} + \frac{3}{2^2 \cdot 4^2 \cdot 6 \cdot 8} \cdot \frac{\pi}{2} + \\ \frac{3^2 \cdot 5}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8 \cdot 10} \cdot \frac{\pi}{2} + \frac{3^2 \cdot 5^2 \cdot 7}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10 \cdot 12} \cdot \frac{\pi}{2} + \&c. \\ = \frac{4}{2^2 \cdot 4^2 \cdot 6} \cdot \frac{\pi}{2} + \frac{3 \cdot 6}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} \cdot \frac{\pi}{2} + \\ \frac{3^2 \cdot 5 \cdot 8}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} \cdot \frac{\pi}{2} + \frac{3^2 \cdot 5^2 \cdot 7 \cdot 10}{2^3 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdot 12} \cdot \frac{\pi}{2} + \&c. \dots (3)$$

Add (2) and (3), and divide by $\frac{\pi}{2}$, then

$$\frac{1}{48} = \frac{1 \cdot 7}{2^2 \cdot 4^2 \cdot 6} + \frac{3 \cdot 11}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5 \cdot 15}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \frac{3^2 \cdot 5^2 \cdot 7 \cdot 19}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdot 12} + \&c.$$

Cor. Subtract (2) from (3) and divide by $\frac{\pi}{2}$ and

$$\frac{11}{48} - \frac{16}{45} \cdot \frac{2}{\pi} = \frac{1}{2^2 \cdot 4^2 \cdot 6} + \frac{3}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \frac{3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10} + \frac{3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdot 12} + \&c.$$

This question was answered by Messrs. BAINES, DAVIES and THOMPSON.

VIII. QUESTION 538, by M.

Let AB be the diameter of a circle, AM always equal to $\frac{1}{2}$ BM, MQ, M'Q' perpendicular to AB; let AQ and BQ' (produced if necessary) meet in P: find the equation to the curve traced out by P. Trace its whole extent and shew how all its parts are connected with the circle.

FIRST SOLUTION, by — MASON, of Scoulton, near Hingham, Norfolk.

Let AN = x, NP = y, AB = 2a, AM = $\frac{1}{2}$ BM' = z $\therefore \frac{y^2}{x^2} = \frac{2a}{z} - 1$ and $\frac{y^2}{(2a-x)^2} = \frac{2a-2z}{2z} = \frac{a}{z} - 1$. Hence $y = \pm \frac{x(2a-x)}{\sqrt{4a^2-4ax-x^2}}$. Now when $x = 0, y = 0 \therefore A$ is a point

in the curve, $x = 2a, y = 0$. $\therefore$ B is a point connected with the curve. $x = \pm 2a \sqrt{2} - 2a, y = \text{infinity}$: take $\therefore AD = 2a(\sqrt{2} - 1)$ and draw ZDZ perpendicular to AB , and this line is an asymptote to the two branches AZ and AZ' which are similar and equal.

If x be greater than $2a(\sqrt{2} - 1)$ or greater than AD , y is impossible, except when $x = 2a$. Hence there is no part of the curve extending beyond D in the direction AD ; the point B, at which $y = 0$ for $x = 2a$, is a single point only, and is called a conjugate point.

If x be negative and less than $2a(\sqrt{2} + 1)$ there are two equal values of y positive and negative and $\therefore$ there are two other branches of the curve extending from A towards $z'' D' z'''$ drawn through D' perpendicular to BA , AD' being $= 2a\sqrt{2} + 2a$, $z'' D' z'''$ being an asymptote to these branches.

Further, since $d^2_x y = \frac{24a^2 y}{(4a^2 - 4ax - x^2)^2}$ it is manifest that y and $d^2_x y$ have always the same sign, hence the curve is convex to the axis of x through its whole extent, and consequently at A there is a point of contrary flexure in each branch, the point A is also a double point, and the branches cut the axis of x at A at angles $= 45^\circ$, and each other at right angles.

SECOND SOLUTION, by Mr. DAVIES, Bath.

Let AB the diameter be taken as axis of x , and A as origin, and $AY \perp AB$ be taken as axis of y . Let $AB = 2a$ and the co-ordinates of Q, Q' be x, y , and x'', y'' . Then the equation of the circle is

$$y^2 = 2ax - x^2 \dots (1)$$

$$\text{Hence } y'^2 = 2ax' - x'^2 \dots (2)$$

$$\text{and } y''^2 = 2ax'' - x''^2 \dots (3)$$

But by the question $BM' = 2AM = 2x$, and hence

$$x'' = 2(a - x) \dots (4)$$

And the equations of AQ and BQ' become respectively,

$$xy = x'y \dots (5)$$

$$y''(2a - x) = y(2a - x'') \dots (6)$$

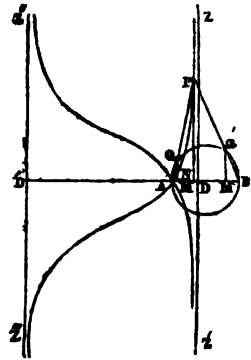
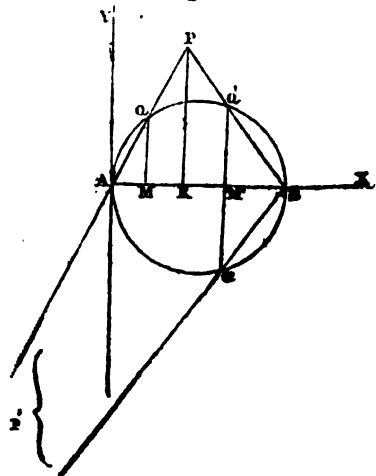


Fig. 1.



Insert (4) in (3) and (6); then

$$y_{//}^2 = 4x(a-x) \dots (7)$$

$$y_{//} = \frac{2x, y}{2a-x} \dots (8)$$

Equating (7) and (8) and reducing, we obtain

$$x = \frac{(2a-x^2)x}{y^2 + (2a-x)^2} \dots (9)$$

Comparing (2) and (5) we have

$$x = \frac{2ax^2}{x^2 + y^2} \dots (10)$$

Equating (9) and (10) we find

$$(2a-x)^2(y^2-x^2) = 2x^2y^2 \dots (11)$$

or resolving for y we have ultimately,

$$y = \pm \frac{x(2a-x)}{\sqrt{4a^2-4ax-x^2}} \dots (12)$$

which is the equation of the curve sought.

We may now proceed to examine its course:—

1. In the first place, whatever be possible values of x , we find the curve has two corresponding values of y , above and below the axis AB ; or in other words the curve is composed of branches symmetrical with respect to the axis of x .

2. That a point may belong to this curve, we must have the denominator $\sqrt{4a^2-4ax-x^2}$, real, or which comes to the same thing, the possible values of x must be included between the roots of $4a^2-4ax-x^2=0$: that is they must be included between

$$+2a\{\sqrt{2}-1\} \text{ and } -2a\{\sqrt{2}+1\}.$$

At these points, since the numerator of (12) is finite, we shall have y infinite, or the ordinates now become asymptotes to the branches of the curve.

Let O Fig. 2, be the centre of the circle: make AK , to the left, or negative side of A , equal to $2a(\sqrt{2}+1)$, and AN to the right, equal to $2a(\sqrt{2}-1)$, and draw the perpendiculars LKL' , RNR' . These will be asymptotes to the curve and will wholly enclose it.

3. To find where the curve cuts the axis, put y equal to zero, which gives the three conditions

$$\begin{aligned} \pm \sqrt{4a^2-4ax-x^2} &= \pm \frac{1}{0} \\ 2a-x &= 0 \\ x &= 0 \end{aligned}$$

The first and second of these exceed the limits just laid down as the boundaries of the curve: but the third is consistent, and gives

$$y = 0.$$

whence the curve passes through the origin A .

4. The equation of the tangent is found from (12) to be

$$\frac{dy}{dx} = \pm \frac{8a^2 - 16a^2x + 10ax^2 - 2x^3}{(4a^2 - 4ax - x^2)^{\frac{3}{2}}}$$

which gives for the case $x = 0$, or the inclination of the curve to axis of x at its origin A ,

$$\frac{dy}{dx} = \pm 1,$$

or the intersection of the curve is at an angle of $\frac{\pi}{4}$, or again the branches intersect at right angles.

5. If we seek the values of x and y which give $\frac{dy}{dx} = 0$ and $\frac{dx}{dy} = 0$, we shall find the latter to be when $x = \pm 2a(\sqrt{2} \mp 1)$ or at N and K ; and the former will give

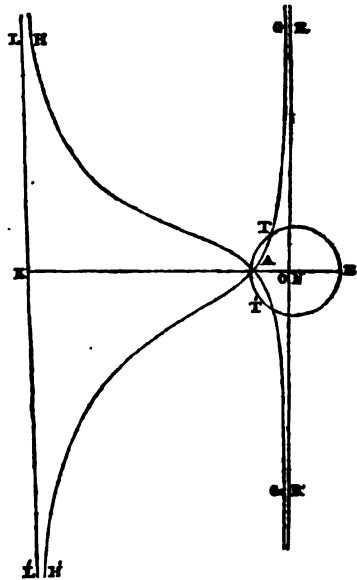
$$x^3 - 10ax^2 + 16a^2x - 4a^3 = 0$$

which must be examined in detail.

Fig. 2.

The roots of this equation are real,* but they all transcend the limits of x assigned in (2), and hence there is no tangent parallel to the axis of x .

6. The two branches lying to the right of A from the path of P when y , and $y_{//}$, are affected by the same sign: viz. AG when we take $+y$, and $+y_{//}$, and AG' when we take $-y$, and $-y_{//}$. The two to the left are the locus of P , when we take y , and $y_{//}$, with opposite signs: viz. AH when we take $+y$, and $-y_{//}$, and AH' when we take $-y$, and $+y_{//}$. By fixing upon $+y_{//}$, then we trace the



* The three roots are 1.6019a, - .2684a, and - 8.0568a.

continuous branches QA, AH' by taking $+y$, and $-y$, for that value of y , and we trace the others by fixing upon $-y$, and taking both $+y$, and $-y$. We thus see how all "the parts are connected with the circle."

7. It cuts the circle in two points τ, τ' , when we make

$$2ax - x^2 = \frac{x^2(2a - x)}{4a^2 - 4ax - x^2}, \text{ or } x = \frac{2a}{3}$$

as by the genesis it ought to do. Then also, $y = \pm \frac{2a}{3} \sqrt{2}$

8. For the area: we have $ydx = \pm \frac{(2a - x)xdx}{\sqrt{4a^2 - 4ax - x^2}}$. Or putting the quantity under the radical $= x$, it is

$$ydx = \pm \left\{ \frac{2axdx}{\sqrt{x}} - \frac{x^2dx}{\sqrt{x}} \right\}.$$

These give, by the usual methods, abating the corrections, which are contained under the integral signs which remain in the result,

$$\begin{aligned} 2a \int \frac{x dx}{\sqrt{x}} &= -2a\sqrt{x} - 4a^2 \int \frac{dx}{\sqrt{x}} \\ - \int \frac{x^2 dx}{\sqrt{x}} &= + \left(\frac{x}{2} - 3a \right) \sqrt{x} - 8a^2 \int \frac{dx}{\sqrt{x}} \end{aligned}$$

Collecting these, and integrating $\int \frac{dx}{\sqrt{x}}$, we have

$$\int y dx = \left(\frac{x}{2} - 3a \right) \sqrt{4a^2 - 4ax - x^2} - 12a^2 \sin^{-1} \frac{x + 2a}{2a\sqrt{2}} + C.$$

But when $x = 0$, we have the area $= 0$, and hence

$$\begin{aligned} C &= \mp (10a^2 + 12a^2 \sin^{-1} \frac{1}{\sqrt{2}}) \\ &= \pm a^2 \{10 + 3\pi\} \end{aligned}$$

where $\pm$ refer to the positive and negative values of y .

Taking this between the limits of possibility, we shall have for the portion of the area lying above the axis AX ,

$$\int_{-2a(\sqrt{2}+1)}^{+2a(\sqrt{2}-1)} y dx = 12a^2\pi, \text{ and hence the area enclosed}$$

between the four branches and the asymptotes

$$= 24a^2\pi = 12 \text{ times the generating circle } AQ'BQ''.$$

I have not remarked any other relations to the generating circle than these now mentioned.

This question was answered by Messrs. BAINES, GODWARD, LAWS and THOMPSON.

IX. QUESTION 539, by GALLICUS.

Any conoid being cut by a plane, and the section seen from either focus, it will appear a circle.

FIRST SOLUTION, by Mr. T. S. DAVIES, Bath.

This question is the converse of a particular case of a theorem to which a geometrical demonstration has been given in the Gentleman's Diary, 1830, Prop. 1256; and it would be extremely easy to prove it, *ex absurdo*, by means of that theorem. Nor should we find much difficulty in giving a *direct proof* by means of principles very analogous to those employed on that occasion, nor, indeed, more simply by methods a little different: but as my paper on the "Geometry of Three Dimensions" contains a pretty ample discussion of the properties of this figure, I shall, instead of repeating those processes, merely refer to it, and give a demonstration by the "analytical" method.

Let the major axis of the generating conic section be taken for axis of z , and the system be referred to rectangular co-ordinates originating at the focus from which the section is viewed. We shall, to avoid the use of the double sign prove it for the ellipsoid only, as it is instantly seen that the same method will prove it in the hyperboloid of two sheets.

Let a, b , be the semi-axes of the generatrix; then the ellipsoid itself is expressed by

$$a^2(x^2 + y^2) + b^2(z^2 + az)^2 = a^2b^2 \dots (1)$$

Denote now the equation of the cutting plane, referred to the same axes, by

$$a'x' + b'y' + c'z' = d' \dots (2)$$

Then (1) (2) are the equations of the directrix of the visual cone, whose vertex is at the origin of co-ordinates. Let one of its edges be

$$x' = az' \dots (3)$$

$$y' = \beta z' \dots (4)$$

From (2, 3, 4) we obtain

$$\left. \begin{aligned} z' &= \frac{d'}{a'a + b'\beta + c'} \\ y' &= \frac{\beta d'}{a'a + b'\beta + c'} \\ x' &= \frac{ad'}{a'a + b'\beta + c'} \end{aligned} \right\} \dots (5)$$

Insert the values of $x'y'z'$ from (5) in (1), and reduce: thence we obtain

$$a^2(a^2 + \beta^2)d'^2 + b\{(a'a + b'\beta + c')ac + d'\}^2 = a^2b^2(a'a + b'\beta + c')^2 \dots (6)$$

But from (3) (4) we have

$$a = \frac{x}{z} \text{ and } \beta = \frac{y}{z}; \text{ and this inserted in (6) gives}$$

$$a^2d'^2(x^2 + y^2) + b^2\{acc' + d'\}z + acb'y + a'acx\}^2 = a^2b^2(a'x + b'y + c'z)^2 \dots (7)$$

which further reduces to

$$x^2(a^2d'^2 - a'^2b') + y^2(a^2d'^2 - b'^2b') + z^2(ac + d')^2 - a^2c'^2b'^2\} - 2xyz(b^2b'c' - ab'd'e) - 2xz(b^2a'c' - aa'd'e)b'^2 - 2xy.b^4a'b'\} = 0 \dots (8)$$

Write this for the moment in the form

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'xz + 2C'xy = 0 \dots (9)$$

Put the values of $A, A', \dots$ in the following formulæ, and we get

$$\frac{AA' - B'C'}{A'} = \frac{BB' - C'A'}{B'} = \frac{CC' - A'B'}{C'} \dots (10)$$

(which, Hamilton's Anal. Geom. Art. 402, is a test of the cone being of revolution) and we find them verified.

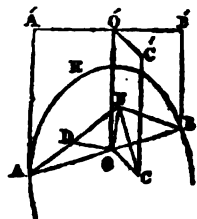
By following a precisely similar course with respect to the paraboloid of revolution, employing the equation

$$x^2 + y^2 = 4a(a + z)$$

we should find the same truth established as we have already found for the ellipse; but it is not necessary to dilate on so simple a subject.

SECOND SOLUTION, by Mr. GODWARD, London.

Let F be the focus of the conoid, and AEB the principal section perpendicular to the cutting plane AB . Join AF, BF , draw FO bisecting the angle AFB and OD perpendicular to OF ; then, as is well known, AF, FD , and BF , are, in harmonical proportion. Again, draw OC perpendicular to the plane AFB meeting the conoid in C , join CF , and draw AA', OO', CC', BB' , perpendicular to the directrix plane; then $AA' : BB' :: (AF : BF :: AO : OB :: AA' - OO' : OO' - BB' ::) AA' - CC' : CC' - BB'$; but, by a property of the conoid, $AA' : CC' : BB' :: AF : CF : BF$ and therefore $AF : BF :: AF - CF : CF - BF$, implying that AF, CF , and BF are in harmonical proportion. Hence, as AF, FD , and BF are also in the same proportion, it manifestly follows that $FD = FC$ and $OC = OD$.



From this we readily infer that the section perpendicular to the axis of the radiant cone is a circle, and consequently that the section when viewed from the focus will appear a circle.

Mr. JOHN BAINES, also answered the question.

X. QUESTION 540, by TINTORETTA.

If spheres any how scattered in space, be put in perspective, the major axes of the representations all tend to one point in the picture, determine that point, both by Geometrical and Analytical considerations.

FIRST SOLUTION, GEOMETRICALLY, by Mr. GODWARD.

If planes be drawn through the axes of the radiant enveloping cones of the spheres, perpendicular to the plane of the picture, their intersection with the elliptic representations will be the major axes; and these axes will therefore tend to the point where the common intersection of the planes passing through the axes of the radiant cones meets the plane of the picture, and which point is obviously the same as the intersection of the perpendicular from the eye with the said plane.

SECOND SOLUTION, ANALYTICALLY, by Mr. T. S. DAVIES.

The geometrical methods of demonstrating this property are sufficiently obvious without entering into specific statements of them : and hence I shall pass at once to the "*analytical*" method.

The visual cone is a cone of revolution, as it is easy to shew analytically. Let the circle of contact be taken for the plane of xy , and the axis of the cone for that of z . Take the centre of this circle of origin of co-ordinates and so dispose of the axes of x and y that xz may be perpendicular to the plane of the picture. Then the perpendicular from the vertex of the cone upon the picture is in xz , and cuts the picture in its common intersection with xz .

Denote by α the angle which the picture makes with the plane of xy ; and put p' for the distance of the circle of contact from the vertex, and r' for the radius of that circle. Place the new origin at the intersection of the axis of the cone with the picture; put r for the radius of its circular section at that point,

and p for its distance from the vertex. Then $\frac{r'}{p'} = \frac{r}{p}$ and the

equation of the section itself (referred to the trace of xz upon the picture, and a perpendicular to it in the picture) is well known to be

$$(p^2 \cos^2 \alpha - r \sin^2 \alpha) y^2 + p^2 x^2 + 2 p r^2 \sin \alpha, y = r^2 p^2 \dots (1).$$

Now this is an equation of the second degree referred to *rectangular* co-ordinates which (with respect to the line in question) are *conjugate* to one another—that is, to such axes as are parallel to the principal diameters. But the absence of x of the first degree in the equation indicates that the centre is in the axis of x , or in other words, that the axis of x coincides with the major axis of the section, or of the delineation of the sphere upon the plane of the picture.

But this is the line in which we have already noticed that the perpendicular from the vertex (or eye) upon the picture falls; and as this, wherever the spheres be placed is a fixed point in the picture it is clear that how many so ever they be in number or however they be situated, the axes majores of the delineations pass through this point. The alledged property is thus established.

However, though this process is sufficient to establish the property it will be necessary, in case of our discussing such delineation in connection with any other upon the same picture, to refer it to other co-ordinates than those we have employed in the present instance. We might, it is true, transform the equation by the common method, but it is highly probable that the compu-

tations to which it would lead us, would be much more tedious than the discussion of the formula derived at once from the general conditions of the problem. I have gone through the latter process, and as the formula is tolerably simple, I shall put down the investigation by way of appendix to the solution above given.

Take the plane of the picture for that of xy , and the origin at any point in that plane; and let the axis of z be perpendicular to the picture. In reference to this system of co-ordinates let $a'b'c'$ be the co-ordinates of the centre of the sphere, r its radius, and abc the co-ordinates of the eye. Then the equation of the sphere is

$$(x - a')^2 + (y - b')^2 + (z - c')^2 = r^2 \dots (2)$$

The equation of the plane of contact of this sphere with the visual cone, whose vertex is abc , will be found by the usual processes (as Monge *Anal. Appl.* p. 14, or Young's *Analyt. Geom.* p. 169,) to be

$$(a - a')(x - a') + (b - b')(y - b') + (c - c')(z - c') = r^2 \dots (3)$$

To abbreviate the writing in the remainder of this investigation, we shall put $a - a' = A$; $b - b' = B$; $c - c' = C$; $r^2 - B^2 - C^2 = R_a^2$; $r^2 - A^2 - C^2 = R_b^2$; and $r^2 - A^2 - B^2 = R_c^2$.

Denote one of the edges of the visual cone by

$$\left. \begin{aligned} x - a &= \alpha (z - c) \\ y - b &= \beta (z - c) \end{aligned} \right\} \dots (4)$$

From (4) we get

$$\left. \begin{aligned} x - a' &= \alpha (z - c) + A \\ y - b' &= \beta (z - c) + B \\ z - c' &= (z - c) + C \end{aligned} \right\} \dots (5)$$

Put these values of $x - a'$ and $y - b'$ in (2), and we shall find

$$\left. \begin{aligned} z - c' &= \frac{R_c^2 + A\alpha + B\beta}{A\alpha + B\beta + C} \\ y - b' &= \frac{R_b^2\beta + AB\alpha + BC}{A\alpha + B + C} \\ x - a' &= \frac{R_a^2\alpha + AB\beta + AC}{A\alpha + B\beta + C} \end{aligned} \right\} \dots (6)$$

But $\alpha = \frac{x - a}{z - c}$, and $\beta = \frac{y - b}{z - c}$; which substituted in (6)

gives

$$\left. \begin{aligned} z - c' &= \frac{R_c^2 (x - c + AC(x - a) + BC(y - b))}{A(x - a) + B(y - b) + C(z - c)} \\ y - b' &= \frac{R_b^2 (y - b) + AB(x - a) + BC(z - c)}{A(x - a) + B(y - b) + C(z - c)} \\ x - a' &= \frac{R_a^2 (x - a) + AB(y - b) + AC(z - c)}{A(x - a) + B(y - b) + C(z - c)} \end{aligned} \right\} \dots (7)$$

Making these substitutions in (2) and reducing, we get,

$$\left. \begin{aligned} &+ R_c^2 (x - c)^2 + 2AB(x - a)(y - b) \\ &+ R_b^2 (y - b)^2 + 2AC(x - a)(z - c) \\ &+ R_a^2 (x - a)^2 - 2BC(y - b)(z - c) \end{aligned} \right\} = 0 \dots (8)$$

which is the equation of the visual cone.

But where the cone of rays cuts the picture, we have $z = 0$, and the trace of the cone upon the plane xy is the delineation whose equation we have been seeking. This we see at once is

$$\left. \begin{aligned} &R_a^2 (x - a)^2 + 2AB(x - a)(y - b) + R_b^2 (y - b)^2 + R_c^2 c^2 \\ &- 2ACC(x - a) - 2BCC(y - b) \end{aligned} \right\} = 0 \dots (9)$$

No further discussion of the subject seems to be necessary here; though it may be remarked that if we determine the equation of the greatest diameter of this curve we shall find it will be fulfilled by the quantities abc . We should thus furnish a demonstration of the question at the head of this, *altogether free from the suspicion of tacitly adopting properties derived from a contemplation of the figure to prove a theorem which we were required to establish by means of its equation*. This, however, leads to calculations of some complexity, and therefore may, perhaps, for the present be dispensed with.

Mr. THOMPSON sent a Solution to this Question.

XI. QUESTION 541, by NEMO.

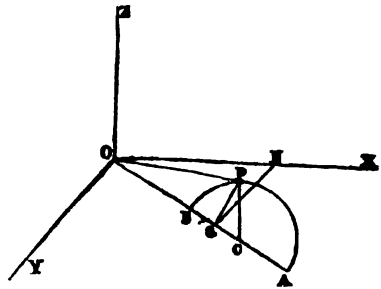
A ring is cut by a plane in a given direction: what is the equation of the section?

FIRST SOLUTION, by Mr. T. S. DAVIES, Bath.

Let a given sphere so move that its centre shall describe a given circle, then the envelope of the space it occupies during that time is the common ring, or torus. Its equation may be found by the usual method of investigating envelopes of three dimensions, but in the present very simple case we may obtain

it by more elementary considerations. The intersection of two consecutive spheres is a great circle, and its plane is elevated at right angles to the plane of the directrix and passing through its centre. This great circle is the generatrix of the torus.

Let $OC = a$ be the radius of the directrix, $CB = CA = r$, that of the generatrix. Refer the system to rectangular co-ordinates having O , the centre of the directrix for origin, and this plane taken as that of xy . Then the axis of z will be at right angles to the plane of the paper: and the plane of the generatrix revolves round Oz .



Let P be a point in the torus, OAP the corresponding position of the plane of the generatrix: then we shall have

$$OP^2 = OC^2 + CP^2 \pm 2OC \cdot CQ$$

where OP , OQ , QP are the co-ordinates xyz , or

$$x^2 + y^2 + z^2 \pm 2a\sqrt{r^2 - z^2} = a^2 + r^2 \dots (1)$$

(See Jephson's Fluxions, vol. I. p. 325.)

Let the equation of the cutting plane be written

$$p = x \cos \alpha + y \cos \beta + z \cos \gamma \dots (2)$$

Then p is the perpendicular from the origin upon the plane, and α, β, γ , are the angles made by this perpendicular and the co-ordinate axes respectively. The point of intersection of this perpendicular and the plane is

$$x' = p \cos \alpha, y' = p \cos \beta, z' = p \cos \gamma \dots (3)$$

Transpose the origin to the point $x'y'z'$, retaining the parallelism of the axes: then we have, inserting (3) in (1),

$$(x + p \cos \alpha)^2 + (y + p \cos \beta)^2 + (z + p \cos \gamma)^2 \pm 2a\sqrt{r^2 - (z + p \cos \gamma)^2} = a^2 + r^2 \dots (4)$$

and the plane (2) referred to the same co-ordinates is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0 \dots (5)$$

The trace (t) of this upon the new plane of xy is denoted by

$$\cos \alpha + y \cos \beta = 0 \dots (6)$$

and the angle (t) therefore, becomes $\tan^{-1} - \frac{\cos \alpha}{\cos \beta} = \kappa \dots (7)$

Also r is the inclination of the cutting plane to that of xy , and these values substituted in Euler's formulæ, (Hamilt. An. Geom. 251, Young's An. Geom. p. 278,) we obtain the equation of the section sought. We thus obtain,

$$\begin{aligned} & (x \cos \gamma + y \sin \kappa \cos \kappa + p \cos \alpha)^2 + \\ & (x \sin \beta - y \cos \kappa \cos \kappa + p \cos \kappa)^2 \\ & + (y \sin \gamma + p \cos \gamma)^2 \pm 2a \sqrt{r^2 - (y \sin \gamma + p \cos \gamma)^2} \\ & = a^2 + r^2 \dots (8) \end{aligned}$$

This equation is in the most general form, and such as would have been required for comparison of the section made in the given ring by the given plane, with any other figure given in reference to the same co-ordinates. If, however, our object had been to ascertain any properties of the section considered alone, we might simplify in some slight degree by means of the arbitrary values of κ . The symmetry of the *Torus* enables us to take its axes in *any direction from* o in the plane of the directrix. Let us, then, take the axis of y parallel to the trace of the cutting plane: then

$$\kappa = 0, \sin \kappa = 0, \cos \kappa = 1$$

$$\alpha = \frac{\pi}{2}, \sin \alpha = 1, \cos \alpha = 0$$

$$\beta = 0, \sin \beta = 0, \cos \beta = 1.$$

Inserting these values in (8) and reducing, we obtain

$$\begin{aligned} & x^2 + y^2 - 2py \cos \gamma (1 - \sin \gamma) + p^2 - r^2 - a^2 = \\ & \pm 2a \sqrt{r^2 - (y \sin \gamma + p \cos \gamma)^2} \dots (9) \end{aligned}$$

or interchanging the axes of x and y in equation (9), we find

$$\begin{aligned} & y = \pm \sqrt{r^2 + a^2 - p^2 - x^2 + 2px \cos \gamma (1 - \sin \gamma)} \\ & \pm 2a \sqrt{r^2 - (x \sin \gamma + p \cos \gamma)^2} \dots (10) \end{aligned}$$

The curve, then, is composed of two separate branches, each of which is symmetrical with respect to the axis of x .

That any one of the four roots should be possible, the second radical must be real, which furnishes the following limits of x ;—

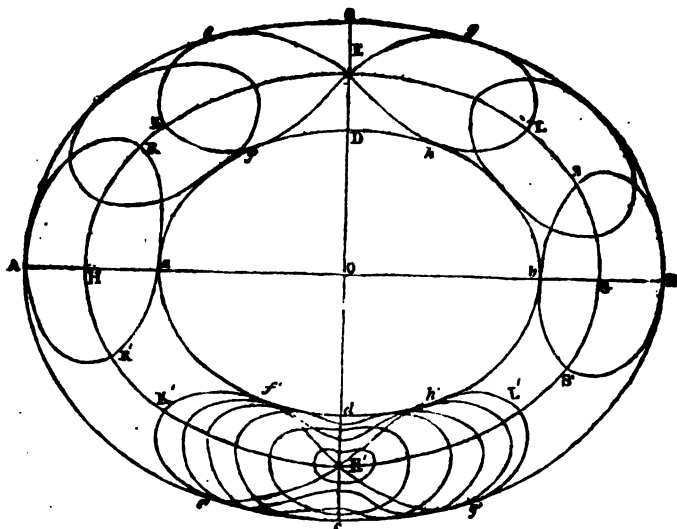
$$\begin{aligned} & x = + \frac{r - p \cos \gamma}{\sin \gamma} \left\{ \dots (11) \right. \\ & x = - \frac{r + p \cos \gamma}{\sin \gamma} \left. \right\} \end{aligned}$$

which render the radical $= 0$:

$$\text{and } x = -p \cot \gamma \dots (12)$$

which renders it greatest, viz. $\pm 2ar$.

The ring being given, then, the two conditions which determine the section are the magnitudes of p and γ . If γ remain constant, and p take all possible values, we obtain a series of curves which projected on the plane through the centre of the torus, and parallel to the cutting plane assume forms like the following.



If we conceive the ring to stand before us, so that the half whose projection is $\triangle CB bca$ is above the plane of the paper, and of course the other half below, we shall be able to readily comprehend the meaning of the figure.

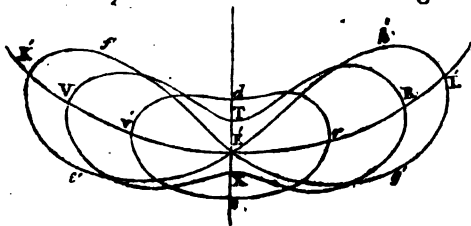
There are two positions of the cutting plane where it touches the generating sphere whose centre is E , and these will also be projected into E . The section will then resemble $\epsilon\kappa f \epsilon g h L$, not unlike in form to the figure which the lemniscata of Bernoulli would become if its axis were bent into an elliptic form.

As we move the cutting plane down towards the plane of the paper, the sections become curves with *pairs of isolated branches* of the oval kind, being more acute at the extremities nearer to E than at the more remote ones. These also resemble the more general case of the lemniscates, that is the ellipse of Cassini in appearance, with the same kind of difference as before, viz. that the axis of abscissæ is bent into an elliptic form.

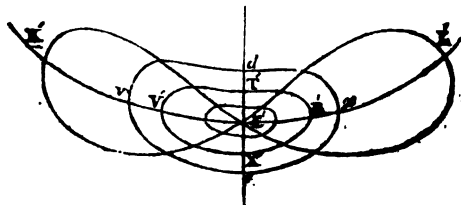
When we arrive at the plane of the paper, we obtain a pair of isolated symmetrical branches, taken with respect to AB, and symmetrical with respect to CD; viz. those which in the annexed figure are marked RAR'a and SBE'b.

Continuing the progress of the cutting plane *below* the plane of the paper, we obtain a series of sections similar to those already determined, till we arrive at the point of contact with the *upper* side of the generating sphere in the position E'; viz. e'x'f'x'g'h'l' in the lower part of the figure

Proceeding in the same way we obtain a series of figures such as those in the annexed figure, viz. RTVXR, till the plane arrives at the centre of the said sphere, whose centre is projected into x'. We then obtain the section docr.



Still lower we obtain a series of curves, similar to that traced in the figure, viz. T'v'x'R'; and these continually diminish till at last the curve merges in the single point of contact of the plane, and the outer part of the ring.



The same circumstances take place *above* $EEKfEGLh$, as we have described below $E'e'K'f'E'g'L'h'$. In figure (a) we see both series which arise from cutting the sphere in its position E'; shewing how the curves of figures, (3 and 4) are related in position to one another.

Nothing remarkable results from varying γ whilst p is constant: and the only values of γ that require our especial notice are 0 and $\frac{\pi}{2}$.

1. Let $\gamma = 0$; then $\sin \gamma = 0$, $\cos \gamma = 1$, and we have (10) converted into

$$y = \pm \sqrt{r^2 + a^2 - p^2 - x^2 + 2px \pm 2a\sqrt{r^2 - p^2}} \dots (13)$$

which is real so long as $r^2 - p^2$ is positive.

This indicates two circles whose centres are orthogonally projected into the point o of the directrix.

When $r = p^2$ then the expression becomes

$$y = \pm \sqrt{a^2 - x^2 + 2px} \dots (14)$$

which shews that the two circles coalesce, and that the plane is now tangential to the torus.

2. Let $\gamma = \frac{\pi}{2}$. Then $\sin \gamma = 1$, $\cos \gamma = 0$, and (10) is converted into restoring the old axes of x and y .

$$x = \pm \sqrt{\gamma^2 + a^2 - p^2 - y^2 \pm 2a\sqrt{r^2 - y^2}} \dots (15)$$

which denotes the *Ellipse of Cassini*, having for square of its semibase, and for rectangle of the radii-vectores, the respective quantities,

$$r^2 + a^2 - p^2, \text{ and } 4ar,$$

in which it appears that the rectangle of the radii-vectores is constant whichever of the vertical sections be taken.

Let $p = a + r$, then

$$x \pm \sqrt{-y^2 - 2ar \pm 2a\sqrt{r^2 - y^2}}$$

which is only possible when $y = 0$, and then also $x = 0$. The curve is then reduced to a single point.

Next let $p = a - r$: then

$$x = \pm \sqrt{2ar - y^2 \pm 2a\sqrt{r^2 - y^2}} \dots (17)$$

which denotes the *lemniscata of Bernoulli*, and the cutting plane is now a tangent to the inner equatoreal surface of the ring.*

If $p > r + a$, then (15) becomes imaginary, and hence no curve can then be formed, as no intersection takes place.

Lastly, let $p = 0$, or the plane be a meridional one. Then the expression (15) becomes

$$r^2 - y^2 = (x \pm a)^2 \dots (18)$$

which designates two circles to radius r , and whose centres are at the distance $\pm a$ from the origin of co-ordinate, reckoned along the axis of x .

When, also, we take in the general equation (10) $p = 0$, we find a neat result, viz,

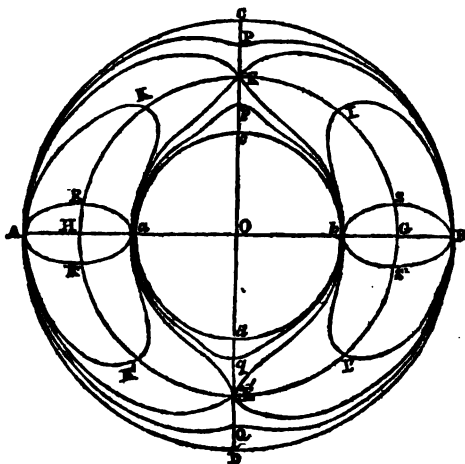
$$y = \pm \sqrt{r^2 + a^2 - x^2 \pm 2a\sqrt{r^2 - x^2} \sin 2\gamma} \dots (19)$$

* This particular case is selected by M. Hachette as an example of the tangent planes of the second species, (that is, which both touch and cut the surface to which they are applied,) and he has therefore entered into some details respecting it, which are both curious and instructive. Vid. *Seconde Supp. à la Géométrie Descrptive de Monge*, p. 17, and *Hachette Traité de Géom. Descrpt.* p. 65.

which expresses all the diametral sections of the ring, by giving to γ all values from 0 to $\frac{\pi}{2}$. There are *some* values of x which

render this real for *all* values that can be given to γ : and when this is the case the sub-radical is real. Also between certain of these values of x in the sub-radical, the principal radical is also real : and hence there are in all such cases four real values of y , or in other words four distinct semi-branches of the curve, symmetrically situated two and two with respect to the axis of x .

The following figure contains a delineation of the several species of forms, all the cutting planes, impressed with the section which they formed in the Torus being turned round upon AB till they coincided with the plane of the generatrix, and fixed permanently there. The figure $EBE'a$, $BBB'aE$ being that formed when the plane touches the generating spheres in the positions whose centres are $E'E'$. The curves $AKAK'$, $BLBL'$ are two branches (symmetrical and equal) produced by the plane



passing farther from the centres of the said spheres ; and $APBQ$, $apbq$ those formed when the plane passes near to the said centres.

Schol. In the *Mémoires de l'Académie Royale des Bruxelles*, 1824-5, a paper on *Surfaces Spiriques* (the same as the Torus), by M. Pagani was inserted ; and a short notice of it appeared in Quetelet's *Correspondance*, vol. II. p. 237-42, referring chiefly to the history of these surfaces. Indeed there is only the equation of the surface put down in Quetelet, and a slight remark upon another form of that equation in Vol. IV. p. 291 . and as I have not yet been able to procure the volume of the *Mémoires* which contains the dissertation, I am unable to say to what extent, or in what direction, the researches of M. Pagani have proceeded. Had I been aware when I undertook the investigation, that it had already been discussed in a lengthened *Mémoire*, I should

probably have laid mine aside till I had seen that dissertation; as it is, and the time is expired, I send it—though well assured from the high talents of Pagani, and the honorable position in which his paper is situated, that the impossibility of furnishing a copy of his, is the only apology that can be offered for giving my own. Should I be so fortunate as to procure the said volume, I shall give an analysis of the paper in question in the Repository.

Several curious discussions respecting the perspectives of the Torus may be seen at the end of the second volume of Hamilton's *Stereography*; and a considerable number of propositions relative to the sections of a ring in Tacquet *Cylindrica et Annularis*, occupying 142 pages in folio in his *Opera Mathematica*. The Reader may also consult Montucla *Histoire des Mathematiques*, and Quetelet's *Correspondance* already referred to.

SECOND SOLUTION, by Mr. GODWARD.

Let the centre of the ring be the origin of co-ordinates, and the principal section passing through this centre and that of the generating circle be the plane xy . Take the axis of y parallel to the intersecting plane, and put x = the mean radius of the ring, and r = the radius of the generating circle; then the equation to the surface of the ring will obviously be,

$$\{R - \sqrt{(x^2 + y^2)}\}^2 + z^2 = r^2 \dots (1)$$

Transferring the origin to the point where the intersecting plane meets the axis x , the distance of this point and the centre of the ring being represented by a , the equation becomes,

$$\{R - \sqrt{(x + a)^2 + y^2}\}^2 + z^2 = r^2 \dots (2)$$

Let θ be the angle which the intersecting plane makes with xy ; hence substituting $x \cos \theta$ for x , and $x \sin \theta$ for z in (2) (Hymer's Anal. Geom. Art. 41), we obtain

$\{R - \sqrt{(x \cos \theta + a)^2 + y^2}\}^2 + x^2 \sin^2 \theta = r^2$, the equation of the section.

If $\theta = 90^\circ$, the equation is $\{R - \sqrt{(a^2 + y^2)}\}^2 + x^2 = r^2$.

If $\theta = 90^\circ$, and $a = 0$, $\{R - y\}^2 + x^2 = r^2$, an equation to a circle.

Mr. BAINES answered this Question.

XII. QUESTION 542, by Mr. W. S. B. WOOLHOUSE.

Suppose with an organ the natural diatonic octave C, D, E, F, G, A, B, C, have C and D tuned in the ratio of 10 to 9; and that

E, F and G are tuned a perfect third major, a perfect fourth and a perfect fifth respectively from the key note C; that the intervals of the tetrachord G, A, B, C are made the same as those of C, D, E, F; and that the semitones are taken exactly half way between the successive notes CD, DE, FG, GA, AB. It is required to shew the defects in the diatonic scales of each of the other eleven major keys, viz. G one sharp, D two sharps, A three sharps, E four sharps, B five sharps: F one flat, B $\flat$ two flats, E $\flat$ three flats, A $\flat$ four flats, D $\flat$ five flats, and G $\flat$ six flats.

SOLUTION, by Mr. W. S. B. WOOLHOUSE.

By dividing the octave into 730 equal intervals called *degrees*, the number of these degrees in the several true diatonic intervals are*, from C ascending

to D .. 111, to E .. 235, to F .. 303, to G .. 427,
to A .. 538, to B .. 662, to C .. 730.

I have here taken CD the minor tone to make the tetrachords CDEF, GABC alike as proposed.

Hence when the semitones are taken exactly half way our chromatic series reckoned from C in degrees will be as hereunder :

to C $\sharp$ or D $\flat$ $55\frac{1}{2}$ degrees.	to G 427 degrees.
.. D 111 ..	.. G $\sharp$ or A $\flat$ $482\frac{1}{2}$..
.. D $\sharp$ or E $\flat$ 173 ..	.. A 538 ..
.. E 235 ..	.. A $\sharp$ or B $\flat$ 600 ..
.. F 303 ..	.. B 662 ..
.. F $\sharp$ or G $\flat$ 365 ..	.. C 730 ..

And by differencing, we find the succession of intervals on each of the other eleven major keys, and their respective defects to be as in the following arrangement :—

Perfect Key.	One Sharp.	Errors.	Two Sharps.	Errors
C to D .. 111	G to A .. 111	..	D to E .. 124	+ 13
E .. 235	B .. 235	..	F $\sharp$.. 254	+ 19
F .. 303	C .. 303	..	G .. 316	+ 13
G .. 427	D .. 414	- 13	A .. 427	..
A .. 538	E .. 538	..	B .. 551	+ 13
B .. 662	F $\sharp$.. 668	+ 6	C $\sharp$.. $674\frac{1}{2}$	+ $12\frac{1}{2}$
C .. 730	G .. 730	..	D .. 730	..

* See Art 16, of my small tract on Musical intervals, harmonics and temperament, just published.

Three Sharps.			Four Sharps.	
111	A to B..124	+ 13	E to F#.130	+ 19
235	C#.247½	+ 12½	G#.247½	+ 12½
303	D..303	..	A..303	..
427	E..427	..	B..427	..
538	F#.557	+ 19	C#.550½	+ 12½
662	G#.674½	+ 12½	D#.668	+ 6
730	A..730	..	E..730	..
Five Sharps.			One Flat.	
111	B to C#.123½	+ 12½	F to G..124	+ 13
235	D#.241	+ 6	A..235	..
303	E..303	..	Bb.297	- 6
427	F#.433	+ 6	C..427	..
538	G#.550	+ 12½	D..538	..
662	A#.668	+ 6	E..662	..
730	B..730	..	F..730	..
Two Flats.			Three Flats.	
111	Bb to C..130	+ 19	Eb to F..130	+ 19
235	D..241	+ 6	G..254	+ 19
303	Eb.303	.	Ab.309½	+ 6½
427	F..433	+ 6	Bb.427	..
538	G..557	+ 19	C..557	+ 19
662	A..668	+ 6	D..668	+ 6
730	Bb.730	..	Eb.730	..
Four Flats.			Five Flats.	
111	Ab to Bb.117½	+ 6½	Db to Eb.117½	+ 6½
235	C..247½	+ 12½	F..247½	+ 12½
303	Db.303	..	Gb.309½	+ 6½
427	Eb.420½	- 6½	Ab.427	..
538	F..550½	+ 12½	Eb.544½	+ 6½
662	G..674½	+ 12½	C..674½	+ 12½
730	Ab.730	..	Db.730	..
Six Flats.				
111	Gb to Ab.117½	+ 6½		
235	Bb.235	..		
303	Cb.297	- 6		
427	Db.420½	- 6½		
538	Eb.538	..		
662	F.668	+ 6		
730	Gb.730	..		

The errors which are too sharp are marked +, and those too flat —. It appears that the notes are generally inclined to be

sharp, and that the greatest deviation from a true melody is 19 degrees, or about $\frac{1}{3}$ th of a mean tone.

Mr. LAWS, of Newcastle, answered this Question.

XIII. QUESTION 543, by Mr. W. S. B. WOOLHOUSE.

Let there be assumed any two opposite equilateral hyperbolas; let there also be assumed any ellipse, and let it be placed with its centre any where on either of the branches of the hyperbolas, in such a manner that its principal diameters shall be parallel to the asymptotes—if it be intersected by the hyperbolas in four points, shew that four normals drawn from thence will all meet in a point.

FIRST SOLUTION, by Mr. WOOLHOUSE.

We shall assume the centre of the ellipse on the hyperbolic arc for the origin, and the two semi-diameters a , b for axes of co-ordinates. Let $x'' y''$ be the common centre of the hyperbolas and their equation will be $(x-x'')(y-y'') = a'' y''$ or $xy - y''x - x''y = 0$; and if $x_1 y_1, x_2 y_2$ be any two of the four points of intersection we shall thence have

$$x_1 y_1 - y'' x_1 - x'' y_1 = 0, \quad x_2 y_2 - y'' x_2 - x'' y_2 = 0 \dots (1)$$

Let u, v be the ordinates of the point where the normals from $x_1 y_1, x_2 y_2$ intersect, and the normal equation applied to both will give

$$b^2 \frac{x_1 - u}{x_1} - a^2 \frac{y_1 - v}{y_1} = 0, \quad b^2 \frac{x_2 - u}{x_2} - a^2 \frac{y_2 - v}{y_2} = 0 \dots (2)$$

The values of uv determined from these are

$$u = \left(1 - \frac{b^2}{a^2}\right) (y_1 - y_2) \div \left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)$$

$$v = \left(1 - \frac{a^2}{b^2}\right) (x_1 - x_2) \div \left(\frac{x_1}{y_1} - \frac{x_2}{y_2}\right)$$

But the equations (1) give

$$y_1 - y_2 = x'' \left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right), \quad x_1 - x_2 = y'' \left(\frac{x_1}{y_1} - \frac{x_2}{y_2}\right).$$

$$\therefore u = \left(1 - \frac{a^2}{b^2}\right) x'', \quad v = \left(1 - \frac{a^2}{b^2}\right) y''.$$

Hence as these values of the ordinates of the intersection of two normals from $x_1 y_1, x_2 y_2$ is independent of those points,

it follows that normals from any two of the four intersections will meet in this same point, and consequently that they will all of them pass through it. Since $(u - x'')(v - y'') = x''y''$ it also follows that this point $u v$ where the normals meet is also on one of the branches of the hyperbolas.

SECOND SOLUTION, by Mr. WM. GODWARD.

Let $(\bar{x} \bar{y})$ be the intersection of the normals through the points $(x' y')$ $(x'' y'')$ of the intersections of the curves, the centre of the ellipse being the origin, and its principal semi-diameters, the axes of co-ordinates.

The equations to the normals through the points $(x' y')$ $(x'' y'')$ are

$$\bar{y} - y' = \frac{a^2 y'}{b^2 x'} (\bar{x} - x'),$$

$$\bar{y} - y'' = \frac{a^2 y''}{b^2 x''} (\bar{x} - x'');$$

and from these we obtain

$$\bar{x} = \left(1 - \frac{b}{a^2}\right) \frac{x' x'' (y' - y'')}{x'' y' - x' y''}, \text{ and } \bar{y} = \left(\frac{a^2}{b^2} - 1\right) \frac{y' y'' (x' - x'')}{x'' y' - x' y''},$$

Again, let (x, y) be the centre of the equilateral hyperbolas and as $(x' y')$ $(x'' y'')$ are points in the curve, the equation gives

$$(x' - x)(y' - y) = x y,$$

$$(x'' - x)(y'' - y) = x y,$$

from which we get $x_1 = \frac{x' x'' (y' - y'')}{x'' y' - x' y''}$, and $y_1 = -\frac{y' y'' (x' - x'')}{x'' y' - x' y''}$

Hence $\bar{x} = \left(1 - \frac{b^2}{a^2}\right) x$, and $\bar{y} = -\left(\frac{a^2}{b^2} - 1\right) y$; and as these

are independent of the assumed points $(x' y')$ $(x'' y'')$, it manifestly follows that the four normals will pass through the point $(\bar{x} \bar{y})$.

Mr. THOMPSON, of Newcastle, answered this Question.

XIV. QUESTION 544, by Mr. T. S. DAVIES.

Find the equation of a great circle, which passes through two given points on the surface of the sphere, in terms of the

variables, lat. and long. Also find the co-ordinates of the poles of that circle.

FIRST SOLUTION, by Mr. JOHN BAINES, Thornhill.

Suppose x' and y' to represent the longitude and latitude of one of the given places, and x'' and y'' the other, then supposing the great circle passing through the two given places to make an angle $= \theta$ with the equator, and to cut it at the distance a from the first meridian, we shall have $a + x$ and y for the variable co-ordinates (arcs of great circles) on the sphere. From the nature of the question, and the principles of spherical trigonometry, we have three independent equations, viz. $\tan y = \sin(a + x) \tan \theta \dots (1)$, $\tan y' = \sin(a + x') \tan \theta \dots (2)$, and $\tan y'' = \sin(a + x'') \tan \theta \dots (3)$, in which x' , y' , x'' and y'' are given quantities, and a and θ are to be eliminated.

Nos. (2) and (3) give $\tan \theta = \frac{\tan y'}{\sin(a + x')} = \frac{\tan y''}{\sin(a + x'')}$, and

hence $\tan y' \sin(a + x'') = \tan y'' \sin(a + x')$, or $\tan y' (\sin a \cos x'' + \cos a \sin x'') = \tan y'' (\sin a \cos x' + \cos a \sin x') \dots (4)$. Transpose No. (4) and $\sin a (\tan y' \cos x'' - \tan y'' \cos x') = \cos a (\tan y'' \sin x' - \tan y' \sin x'') \dots (5)$, Square No. (5), transpose and reduce, then

$$\sin a = \frac{\tan y'' \sin x' - \tan y' \sin x''}{\{ \tan^2 y' + \tan^2 y'' - 2 \tan y' \tan y'' (\sin x' \sin x'' + \cos x' \cos x'') \}^{\frac{1}{2}}}$$

and $\cos a = \frac{\tan y' \cos x'' - \tan y'' \cos x'}{\text{denominator}}$. But $\tan \theta =$

$\frac{\tan y'}{\sin(a + x')}$, therefore $\tan y = \sin(a + x) \tan \theta$, the general

equation to the great circle, becomes $\tan y = \tan y' \frac{\sin(a + x)}{\sin(a + x')}$

$= \tan y' \frac{\sin a \cos x + \cos a \sin x}{\sin a \cos x' + \cos a \sin x'}$, and this by substituting the values of $\sin a$ and $\cos a$ found above gives

$$\tan y = \tan y' \frac{-\tan y' (\sin x'' \cos x - \cos x'' \sin x) - \tan y'' (\sin x \cos x - \cos x' \sin x)}{\tan y' (\sin x'' \cos x' - \cos x'' \sin x')}, \text{ or}$$

which is the same

$\tan y = \frac{\tan y' (\sin (x'' - x) - \tan y'' \sin (x' - x))}{\sin (x'' - x')} =$ the equation of the great circle required.

The co-ordinates of the poles will obviously be when $a + x = 90^\circ$, and $y = \theta$; hence they are 90° and $90^\circ \mp \theta$, but

$\tan \theta = \frac{\tan y'}{\sin (a + x')} = \frac{\tan y'}{\sin a \cos x' + \cos a \sin x'}$, by substituting the values of $\sin a$ and $\cos a$, and reducing,

$$\frac{\sqrt{\{\tan^2 y' + \tan^2 y'' - 2 \tan y' \tan y'' \cos (x'' - x')\}}}{\sin (x'' - x')}.$$

Consequently 90° and

$$90^\circ \mp \tan^{-1} \frac{\sqrt{\{\tan^2 y' + \tan^2 y'' - 2 \tan y' \tan y'' \cos (x'' - x')\}}}{\sin (x'' - x')}$$

are the co-ordinates of the poles required.

SECOND SOLUTION, by Mr. T. S. DAVIES, Bath.

Let P be the pole of reference, and PP' the prime meridian. Let the spherical centre of the great circle sought be L, so that P'PL = κ and PL = λ ; and let M be a point in the circle sought.

Then in a great circle we have $LM = \frac{\pi}{2}$. Put P'PM = θ , and PM = ϕ . Then LPM = $\theta - \kappa$.

Also LPM is a quadrantal triangle, and we have at once from it the following general equation of a circle between the variables ϕ , θ , and the constants κ , λ : viz.

$$\cot \phi = -\tan \lambda \cos \overline{\theta - \kappa} \dots (1)$$

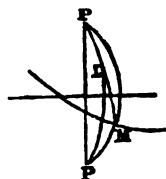
If the two given values of ϕ be α , α'' , and those of θ be β , β'' , we shall have, because the circle passes through these,

$$\cot \alpha = -\tan \lambda \cos \overline{\beta - \kappa} \dots (2)$$

$$\cot \alpha'' = -\tan \lambda \cos \overline{\beta'' - \kappa} \dots (3)$$

Divide (2, 3) and we get

$$\begin{aligned} \frac{\cot \alpha}{\cot \alpha''} &= \frac{\cos \overline{\beta - \kappa}}{\cos \overline{\beta'' - \kappa}} = \frac{\cos \beta, \cos \kappa + \sin \beta, \sin \kappa}{\cos \beta'', \cos \kappa + \sin \beta'', \sin \kappa} \\ &= \frac{\cos \beta + \sin \beta, \tan \kappa}{\cos \beta'' + \sin \beta'', \tan \kappa} \end{aligned}$$



$$\text{From which } \tan \kappa = - \frac{\cot \alpha_{//} \cos \beta_{//} - \cot \alpha \cos \beta_{//}}{\cot \alpha_{//} \sin \beta_{//} - \cot \alpha \sin \beta_{//}} \dots (4)$$

$$\sin \kappa = \mp \frac{\cot \alpha_{//} \cos \beta_{//} - \cot \alpha \cos \beta_{//}}{\sqrt{\cot^2 \alpha_{//} - 2 \cot \alpha_{//} \cot \alpha \cos \beta_{//} - \beta_{//} + \cot^2 \alpha}} \dots (5)$$

$$\cos \kappa = \pm \frac{\cot \alpha_{//} \sin \beta_{//} - \cot \alpha \sin \beta_{//}}{\sqrt{\cot^2 \alpha_{//} - 2 \cot \alpha_{//} \cot \alpha \cos \beta_{//} - \beta_{//} + \cot^2 \alpha}} \dots (6)$$

These give the tangent, sine and cosine of the *longitude* of the pole of the great circle sought.

Again, from equations (2) and (3) we have

$$\cos^{-1} (-\cot \alpha_{//} \cot \lambda) - \cos^{-1} (-\cot \alpha \cot \lambda) = \beta_{//} - \beta,$$

or taking cosine of each side, and transposing we obtain

$$\cos \beta_{//} - \cos \beta - \cot \alpha \cot \alpha_{//} \cot^2 \lambda = \sqrt{(1 - \cot^2 \alpha_{//} \cot^2 \lambda) (1 - \cot^2 \alpha \cot^2 \lambda)}$$

or squaring and otherwise reducing, it is

$$\cot \lambda = \frac{\pm \sin \beta_{//} - \sin \beta}{\sqrt{\cot^2 \alpha_{//} - 2 \cot \alpha_{//} \cot \alpha \cos \beta_{//} - \beta_{//} + \cot^2 \alpha}} \dots (7)$$

which gives the polar distance or *co-latitude* of the pole of the great circle sought.

Expand (1): then we have

$$\cot \varphi = -\tan \lambda (\cos \theta \cos \kappa + \sin \theta \sin \kappa) \dots (8)$$

Insert the values of $\tan \lambda$, $\cos \kappa$, $\sin \kappa$ in (8), which gives

$$\cot \varphi = - \left\{ \begin{array}{l} \cos \theta (\cot \alpha_{//} \sin \beta_{//} - \cot \alpha \sin \beta_{//}) \\ \mp \sin \theta (\cot \alpha_{//} \cos \beta_{//} - \cot \alpha \cos \beta_{//}) \end{array} \right\} \frac{1}{\sin \beta_{//} - \sin \beta} \dots (9)$$

$$\text{or } \cot \varphi = - \frac{\cot \alpha \sin (\beta_{//} \mp \theta) - \cot \alpha_{//} \sin (\beta \mp \theta)}{\sin \beta_{//} - \sin \beta} \dots (10)$$

Of these two equations, the former (9) though the more complex in appearance is commonly most useful in subsequent enquiries. The latter in point of algebraical elegance is, however, far preferable.

Since this question was proposed in the Repository, I have so amply discussed the whole class of collateral enquiries, in a paper in the Edinburgh Transactions, that it will be unnecessary to resume it here. I had, indeed, intended, as the Edinburgh Transactions are not likely to fall in the way of the majority of the readers of the Repository, to give a slight sketch of my general method of treating these subjects,—

rather to *excite* curiosity than to *settle* it—to shew the *character* of the inquiry, rather than to give any idea of its *power*: but as this purpose will be better answered by the supplement which Mr. Young has obligingly engaged to annex to the forthcoming second edition of his Analytical Geometry, I shall omit it here entirely.

I may here state that besides the paper already printed in the Transactions and the paper on the Antique Hour-lines, out of which it mainly grew, there is now lying ready for transmission to the Royal Society of Edinburgh, a second, which will probably be read at the opening of its next session. This will, so far as I at present see, complete my researches in this department of pure Geometry. Respecting the views which I there take, I cannot, with propriety, at present, make any statement. My object is merely to intimate the existence of such a document.

The problem above solved, bears date May 30, 1829, but the solution in my MS. is something different from that now given, so far as detail is concerned, but not in its essential principles. I had thought of inserting it here, but it would occupy a page which can be more usefully devoted to other subjects. That above given is taken almost *verbatim et literatim* from my paper before referred to, except as to the method of finding equation (1).

July 2, 1832.

THIRD SOLUTION, by Mr. GODWARD.

Let A be that point in the ecliptic AB, from which longitudes are measured, and CP a great circle of the sphere intersecting AB in C. Draw PB perpendicular to AB, and put $\angle C = a$, $CA = b$, $AB = x$, and $PB = y$. Then by spherics, $\tan PB = \tan C \sin CB$, or



$$\tan y = \tan a \sin (b + x) \dots (1), \text{ an equation to CP.}$$

Let (x', y') and (x'', y'') be the co-ordinates corresponding to the given point, then

$$\tan y' = \tan a \sin (b + x') \dots (2)$$

$$\tan y'' = \tan a \sin (b + x'') \dots (3).$$

Eliminating $\tan a$ from equations (1) and (2) and from (2) and (3) respectively, we have

$$\tan y \sin (b + x') = \tan y' \sin (b + x) \dots (4)$$

$$\tan y' \sin (b + x'') = \tan y'' \sin (bx') \dots (5)$$

from (4) we obtain $\tan b = \frac{\tan y' \sin x - \tan y \sin x'}{\tan y \cos x' - \tan y' \cos x}$, and from

$$(5) \tan b = \frac{\tan y'' \sin x' - \tan y' \sin x''}{\tan y' \cos x'' - \tan y'' \cos x'} \dots (6)$$

Equating these values of $\tan b$ we obtain after due reduction

$$\tan y = \frac{\tan y' \sin (x'' - x) - \tan y'' \sin (x' - x)}{\sin (x'' - x')}$$
 the equation required.

Again, since b has been determined (6), a becomes known from (2) or (3). Let CD , CE be quadrantal arcs, then the arc DE of the great circle passing through D , E , will, if produced, obviously pass through the poles of CB , CP . Hence, as $DE = \angle C = a$, the co-ordinates of the upper and lower poles of CP , are $(90^\circ - b, 90^\circ + a)$, and $(90^\circ - b, 90^\circ - a)$.

FOURTH SOLUTION, by Mr. WOOLHOUSE.

Let $x' x''$ be the longitudes, and $y' y''$ the latitudes of the two points, c the longitude of the intersection of the circle with the ecliptic, and i its inclination. Then $x' - c$, $x'' - c$ are the abscissæ from this intersection, and by spherics

$$\tan y' = \tan i \sin (x' - c), \quad \tan y'' = \tan i \sin (x'' - c)$$

from which find the values of the constants i, c , and the general equation of the circle in its most simple form, viz.

$$\tan y = \tan i \sin (c - x),$$

will be known.

The poles are obviously determined by the common abscissa $\frac{\pi}{2} + c$ and the ordinates $(\frac{\pi}{2} + i)$, $-(\frac{\pi}{2} - i)$:

XV. QUESTION 545. by Mr. W. S. B. WOOLHOUSE.

Suppose a combustible straight line of a given length and a given uniform temperature to be lighted at one extremity, and to commence burning by a flame with a given velocity. It is required to determine the time in which it will be totally destroyed, supposing the combustion to be affected by two conditional circumstances;—1st. The velocity and size of the flame when arrived at any point be proportional to the resulting temperature at that point; and 2ndly, that the temperature of each point during the burning be continually increasing at a rate proportional to the magnitude of the flame, and reciprocally as a given constant value added to the square of its distance.

SOLUTION, by Mr. W. S. B. WOOLHOUSE.

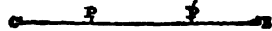
In the analytical consideration of this question, four constants must be assumed either as given or determinable by experiments. For, by the conditions, we must have

1st. Magnitude of the flame = its velocity $\times c_1$.

and. Rate of increase of temperature at each point during the combustion = $c_3 \cdot \frac{\text{mag. flame}}{c_2 + \text{dist.}^2}$.

3rd. Velocity of the flame = the resulting temperature, or the temperature close to the point of burning $\times c_4$.

Let CB be the combustible line, P the place of the flame at any instant, and P' any other point at which it has not arrived. Denote CP



by x and CP' by x' ; then is $\frac{dx}{dt}$ the velocity of the flame at P, and consequently the magnitude of the flame at this point = $c_1 \frac{dx}{dt}$. Hence also the rate of increase of temperature at

the point P' at this instant = $c_3 \frac{c_1 \frac{dx}{dt}}{c_2 + PP'^2} = \frac{c_1 c_3 \frac{dx}{dt}}{c_2 + (x' - x)^2}$ and the increase during the instant dt , or during the generation of dx ,

$$= \frac{c_1 c_3 \frac{dx}{dt}}{c_2 + (x' - x)^2} dt = c_1 c_3 \frac{dx}{c_2 + (x' - x)^2}.$$

Let τ be the original uniform temperature of the line and the integral of this, keeping x' fixed while x progresses from c to P , and made to begin with τ when $x = 0$, will express the temperature at any point P' when the flame has arrived at another point P. The variable part of this integral is

$$= \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x' - x}{\sqrt{c_2}}, \text{ which so modified gives}$$

Temperature at P'

$$\begin{aligned} &= \tau + \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x'}{\sqrt{c_2}} - \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x' - x}{\sqrt{c_2}} \\ &= \tau + \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x \sqrt{c_2}}{c_2 + x'(x' - x)}, \end{aligned}$$

the temperature gained by exposure to the flame being

$$\frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x \sqrt{c_2}}{c_2 + x'(x' - x)}.$$

Let now x become equal to x' and we get the resulting temperature at P' on the arrival of the flame

$$= \tau + \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x'}{\sqrt{c_2}}.$$

By our last condition therefore

$$\frac{dx'}{dt} = c_4 \left(\tau + \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x'}{\sqrt{c_2}} \right)$$

At the commencement $\frac{dx}{dt} = v$ is given, and

$$\therefore v = c_4 \tau \therefore c_4 = \frac{v}{\tau}.$$

Consequently

$$dt = \frac{dx'}{c_4 \left(\tau + \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x'}{\sqrt{c_2}} \right)} = \frac{\tau}{v} \cdot \frac{dx'}{\tau + \frac{c_1 c_3}{\sqrt{c_2}} \tan^{-1} \frac{x'}{\sqrt{c_2}}},$$

the integral of which, from $x' = 0$ to $x' = CB$, will give the time sought. This integration will be very simple by assuming

$\frac{c_1 c_3}{\tau \sqrt{c_2}} = c''$ and $\frac{x'}{\sqrt{c_2}} = y$, for then

$$dt = \frac{\sqrt{c_2}}{v} \cdot \frac{dy}{1 + c'' \tan^{-1} y},$$

which may be expanded by putting $y - \frac{y^3}{3} + \frac{y^5}{5}$, &c. for

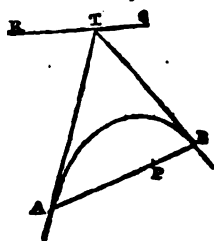
$\tan^{-1} y$. It appears that only two constants c_2 , c'' are indeterminate and must be found from two or more experiments.

XVI. QUESTION 546, by Mr. T. S. DAVIES.

PROBLEM. A circle and a point within it are given, as well as the position of a plane parallel to the picture, then another plane may be determined, in any point of which the eye being placed, the perspective representation of the point shall be the centre of the perspective representation of the circle.

FIRST SOLUTION, *by Mr. WOOLHOUSE.*

This question may be very easily resolved for any conic section and a given point P , by supposing chords AB to be drawn through the point and the pairs of tangents AT , BT meeting in the points T ; these points T of intersection of the tangents will be all on a given line RS , ordinately applied to the diameter which passes through the point P . Suppose the figure to be projected



upon the plane of the picture, and the perspective of RS will be determined by a plane passing through it and the eye. Now when the perspective representation has P for the centre of the curve, the lines AB will be represented as diameters, and the points T and consequently the perspective of the line RS will be infinitely removed from the curve; and this must evidently be the case when the plane through RS and the eye is parallel to the picture. Hence we see that a plane through RS parallel to the plane of the picture is the one which will answer the proposed condition. When the curve is a circle, as in the question, the line RS is perpendicular to the diameter which passes through the point.

Note.—It ought to be here observed as a condition absolutely necessary, although not mentioned in the question, that the plane of the picture, and consequently its intersection with the plane of the real curve must be parallel to the line RS , for otherwise no plane through RS can be parallel to that of the picture, and therefore a solution of the question would not be possible.

SECOND SOLUTION, *by Mr. T. S. DAVIES, the Proposer.*

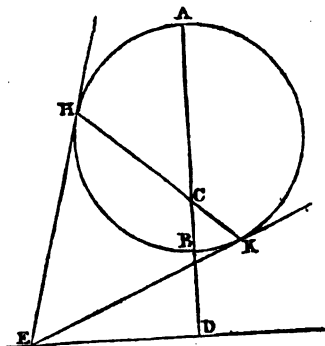
This porism is not confined to the circle, but belongs to all the conic sections; and the substitution of the word conical projective for perspective, will cause the point to be arbitrarily situated with respect to the given conic section. We may state it, then, thus:—

A conic section, and a point being given upon a plane, a straight line may be found such that a plane being drawn through it in any assigned angle with the former plane; then any point being taken in this second plane as the centre of the conical projection of the conic section and the point, the intersection of the cone and line by any plane parallel to a plane which can be found, will give a conic section and a point, such that the latter shall be the centre of the former.

I shall, however, only demonstrate the porism as communicated, reserving the details upon several collateral questions, which I intended to give here for my work on *Projection and Descriptive Geometry*, to which, as they naturally appertain, it is sufficient to confine them for the present. Nor is it necessary here to give the analysis of the porism by which the following construction is obtained.

Let $AHBK$ be the given circle, and c the given point. Draw the diameter AB through C , and take $AD : DB :: AC : CB$. Through D , draw DE at right angles to DA .

Then any plane (P) passing through ED will be such that from any point in it the circle AHB and point C being seen, and the visual cone cut by a plane parallel to a plane which is parallel to (P), the perspective of the point will be the centre of the perspective of the circle.



Dem. Draw any line HCK through C , terminated both ways at the circumference at H , and K . Draw the tangents HE , KE . These will meet at E in the fore-named straight line DE , by a known property of this figure. Our construction gives DE as the *directing line* of the picture; and hence the tangents at H and K have the same *directing point* E , and same *direction*. But by a fundamental theorem in perspective, all lines in an objective plane which have the same director, have parallel representations on the picture. The pictures, then, of HE , KE are parallel: and as they are parallel tangents to the picture of the circle, the picture of the line HK is a diameter of the picture of AHB . It hence passes through the *centre* of that picture.

Again, any other pair of tangents $H'E$, $K'E$, the chord joining the points of contact of which passes through C , will be projected into a diameter of the picture, and as their intersection C will be projected into the intersection of the two diameters, that is of the centre of the pictures of AHB , the proposition is fully established.

Q. E. D.

XVII. QUESTION 547, by R. N.

Prove that
$$\frac{\sin \theta}{1 + 2 \cos \theta} = \frac{1}{4} \cot \frac{\theta}{2} - \frac{3}{4} \cot \frac{3\theta}{2}.$$

FIRST SOLUTION, *by* — MASON, of Scoulton.

$$\begin{aligned}
 \frac{\sin \theta}{1 + 2 \cos \theta} &= \frac{2 \sin \theta (1 - \cos \theta)}{2 (1 + \cos \theta + \cos \theta) (1 - \cos \theta)} \\
 &= \frac{2 \sin \theta - \sin 2\theta}{2 + 2 \cos \theta - 4 \cos^2 \theta} = \frac{\sin \theta + \sin \theta - \sin 2\theta}{2 \cos \theta - 2 \cos 2\theta} \\
 &= \frac{\sin \frac{3\theta}{2} \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \sin \frac{\theta}{2} - (\sin 2\theta - \sin \theta)}{2 (\cos \theta - \cos 2\theta)} \\
 &= \frac{\sin \frac{3}{2} \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \sin \frac{\theta}{2} - 2 \cos \frac{3\theta}{2} \sin \frac{\theta}{2}}{4 \sin \frac{3\theta}{2} \sin \frac{\theta}{2}} = \\
 &\quad \frac{1}{4} \cot \frac{\theta}{2} - \frac{3}{4} \cot \frac{3\theta}{2} \dots \text{Q. E. D.}
 \end{aligned}$$

SECOND SOLUTION, *by* Mr. GODWARD.

$$\begin{aligned}
 \frac{1}{4} \cot \frac{\theta}{2} - \frac{3}{4} \cot \frac{3\theta}{2} &= \frac{1}{4} \cot \frac{\theta}{2} - \frac{3}{4} \left(\frac{\cot^3 \frac{\theta}{2} - 3 \cot \frac{\theta}{2}}{3 \cot^2 \frac{\theta}{2} - 1} \right) \\
 &= \frac{2 \cot \frac{\theta}{2}}{3 \cot^2 \frac{\theta}{2} - 1} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{3 \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}} = \frac{\sin \theta}{4 \cos^2 \frac{\theta}{2} - 1} = \\
 &\quad \frac{\sin \theta}{1 + 2 (2 \cos^2 \frac{\theta}{2} - 1)} = \frac{\sin \theta}{1 + 2 \cos \theta}.
 \end{aligned}$$

According to one or other of these methods, the question was answered by Messrs. BAINES, COHEN, DAVIES and LAW.

XVIII. QUESTION 548, *by* M.

Find the integral of $\frac{dx}{1+x^2}$ between the limits of $x=0$ and $x=\infty$.

SOLUTION, by — MASON, of Scoulton, near Hingham,
Norfolk.

This problem is only a particular case of the more general one which follows. The integral of $\frac{dx}{1+x^n}$ between the limits $x=0$ and $x=\infty$.

The general quadratic factor of x^n+1 is $(x^2-2x \cos \frac{2m+1}{n} \pi + 1)$

where $m=0, 1, 2, \dots, \left(\frac{n}{2}-1\right)$ when n is even: besides which general quadratic factor, when n is odd there is the simple factor $(x+1)$, and the values of m in the case of n odd are $0, 1, 2, \dots, \left(\frac{n-1}{2}\right)$.

$$\begin{aligned} \text{Hence } \int \frac{1}{x^{1+x}} &= \frac{a}{n} \int \frac{1}{1+x} + \frac{1}{n} \int \frac{-2x \cos \frac{2m+1}{n} \pi + 1}{(x^2 - 2x \cos \frac{2m+1}{n} \pi + 1)} \\ &= \frac{a}{2n} \log_e (x^2 + 2x + 1) - \\ &\quad \frac{\cos \frac{2m+1}{n} \pi}{n} \log_e (x^2 - 2x \cos \frac{2m+1}{n} \pi + 1) + \\ &\quad \frac{2 \sin \frac{2m+1}{n} \pi}{n} \tan^{-1} \frac{x - \cos \frac{2m+1}{n} \pi}{\sin^2 \frac{2m+1}{n} \pi} + \text{constant, where} \end{aligned}$$

$a=1$ when n is odd, and $a=0$ where n is even.

$$\text{Now } \int_x \frac{1}{1+x^n} = 0 \text{ when } x=0.$$

$$\begin{aligned} \therefore 0 &= \frac{a}{n} \sin \frac{2m+1}{n} \pi \times \tan^{-1} (-\cot \frac{2m+1}{n} \pi) + C, \\ \therefore \int_x \frac{1}{1+x^n} &= \frac{a}{2n} \log_e (x^2 + 2x + 1) - \frac{1}{n} \cos \frac{2m+1}{n} \pi \cdot \\ &\log_e (x^2 - 2x \cos \frac{2m+1}{n} \pi + 1) + \frac{2}{n} \sin \frac{2m+1}{n} \pi \end{aligned}$$

$$\left\{ \tan^{-1} \left(\frac{x - \cos \frac{2m+1}{n} \pi}{\sin \frac{2m+1}{n} \pi} \right) - \tan^{-1} \left(-\cot \frac{2m+1}{n} \pi \right) \right\} =$$

$$\frac{a}{2n} \log_e (x^2 + 2x + 1) - \frac{1}{n} \cos \frac{2m+1}{n} \pi \cdot \log_e (x^2 - 2x$$

$$\cos \frac{2m+1}{n} \pi + 1) + \frac{2}{n} \sin \frac{2m+1}{n} \pi \tan^{-1} \frac{x \sin \frac{2m+1}{n} \pi}{1 - x \cos \frac{2m+1}{n} \pi}.$$

If we now suppose x so great that x^2 is considerably greater than $2x + 1$ or than $2x \cos \frac{2m+1}{n} \pi + 1$ we shall have

$$\int_x \frac{1}{1+x^n} = \frac{1}{n} \log_e x^2 \left\{ \frac{a}{2} - \cos \frac{2m+1}{n} \pi \right\} + \frac{2}{n} \sin \frac{2m+1}{n} \pi$$

$$\tan^{-1} \left(-\tan \frac{2m+1}{n} \pi \right) \text{ very nearly } = \frac{1}{n} \log_e x^2 \left\{ \frac{a}{2} - \cos \frac{2m+1}{n} \pi \right\}$$

$$+ \frac{2}{n} \sin \frac{2m+1}{n} \pi \left\{ \pi - \frac{2m+1}{n} \pi \right\} \text{ also very nearly, where}$$

$$m = 0, 1, 2, \dots \left(\frac{n}{2} - 1 \right) \text{ and } a = 0, \text{ when } n \text{ is even, and } m =$$

$$0, 1, 2, \dots \left(\frac{n-1}{2} \right) \text{ and } a = 1 \text{ when } n \text{ is odd.}$$

First let n be even, the coefficient of $\log_e x^2 = -\frac{1}{n}$

$$\left\{ \cos \frac{\pi}{2n} + \cos \frac{3\pi}{n} + \cos \frac{5\pi}{n} + \dots \cos \frac{n-1}{n} \pi \right\} = 0 \text{ and}$$

the coefficient of

$$\frac{2\pi}{n} \text{ is } = \left\{ \sin \frac{\pi}{n} + \sin \frac{3\pi}{n} + \sin \frac{5\pi}{n} \dots \sin \frac{n-1}{n} \pi \right\}$$

$$\left\{ -\frac{1}{n} \sin \frac{\pi}{n} - \frac{3}{n} \sin \frac{3\pi}{n} - \frac{5}{n} \sin \frac{5\pi}{n} \dots - \frac{n-1}{n} \sin \frac{n-1}{n} \pi \right\}$$

$$\text{which is } = \frac{1}{\sin \frac{\pi}{n}} - \frac{1}{2 \sin \frac{\pi}{n}} = \frac{1}{2 \sin \frac{\pi}{n}}.$$

Hence when n is even, $\int_x \frac{1}{1+x^n} = \left\{ 0 \times \log_e x^2 \right\} + \frac{2\pi}{n} \cdot \frac{1}{2 \sin \frac{\pi}{n}} \left\} \right\}$

$= \frac{\pi}{n \sin \frac{\pi}{n}}$ accurately between the proposed limits.

Next when n is odd, $\cos \frac{\pi}{n} + \cos \frac{3\pi}{n} + \dots \cos \frac{n-2}{n} \pi$
 $= \frac{1}{2}.$

$\therefore$ coefficient of $\log_e x^2 = \frac{1}{n} \left\{ \frac{1}{2} - \frac{1}{2} \right\} = 0$

$$\begin{aligned} \text{coefficient of } \frac{2\pi}{n} &= \left\{ \sin \frac{\pi}{n} + \sin \frac{3\pi}{n} + \dots \sin \frac{n-2}{n} \pi \right\} \\ &\quad \left\{ -\frac{1}{n} \sin \frac{\pi}{n} - \frac{3}{n} \sin \frac{3\pi}{n} - \dots \frac{n-2}{n} \sin \frac{n-2}{n} \pi \right\} \\ &= \frac{1}{2 \sin \frac{\pi}{n}} + \frac{1}{2} \cot \frac{\pi}{n} - \frac{1}{2} \cot \frac{\pi}{n} \\ &= \frac{1}{2 \sin \frac{\pi}{n}}. \end{aligned}$$

And hence $\int_x \frac{1}{1+x^n} = \frac{2\pi}{n} \cdot \frac{1}{2 \sin \frac{\pi}{n}} = \frac{\pi}{n \sin \frac{\pi}{n}}$ accurately

when n is odd, and the integral taken from $x = 0$ to $x = \infty$.

Hence generally n being a positive integer $\int_x \frac{1}{1+x^n}$ between
the limits of $x = 0$ and $x = \infty$ is $= \frac{\pi}{n \sin \frac{\pi}{n}}.$

Cor. If n be very great $\frac{\pi}{n}$ is very small, and ultimately
 $\sin \frac{\pi}{n} = \frac{\pi}{n}$ when n is increased without limit. Therefore

$\int_x \frac{1}{1+x^n}$ between $x = 0$ and $x = \infty$ when n is also infinite is $= \frac{\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 1$.

Mr. GODWARD also answered this Question.

XIX. QUESTION 549, by Mr. T. S. DAVIES.

Two bodies move in the curve of an ellipse in such a manner as to be always at the same distance from each other; it is required to find the nature of the curve to which the straight line joining the bodies is always a tangent.

Note. This question was proposed as the prize for the last number that was published of the *Gentlemen's Mathematical Companion* (1807), but there was no answer to it printed.

SOLUTION, by Mr. WOOLHOUSE.

Let the co-ordinates of two positions be $u + u'$, $v + v'$ and $u - u'$, $v - v'$, the middle point being uv . Then is the line passing through the two points determined by

$$v - y = \frac{v'}{u'} (u - x) \dots \dots \dots (1).$$

And by the equation of the ellipse

$$\left(\frac{u + u'}{a}\right)^2 + \left(\frac{v + v'}{b}\right)^2 = 1, \quad \left(\frac{u - u'}{a}\right)^2 + \left(\frac{v - v'}{b}\right)^2 = 1,$$

from which we get by addition and subtraction

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{u'^2}{a^2} + \frac{v'^2}{b^2} = 1 \dots \dots \dots (2),$$

$$\frac{uu'}{a^2} + \frac{vv'}{b^2} = 0 \dots \dots \dots (3).$$

We have another condition from the given length of the line; let m be the semi-length and

$$u'^2 + v'^2 = m^2 \dots \dots \dots (4).$$

From (3) we have $\frac{u'}{u} = -\frac{b^2}{a^2} \cdot \frac{u}{y}$ which substituted in (1), it becomes the same as

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = \frac{u}{a^2} x + \frac{v}{b^2} y. \dots\dots\dots(5).$$

The values of u'^2, v'^2 from (3) and (4) are

$$u'^2 = \frac{\frac{v^2}{b^4}}{\frac{u^2}{a^4} + \frac{v^2}{b^4}} m^2, \quad v'^2 = \frac{\frac{u^2}{a^4}}{\frac{u^2}{a^4} + \frac{v^2}{b^4}} m^2,$$

which put in (2) it reduces to

$$\left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right) \left(\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{m^2}{a^2 b^2}\right) = \frac{u^2}{a^4} + \frac{v^2}{b^4}$$

$$\text{or } \left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right) \left\{1 - \left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right)\right\} = \frac{m^2}{a^2 b^2} \left(\frac{u^2}{a^2} + \frac{v^2}{b^2}\right).$$

Assume

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = \sin^2 \phi$$

and we get

$$\frac{u^2}{a^4} + \frac{v^2}{b^4} = \frac{m^2}{a^2 b^2} \tan^2 \phi$$

and from both of these we obtain for u and v

$$u = \frac{a^2 \tan \phi}{\sqrt{(a^2 - b^2)}} \sqrt{(\cos^2 \phi - \frac{m^2}{a^2})}, \quad v = \frac{b^2 \tan \phi}{\sqrt{(a^2 - b^2)}} \sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)}.$$

The equation (5) is thus reduced to

$$\sin^2 \phi = \frac{x \tan \phi}{\sqrt{(a^2 - b^2)}} \sqrt{(\cos^2 \phi - \frac{m^2}{a^2})} + \frac{y \tan \phi}{\sqrt{(a^2 - b^2)}} \sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)}$$

$$\text{or } \cos \phi \sin \phi \sqrt{(a^2 - b^2)} = x \sqrt{(\cos^2 \phi - \frac{m^2}{a^2})} + y \sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)} \dots(6)$$

which is an equation to the line joining the points and involving only one independent variable ϕ . If xy be the point of contact with the curve ϕ only will vary in proceeding to a consecutive position since xy will be the intersection of the lines and common to both. Hence, differentiating with ϕ only, we get

$$(\cos^2 \phi - \sin^2 \phi) \sqrt{(a^2 - b^2)} = - \frac{x \cos \phi \sin \phi}{\sqrt{(\cos^2 \phi - \frac{m^2}{a^2})}} + \frac{y \cos \phi \sin \phi}{\sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)}} \dots(7);$$

and these two last equations (6), (7), give

$$x = \frac{a^2 b^2}{m^2 \sqrt{(a^2 - b^2)}} \cdot \frac{\sqrt{(\cos^2 \phi - \frac{m^2}{a^2})}}{\cos \phi \sin \phi} \left\{ \frac{m^2}{b^2} \sin^2 \phi - \cos^2 \phi \left(\frac{m^2}{b^2} - \cos^2 \phi \right) \right\}$$

$$y = \frac{a^2 b^2}{m^2 \sqrt{(a^2 - b^2)}} \cdot \frac{\sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)}}{\cos \phi \sin \phi} \left\{ \frac{m^2}{a^2} \sin^2 \phi + \cos^2 \phi \left(\cos^2 \phi - \frac{m^2}{a^2} \right) \right\}$$

which fully determine the required curve.

By subtracting the values of u and v we derive

$$x - u = - \frac{a^2 b^2}{m^2 \sqrt{(a^2 - b^2)}} \cot \phi \left(\frac{m^2}{b^2} - \cos^2 \phi \right) \sqrt{(\cos^2 \phi - \frac{m^2}{a^2})},$$

$$y - v = + \frac{a^2 b^2}{m^2 \sqrt{(a^2 - b^2)}} \cot \phi (\cos^2 \phi - \frac{m^2}{a^2}) \sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)};$$

by these we observe that, disregarding algebraic signs, we have always $x < u$ and $y > v$, so that the point of contact xy with the curve is always nearer to the less semi-diameter b and further from the greater semi-diameter a than the middle point uv of the line. Let D be the distance of the contact from the middle point and

$$\begin{aligned} D^2 &= (x - u)^2 + (y - v)^2 = \frac{a^2 b^2}{m^2} \cot^2 \phi \left(\frac{m^2}{b^2} - \cos^2 \phi \right) \left(\cos^2 \phi - \frac{m^2}{a^2} \right) \\ &= \frac{\cot^2 \phi}{m^2} (m^2 - b^2 \cos^2 \phi) (a^2 \cos^2 \phi - m^2) \end{aligned}$$

$$\therefore D = \frac{\cot \phi}{m} \sqrt{(m^2 - b^2 \cos^2 \phi)} \sqrt{(a^2 \cos^2 \phi - m^2)}$$

which admits of a geometrical construction. A concentric and similar ellipse being drawn touching the line in any position at its middle point uv , the value of $\sin \phi$ may be found by dividing any semi-diameter with the similar semi-diameter of the proposed ellipse.

If it be required to compute a series of ordinates of the curve we may take

$$x = \frac{a^2 b^2}{m^2 \sqrt{(a^2 - b^2)}} \cdot \frac{\sqrt{(\cos^2 \phi - \frac{m^2}{a^2})}}{\cos \phi \sin \phi} \left\{ \cos^2 \phi - \frac{m^2}{b^2} (\cos^2 \phi - \sin^2 \phi) \right\},$$

$$y = \frac{a^2 b^2}{m^2 \sqrt{(a^2 - b^2)}} \cdot \frac{\sqrt{(\frac{m^2}{b^2} - \cos^2 \phi)}}{\cos \phi \sin \phi} \left\{ \cos^2 \phi - \frac{m^2}{a^2} (\cos^2 \phi - \sin^2 \phi) \right\},$$

in which we can use $\frac{1}{2} \sin 2\phi$ instead of $\cos \phi \sin \phi$ and $\cos 2\phi$

instead of $\cos^2\phi - \sin^2\phi$. And we must observe that ϕ has two limiting values since we can neither have $\cos \phi < \frac{m}{a}$ nor $> \frac{m}{b}$; these limiting values, when $\cos \phi = \frac{m}{a}$ and $\cos \phi = \frac{m}{b}$, take place when the line is perpendicular to and bisected by the axes of the given ellipse, which points of bisection are vertices to the sought curve.

Answers were also given by Messrs. BAINES, DAVIES, GODWARD, and THOMPSON.

XX. PRIZE QUESTION 530, by ZEPHYR,

A certain curve can be determined from the following property :—

Any five points being denoted by x_1, y_1, x_2, y_2 in order, &c. and the abscissas x_1, x_2, x_3 and x_1, x_3, x_5 being respectively in arithmetical progression, then

$$y_1 - 9y_4 + 16y_3 - 9y_2 + y_1 = 0.$$

FIRST SOLUTION, by Mr. THOMAS STEPHENS DAVIES.

Let $y = u_x$ be the equation of the curve; we have to determine the form of u .

By the conditions of the question we have

$$y_x = u_x; y_2 = u_{x+1}; y_3 = u_{x+\frac{1}{2}}; y_4 = u_{x+2}; y_5 = u_{x+3}.$$

And hence

$$u_{x+3} - 9u_{x+2} + 16u_{x+\frac{1}{2}} - 9u_{x+1} + u_x = 0 \dots\dots\dots(1).$$

Now we have

$$\left. \begin{aligned} u_x &= (1 + \Delta)^0 u_x \\ u_{x+1} &= (1 + \Delta)^1 u_x \\ u_{x+\frac{1}{2}} &= (1 + \Delta)^{\frac{1}{2}} u_x \\ u_{x+2} &= (1 + \Delta)^2 u_x \\ u_{x+3} &= (1 + \Delta)^3 u_x \end{aligned} \right\} \dots\dots\dots(2).$$

Insert (2) in (1) and efface the cancelling terms : then we get

$$\left\{ \frac{9}{388} \cdot \Delta^4 - \dots + \dots \text{ad inf.} \right\} u_x = 0 \dots\dots\dots(3).$$

Now since this is to be independent of x , which can only take place universally by each term of the infinite series (3) becoming separately zero.

That is (putting x to stand successively for all positive integers) when

$$\Delta^{r+3} u_r = 0,$$

or in other words, the curve itself is the cubical parabola, the apollonian parabola or the straight line, its equation being

$$y = A + Bx + Cx^2 + Dx^3.$$

SECOND SOLUTION, by Mr. WILLIAM GODWARD,

Nautical Almanac Offices, Somerset House, London.

Let $y = \phi(x)$ be the equation of the curve; and as the values $x_1 = x - 3h$, $x_2 = x - h$, $x_3 = x$, $x_4 = x + h$, $x_5 = x + 3h$ answer the assumed progressions, the values of y_1, y_2, y_3, y_4, y_5 , will be denoted by the functions $\phi(x - 3h)$, $\phi(x - h)$, $\phi(x)$, $\phi(x + h)$, $\phi(x + 3h)$. Expand these functions by Taylor's Theorem and substitute the results in the proposed equation, and it will be found that the first three differential coefficients

vanish, but not the fourth, viz. $\frac{d^4 \phi x}{dx^4}$. Hence to make all

vanish, as the equation must be generally fulfilled, the process of differentiation must terminate with $\frac{d^4 \phi x}{dx^4} = 0$, and this will

always be the case when ϕ characterizes any algebraic equation of the third degree.

THIRD SOLUTION, by Mr. WOOLHOUSE.

According to the assumed conditions x_1, x_2, x_4, x_5 are always in arithmetical progression and x_3 is the mean, being half way between x_2 and x_4 . Now y being supposed a function of x , the differences of y_1, y_2, y_4, y_5 , in consequence of the above progression, will observe a regular law; if Δ_2, Δ_4 , &c. be the mean 2d, 4th, &c. differences opposite y_2, y_4 , we shall, by the

formula for interpolation, have $y_3 = \frac{1}{2} (y_2 + y_4) - \frac{1}{8} \Delta_2 +$

$\frac{3}{8.16} \Delta_4$, &c.; and this will furnish the proposed equation if

we substitute for Δ_2 its value $\frac{1}{2}(y_1 + y_3 - y_2 - y_4)$ and neglect all succeeding terms. Hence we see that the equation will be rigorously fulfilled when the fourth differences of a continued series vanish, or when y is expressed by any algebraic form not involving x higher than the third degree.

Messrs. THOMPSON, of Newcastle; and — MASON, of Scoulton, answered this question.

A new Solution of Question 531, and some collateral ones, by Mr. THOMAS STEPHENS DAVIES, Bath.

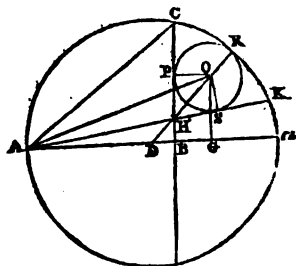
Some years ago my attention was directed to the figure which the ancient Geometers denominated the *Αρβηλον*, or "Cobler's Knife;" and I was led to observe many curious properties of it, which, though they were probably known to the Greeks, are yet not recorded in any of their writings which are now known to exist. I had thought of making a distinct little treatise on the figure and the very little that is known of its history; including not only the properties I had already obtained, but such others as a renewed study of the figure might disclose to me. On this account, I allowed the time of furnishing a solution to the Repository Question to pass by. However, as other occupations have compelled me to defer it so long, that there would not be time to enter upon the investigation in time for the present number, I have thought it better to furnish a few extracts from my notes, that bear a more immediate reference to the class of questions amongst which the proposer seems to have amused himself. If the Editor of the Repository can find me room at the end of the solution-sheet for these, he will confer one little addition to the many great obligations which his kindness has laid me under.

It is possible I may resume the subject more in detail at a time of more leisure than I can now command.

Let a circle AAa and a straight line BP be given in position, and let the diameter Aa of the circle be perpendicular to BP . Then, if any circle PRS be drawn, to touch the given straight line and circle, and a tangent AS be drawn to this new circle from the alternate extremity of the diameter Aa , that tangent will be of a given length, and the locus of the points of contact is a given circle.

Case 1. Let the line PB meet the given circle AAa , and the circle PRS touch it *internally*.

Let O , the centre of PRS , be joined to P, R, S ; prolong RO to meet Aa in D ; and draw OG perpendicular to Aa . Then D is the centre of ACA . Hence $AD = DB$, and $PO = OR = OS = BG$.



$$\begin{aligned}\text{But, Eucl. II. 13. } AO^2 &= AD^2 + DO^2 + 2AD \cdot DG \\ &= AD^2 + (AD - BG)^2 + 2AD(BD + BG) \\ &= 2AD^2 + 2AD \cdot DB + BG^2.\end{aligned}$$

$$\begin{aligned}\text{Hence } AO^2 - BG^2 &= 2AD(AD + DB) \\ &= Aa \cdot AB \\ &= AC^2 \dots \dots \dots (1).\end{aligned}$$

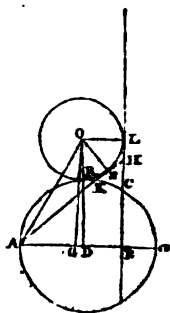
$$\text{But } AO^2 - BG^2 = AO^2 - OS^2 = AS^2 \dots \dots \dots (2).$$

And hence from (1, 2) we have $AS = AC$: and the locus of s a circle to centre A and radius AC . Q. E. 1^{mo}. D.

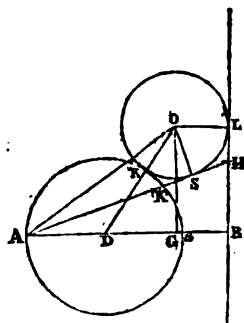
Case 2. Let PB meet the circle as before, but the circle PRS touch ACA *externally*.

$$\begin{aligned}AO^2 &= AD^2 + DO^2 - 2AD \cdot DG \\ &= AD^2 + (AD + BG)^2 - 2AD(CG - DB) \\ &= 2AD(AD + DB) + BG^2 \\ &= \&c. \text{ as before.}\end{aligned}$$

Q. E. 2^{ndo}. D.



Case 3. Let the line BL be entirely *without* the given circle. The contact is here of necessity *external*, where a tangent is drawn from A to LRS .



angle. Hence, by our theorem above,

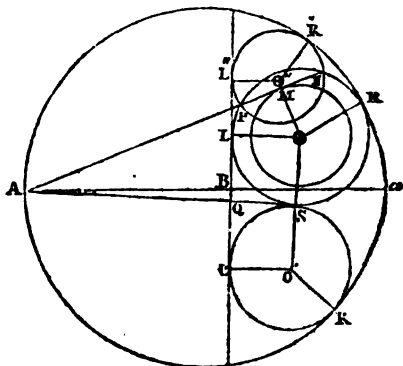
$$AC - AB = AD - AB = BD = \text{radius } OL.$$

Cor. 5. We can also inscribe a second circle in the space to touch any one already described, or which shall cut off from the first a given segment.

To touch :

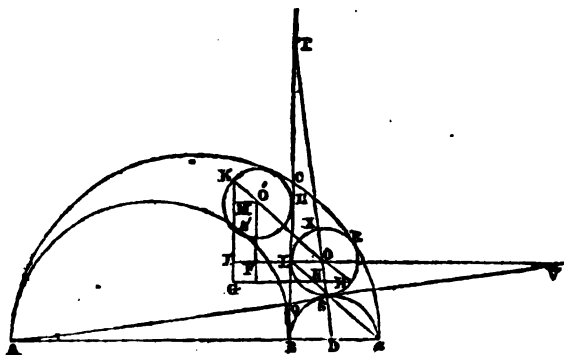
Let LSR be the given circle. Draw the tangent AS cutting the given line in Q . Draw OS and prolong it. Make $QL' = QL$ and draw $L'O'$ parallel to Aa , till it meets OS at O' . Then O' is the centre, and $O'S$ the radius of the circle sought.

NOTE. The *problematical part of Repository Question is a particular case of this.*



To cut off a given segment : draw any chord in the given circle to cut such a segment, and describe a concentric circle to touch it. Draw a tangent to this last circle from A , and let M be its point of contact, and Q' its intersection with the given line. Set off $L''Q' = LQ'$ and draw $L''O''$ parallel to Aa to meet OM in O'' . Then O'' is the centre and $O''L''$ is the radius of the circle sought.

II. Let bsa be a semicircle described on the segment ab of the diameter Aa of the circle ACA ; and LSQ be a circle described within the remaining space $CBsAR$. Then its diameter is a fourth proportional to Aa , AB , BA ; or a third proportional to Aa , BC .



For join AS and prolong it to meet LV in V (L being as before the point of contact, and LV parallel to Aa), and join LS, sa. Prolong also BC and DS to meet in T.

Then, as is well known, Lsa is a straight line, and LQ = QR (I. 3.), and the triangle ABQ equal in all respects to LVQ. Also the right angled triangles TBD, ASD having the common angle ADT and the equal sides BD, DS, we have BT = SA. Hence by parallels

$$AG : AS :: LV (= AB) : VS, \text{ or}$$

$$AG : AB :: AS : VS :: DS : OS :: 2DS : 2OS :: Ba : SX.$$

$$\therefore SX = \frac{AB \cdot AB}{AA} = \frac{BC^2}{AA}. \quad \text{Q. E. D.}$$

In like manner if O'S'L' be described in the corresponding space on the other side of AC we shall find its diameter a fourth proportional to Aa, AB, BA, or a third to Aa and BC.

Cor. 1. Hence the two inscribed circles are equal: which is the first part 521, Repos. and the 11th Question of the series in the Gentleman's Diary, 1831-2. See also Pappi Coll. lib. IV. pp. 12-18: Archimedes, Op. Omn. Ox. p. 357; the Supplement to Borelli's Edition of the Conics of Apollonius, pp. 396-9; and Bland's Geometrical Problems, page 290.

Cor. 2. Since by similar triangles

$$AB : AQ :: AS : SD, \text{ or doubling}$$

$$AB : BL :: AS (= BT) : AB$$

$$\therefore AB \cdot Ba = BL \cdot BT = BC^2 \dots \dots \dots (1)$$

But BT = AS = AC, and therefore

$$\left. \begin{aligned} BL &= \frac{BC^2}{AC} \text{ and in like manner} \\ BL' &= \frac{BC^2}{AC} \end{aligned} \right\} \dots \dots (2).$$

$$\begin{aligned} \text{Hence } (BL \pm BL') &= BC^2 \left\{ \frac{1}{AC} \pm \frac{1}{AC} \right\} \\ &= \frac{BC^2 \{AC \pm AC\}}{AC \cdot AC} \\ &= \frac{BC}{aA} \{AC \pm AC\} \dots \dots \dots (3) \end{aligned}$$

Prolong OO' to H and R; draw OI, HG parallel to Aa, and BC, O'I parallel to BC. Then OF = SX, and we find

$$\begin{aligned}
OO'^2 &= \frac{BC^2}{Aa^2} \{AC - aC\}^2 + \frac{BC^4}{Aa^2} \\
&= \frac{BC^2}{Aa^2} \{AC^2 - 2AC \cdot aC + aC^2 + BC^2\} \\
&= \frac{BC^2}{Aa^2} \{Aa^2 - 2Aa \cdot BC + BC^2\} \\
&= \frac{BC^2}{Aa^2} \{Aa - BC\}^2.
\end{aligned}$$

$$\text{Hence } OO' = \frac{BC}{Aa} \{Aa - BC\}$$

$$= BC - \frac{BC^2}{Aa}$$

$$\text{or } BC - OO' = \frac{BC^2}{Aa} = SX.$$

Cor. 3. We hence learn, too, that $KH = BC$: and hence as the circle upon BC is equal to the space $AS'BaC$ (Archim. O. O. p. 256. Pappi. Coll. Math. lib. IV.) we have the circle upon KH equal to the same space as announced by the proposer of 531, Repos.

Cor. 4. Again, $BL^2 = \frac{BC^4}{aC^2}$ and $BL'^2 = \frac{BC^4}{aC^2}$, we have

$$BL^2 + BL'^2 = BC^4 \left\{ \frac{1}{aC^2} + \frac{1}{aC^2} \right\} + \frac{BC^4 (aC^2 + aC^2)}{aC^2 \cdot aC^2} = BC^4.$$

Cor. 5. $BL \cdot AC = BC^2 = BL' \cdot aC$, therefore

$$BL : BL' :: aC : AC :: AB : aB.$$

Several of these steps may upon the face of them appear to belong to algebra rather than geometry, but a little closer examination will shew that they are only abridged modes of writing processes which are purely geometrical. My object has been to save both room and time.

(To be concluded in our next number.)

ARTICLE IV.

Obituary.
SIR JOHN LESLIE.

Since the publication of our last number, British science has lost one of its brightest ornaments, by the death of Sir John Leslie, professor of Natural Philosophy in the University of Edinburgh. This very distinguished philosopher was born in the County of Fife, in the year 1766. His parents rented a small farm, and he, himself, appeared destined to a life of agricultural labour; but the uncommon talents for calculation, and attachment to mechanical pursuits, which he displayed in his earliest years, attracted the notice of some intelligent individuals, who kindly encouraged his efforts, and by whose advice he was withdrawn from his humble occupation, when he was about twelve years of age, and placed at a neighbouring village school. The rudiments of a good education are within the reach of all the Scottish peasantry; and young Leslie, in the course of a few years, was prepared to be entered a Student in the University of St. Andrews. In this ancient seminary, where a taste for the mathematical sciences has been carefully cherished since the days of the Gregorys, he had the advantage of receiving the instruction of Mr. West, the author of an excellent elementary course of mathematics, a man of original and inventive genius, and, after Dr. Matthew Stewart, one of the greatest masters of the ancient geometry, whom Scotland has produced. From this meritorious individual, who has never had justice rendered to his talents, and who, perhaps from that ignorance of the arts of advancement which is so frequently the lot of the secluded student, never succeeded in surmounting the obstacles of an unfavourable position, Leslie received an impulse to which he owed, in a great degree, all his future success. After a few years residence in St. Andrew's, Mr. Leslie, accompanied by his friend and fellow-student, Mr. Ivory, who now occupies the first place among the mathematicians of this country, removed to Edinburgh, to attend the lectures of Professor John Robison, then in the zenith of his reputation. Robison was

a man of a vigorous and powerful intellect, a good mathematician, and possessed a vast store of practical knowledge which he communicated with extraordinary eloquence. But, unfortunately, his taste in geometry was not very correct: in the demonstrative part of his course he inclined rather to depreciate than to adopt the systematic methods of the modern analysis; and Mr. Leslie accordingly missed the opportunity of acquiring the use of that master-key of the physical sciences, at the only period of life when, perhaps, his mind was sufficiently unoccupied with projects of discovery, to devote the necessary labour to the study of pure mathematics. Having finished the curriculum of philosophy at the University of Edinburgh, Mr. Leslie removed to London as a literary adventurer, carrying with him recommendatory letters from Dr. Adam Smith, the celebrated Author of the *Wealth of Nations*, and others of the Scottish Literati, with whom he had formed a friendship during his residence in Edinburgh. It is said that the first literary employment in which he engaged was that of writing or correcting notes for an edition of the Bible then publishing in numbers; but he soon found a more congenial occupation in the translation of Buffon's *Natural History of Birds*, which extended to nine octavo volumes. This produced him a considerable sum of money, which, as he knew well how to improve it, formed the nucleus of that pecuniary independence which he was fortunate enough to attain in early life. Shortly after completing this undertaking he went to the United States of America, as tutor to a Mr. Randolph; and, on his return, he undertook the same office to Mr. Thomas Wedgwood, with whom he travelled over various parts of the continent. In 1804, he published his *Experimental Inquiry into the Nature and Propagation of Heat*. The following year, on the death of Professor Robison, and the translation of Mr. Playfair to the chair of Natural Philosophy, he was appointed Professor of Mathematics in Edinburgh. This appointment was violently opposed by a party in the Scottish church, on the ostensible ground of his having advanced, in his *Experimental Inquiry*, some metaphysical opinions in regard to cause and effect, that were supposed to have a sceptical or heterodox tendency. The real cause of the opposition, however, was supposed to proceed from the interested views of a faction of the Edinburgh clergy, who, wishing to monopolize the literary professorships of the University, were determined, on any grounds, to resist the appointment of a layman. Unable to oppose him on the score of talent, they stooped to assail his private character and religious principles.

The affair was carried before the ecclesiastical courts, and made much noise at the time. Happily for the cause of science, and the reputation of the country, the intrigues of his opponents were defeated, and Mr. Leslie was allowed to take peaceable possession of his chair. In 1819, he succeeded Mr. Playfair in the chair of Natural Philosophy, a department for which he was pre-eminently qualified. This appointment he continued to hold till the time of his death, which took place at his seat in Fifeshire, on the 5th of November, 1832, after a few days' illness. He had been for several years a corresponding member of the French Institute, and was knighted about six months before his death.

As a mathematician Sir John Leslie did not rank high. He possessed, doubtless, an extensive knowledge of elementary mathematics; but with the higher departments of the new analysis he had a very slender acquaintance, and he was altogether unable to wield it successfully as an instrument of investigation. This arose from no want of the peculiar talent required for mathematical studies; on the contrary, he possessed that talent in a very eminent degree. He seized on the finer and least obvious relations of figure and quantity with astonishing rapidity; he has given decisive proofs of a fertile and inventive genius in geometry; and he was a warm admirer, though no very successful imitator, of the exquisite models left us by the Greeks. We would ascribe his deficiency in pure mathematics to the want of early initiation into the methods of the modern analysis; and more especially to an impatience of prolonged labour, and a strong desire of popular applause, which invariably led him to pursue those paths in which he might hope for most immediate fame. Hence, he never gave to the study of the mathematical calculus that close and sustained attention, without which, in the present state of science, even the finest genius has little chance of attaining to eminence. His *Elements of Geometry*, the first edition of which was published a few years after his appointment to the mathematical chair, was a work well conceived, but executed with little care, and strikingly deficient in that simple elegance and logical precision in which consists at once the excellence and the charm of the geometrical writings of the ancients. The ambitious, oratorical, and figurative style in which he was ever prone to indulge, assorted ill with a work of accurate science, and called forth some severe animadversions from his Colleague, Mr. Playfair, in the *Edinburgh Review*. In the subsequent editions this work received great improvements, and may now be regarded as a valuable text-

book. Indeed we cannot help thinking, with the author himself, that it is chiefly owing to the indolence or prejudices of teachers, that it is not more generally introduced as a class-book in our schools. The substitution of such a work for the more tedious and verbose, though perhaps more strictly logical, Elements of Euclid, especially where a practical course only is aimed at, would save the student much time and irksome labour. The *Treatise on Trigonometry*, was perhaps superior to any production of the same nature that had previously appeared in our language; and the Notes and Illustrations abound with interesting miscellaneous information relative to geodesy, and other practical applications. The *Tract on the Geometrical Analysis*, originally published in the same volume, was a valuable addition to the Elements. It contained a selection of highly interesting geometrical problems, taken chiefly from Pappus, and the writings of Dr. Simson; and was very favourably received by the students of ancient geometry, both in this country and on the continent, where it was translated into French and German. Its merit, however, lay rather in the introduction of a systematic treatise on that elegant branch of analysis into the elements, than in the manner in which it was executed. It added nothing to the stock of known truths; and the demonstrations were not always the best that he might have selected. In some instances he enunciated only particular cases of propositions which had already been generalized, and demonstrated with greater elegance in the Diaries, and other periodical scientific publications. A better treatise might easily be extracted from the early numbers of this Repository. But Leslie never took the trouble to examine such sources. He was indeed, very little versed in the history of the solution of problems. Thus, in his article *Angle* in the Supplement to the *Encyclopedia Britannica*, he ascribes to Newton a proposition which is demonstrated in the Collections of Pappus; and at page 328 of his Geometry, in speaking of the problem to inscribe a regular seventeen-sided polygon in a circle, he says "the investigation of the principle is rather intricate, and the constructions to which it would lead are, in every case, unavoidably and most excessively complicated;" whereas, if he had looked into volume IV. of the Repository, he would have found a construction scarcely more complicated than that of the pentagon. These may be thought trifling faults, but in a professor, and writer on the elements, they cannot be overlooked. The treatise on the *Conic Sections* is strongly marked with the author's peculiarities, exhibiting abundant proofs of acuteness and vigorous thinking; but

disfigured by a loose and inaccurate diction. We scarcely recognize the severe features of the ancient geometry under the garb in which he has thought fit to deck it out. Nevertheless it is a work of intrinsic value, and may be studied with great advantage, particularly as an introduction to the *Principia*. Indeed, many of the propositions, in the second part, are taken, almost literally, from that immortal production. To the *Conic Sections* is appended a short tract on the *Geometry of the Higher Curves*, the object of which seems to have been to prepare his students for the course of Natural Philosophy, by making them acquainted with a few of the leading properties of those curves which occur most frequently in mechanics. But we consider this as the least meritorious portion of the two volumes of his mathematical course. The attempt to explain the properties of the higher curves by the aid of plane geometry alone, is but a clumsy expedient to supersede the use of the differential calculus, which, after all, cannot be dispensed with; and it is, surely, a bootless labour to toil through a page of words, in order to establish some relation that may be demonstrated, with infinitely greater clearness, by a single line of the most elementary calculus. It is like attempting to storm a fortress with a catapult and a battering-ram, instead of applying the more efficient engines of modern artillery. After all, it is only a very few of the most obvious properties of the higher curves which can be demonstrated in this manner. A third volume, containing a *Treatise on the geometry of planes and solids*, and embracing the theory of Perspective, and Spherical Trigonometry, was frequently announced, but has never appeared.

But it is as a cultivator of experimental and physical science that Sir J. Leslie is chiefly distinguished; and it is on his discoveries in physics, and the exquisite instruments he invented for assisting physical researches, that his fame will ultimately rest. In this department he evinced great originality of genius, and powers of invention which have rarely been surpassed; great fertility in devising experiments, as well as acuteness and rapidity in combining the results and pursuing them to their remotest consequences. His physical researches, however, are not marked by that cautious mode of induction which so conspicuously distinguishes the experimental inquiries of Newton, of Cavendish, and Wollaston. He sometimes betrays an impatience of experimental investigation, and adopts conclusions not warranted by the phenomena. In all his speculations there is an evident straining after novelty—a desire to appear out of the beaten track; and he did not hesitate to adopt the most extravagant

idea, provided it was calculated to surprise or astonish. Hence, we find him gravely putting forth such hypotheses as these,—that the interior of the earth is filled with *light*, that the ocean rolls on a bed of *air*, &c. Simplicity formed no part of his scientific character. He had dipped into the writings of the Alchemists, from whom his philosophical ideas and modes of expression had acquired a tinge; and whom he resembled in affecting to surround his discoveries and speculations with a glare of mystery. His first and most elaborate philosophical work was the *Experimental Inquiry into the Nature and Propagation of Heat*. This raised him at once to the highest rank among the experimental philosophers of the day, and procured for him the Rumford medal from the Royal Society. Indeed, whether we regard the ingenuity of the train of investigation pursued in it, the importance and novelty of the results, or the great additions to the then known facts, respecting the laws of heat which it brought to light, we must admit that it is deserving of the highest commendation. It is a work, however, clouded with many faults. It is defective in perspicuity and arrangement, to a degree that renders it difficult and irksome to read. It is filled with discussions and speculations often remotely connected with the subject; and the theory which it attempts to establish, is, to say the least, extremely questionable. Besides, it is to be admired rather for its ingenuity than for the accuracy of the results. When a precise fact is wanted, a reference is not made to Leslie, but to Dulong and Petit. *The short account of Experiments and Instruments depending on the Relations of Air to Heat and Moisture*, forms a supplement to the *Experimental Inquiry*. Like its predecessor, it abounds with acute views and interesting speculation; but it added little or nothing to his previous discoveries. With regard to his instruments, they are all of them simple though ingenious modifications of his *Differential Thermometer*; but this is really an instrument of so much value in all delicate inquiries connected with the variations of heat, that had he been known for nothing else, he would have been entitled, by its invention alone, to be ranked among the benefactors of physical science. We ought not however to disguise the fact, that the invention of this excellent instrument has been disputed, though we know not if on good grounds. At all events he had the merit of bringing it into use. He was also the author of some excellent papers on physical subjects in the Supplement to the *Encyclopædia Britannica*, among which the articles *Climate* and *Meteorology* are productions of conspicuous merit. Of his *Elements of Natural Philosophy* it would, perhaps, be unfair to judge severely, inasmuch as it is an unfinished production. It was intended to

be completed in three volumes, of which one only has appeared. If we regard it as containing the substance of his academical lectures, it certainly does not tend to give us a very high idea of the advanced state of the accurate sciences in the University of Edinburgh. The little geometry employed in it is of the simplest and most elementary kind. It carries us back to the days of Musschenbroeck, Desaguliers, and Keill; and, making allowance for the improved state of physical knowledge, is in every respect inferior to Maclaurin's account of Newton's Principia. It is rather a collection of facts than a scientific exposition of the principles of natural philosophy; and even in this respect, it hardly comes up to the *Géométrie des Arts et Metiers* of Dupin. The last work which he gave the world is the *Dissertation on the Progress of the Physical Sciences*, in the edition now publishing of the *Encyclopædia Britannica*. It was intended as a continuation of the work of Playfair on the same subject, which was brought down only to the death of Newton; and consequently embraced the splendid discoveries that have been made in the various departments of physics from that time to the present day. But this was a task for which Mr. Leslie was in many respects not well qualified. The ardour and impatience of his temperament prevented him from bestowing that minute attention on the examination of the labours or pretensions of others which is indispensably necessary in order to form a correct and impartial judgment; and with the recent improvements in some of the most important departments of mathematical physics, he was in a great measure unacquainted. Accordingly the work is frequently sketchy and superficial; and has the appearance of having been written rather from ancient recollections than from a recent and careful study of the subjects of which he undertakes to treat. Altogether it suffers by a comparison with the more elaborate and finished dissertations with which it is associated.

As a teacher of science, there may be some doubt as to the class in which Sir J. Leslie ought to be ranked. His lectures, like most of his writings, were strikingly deficient in method and order. He was constantly travelling out of his direct course; and he fell into the common fault of men of genius in supposing his auditors were gifted with the same rapidity of conception as himself. His ideas were sometimes fantastic; and, when he attempted a sublime flight, his descriptions were almost burlesque. Yet his lectures possessed no ordinary merit. They abounded with useful and interesting information, and were deeply impressed with that captivating charm which genius alone can confer, and for the want of which no plodding, pains-taking preparation

can compensate. To the few who could follow him they were eminently instructive. Whether the interests of science are better promoted by carrying forward some half-dozen superior minds to the utmost stretch of their capacity, or by laboriously attempting to instil a trifling portion of elementary knowledge into the reluctant mass of dull mediocrity, is a question which we shall leave our readers to decide for themselves. In his private character, though he incurred the imputation of avarice, he exhibited many stirring good qualities of disposition. He manifested great kindness of feeling towards such of his pupils as had distinguished themselves by talents or assiduity; and, when opportunity offered, was ever ready to use his best efforts to promote their views in after life. In the collection of instruments for the purpose of illustrating his course he expended much labour, and considerable sums of money; and he left a sumptuous apparatus of which his successors will have the benefit. It has been truly said, that in him the University of Edinburgh has lost its only European-name.

NOTICES.

THE PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY OF
LONDON, 1832.

Contains the following Papers relating to Mathematics.

Part I.—1. Researches in Physical Astronomy; by J. W. Lubbock, Esq. 2. On the Tides; by the same. 3. On an Inequality of long period in the motion of the Earth and Venus; by G. B. Airy, Esq. 4. On the Theory of Perturbations of the Planets; by James Ivory, Esq. 5. Researches in Physical Astronomy; by J. W. Lubbock, Esq. Part II.—6. Researches in Physical Astronomy; by J. W. Lubbock, Esq. 7. On the correction of a pendulum for the Reduction to a vacuum; together with Remarks on some anomalies observed in pendulum experiments; by Francis Baily, Esq. 8. Account of the Magnetic Experiments made on the Western Coast of Africa, 1830; by Commander Edward Belcher. 9. On the Water Barometer erected in the Hall of the Royal Society; by J. F. Daniell, Esq. 10. Hourly observations on the Barometer; with experimental investigations into the phenomena of its periodical oscillations; by James Hudson, Esq. 11. Note on the Tides in the Port of London; by J. W. Lubbock, Esq. 12. Researches in Physical Astronomy; by the same Gentleman.

EDINBURGH TRANSACTIONS.

Our last notice (p. 59) closed with Vol. X. Since that list was made up, three half volumes more have appeared.

Vol. XI. Contains, connected with Mathematics and Mathematical Physics, the following Papers.

1. Description of some remarkable effects of Unequal Refraction observed at Bridlington Quay, in the summer of 1826; by the Rev. William Scoresby. 2. On the Construction of Polyzonal Lenses, and their combination with plane mirrors for the purposes of illuminating light-houses: by Dr. Brewster. 3. Account of a remarkable peculiarity in the construction of Glauberite, which has one axis of Double Refraction for Violet and two axes for Red Light; by Dr. Brewster. 4. Experimental enquiries concerning the laws of Magnetic Forces; by W. Snow Harris, Esq. 5. On certain new Phenomena of Colour in Labrador Felspar, with observations on the nature and cause of its changeable tints; by Dr. Brewster. 6. Notice regarding a Time-keeper in the Hall of the Royal Society; by John Robison, Esq. 7. Remarks on a Meteorological Journal kept at Carlisle, by the late Mr. William Pitt, for 26 years; by Dr. Barnes.

Vol. XII.—1. Account of observations made in Scotland on the Distribution of Magnetic Intensity; by James Dunlop, Esq. 2. Notice of an Autograph Manuscript by Sir Isaac Newton, containing some Notes upon the Third Book of the Principia, and found among the papers of Dr. David Gregory, formerly Savilian Professor of Geometry in Oxford; by Dr. James Crauford Gregory. 3. Inquiry into the Geometrical Character of the Hour-Lines upon the Antique Sun-Dials; by T. S. Davies, Esq. 4. On a new Analysis of Solar Light, indicating three primary colours, forming coincident spectra of equal length; by Dr. Brewster. 5. Notice regarding some Experiments on the vibrations of Heated Metals; by Arthur Trevelyan, Esq. 6. On the Horary Oscillations of the Barometer near Edinburgh, deduced from 4410 observations; with an enquiry into the law of Geographical Distribution of the Phenomena; by James David Forbes, Esq. 7. A new species of coloured Fringes produced from Reflexion between the lenses of Achromatic or compound Object Glasses; by Dr. Brewster. 8. On the Equations of Loci traced upon the surface of the Sphere, as expressed by Spherical Co-ordinates; by Thomas Stephens Davies, Esq.

TRANSACTIONS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY.

The first half of Vol. IV. has been published since our last, and contains the following papers relative to Mathematics, &c.

1. On the General Equation of Curves of the second Degree; by Augustus De Morgan, Esq. 2. On the nature of

Light in the two rays produced by Double Refraction of Quartz; by G. Airy, Esq. 3. On the Resolution of Algebraical Equations; by R. Murphy, Esq. 4. Mathematical Exposition of some of the Leading Doctrines in Mr. Ricardo's Principles of Political Economy and Taxation; by William Whewell, Esq. 5. An addition to his Paper on the Nature of the Light in Quartz; by G. B. Airy, Esq.

JOURNAL DE L'ÉCOLE POLYTECHNIQUE.

Since our last notice of this valuable work two more Cahiers have been published, the contents of which are as follows.—

CAHIER 20.—1. A Memoir upon the general equations of equilibrium and motion of bodies, solid, elastic, and fluid; by M. Poisson. 2. Memoir upon the system of values which may give to divers elements, determined by a great number of observations, such that the greatest of all errors, abstraction made of the sign, shall become a minimum; by M. Cauchy. 3. Memoir upon the determination of the orbits of planets and comets; by M. J. Binet. 4. Memoir upon the resolution of indeterminate equations of the first degree in whole numbers; by M. J. Binet. 5. Memoir upon the integration of a certain class of equations of partial differentials, and of the phenomena of which the integration discloses the laws in mathematical physics.

CAHIER 21.—1. Memoir upon several questions in Geometry and Mechanics, and upon a new kind of calculus for resolving these equations; by M. Liouville. 2. Memoir upon the Calculus of differences; by the same. 3. Memoir upon the integration of a partial differential equation of the Second Order; by the same. 4. Formulæ relative to the motion of the Ball in the interior of the Cannon; by M. Lagrange. 5. On the Curvature of Surfaces; by M. Poisson. 6. On the influence of the moment of inertia in balancing the steam engine, and of its mean velocity upon the regularity of the rotatory motion given by the piston to the fly-wheel; by M. G. Coriolis. 7. Memoir upon the principle of living forces in instances relative to machinery; by the same. 8. Memoir of Descriptive Geometry, upon the Construction of Tangents in a multiple point, to a curve, either plane or of double curvature; by M. Théodore Olivier. 9. Memoir upon the general equations of the propagation of heat in solid bodies of which the conductivity is not the same in all directions; by M. J. C. Duhamel.

ARTICLE V.

*Solutions to Questions Proposed in No. XXIII.*I. QUESTION 551, by Mr. JOHN BAINES, *Thornhill.*

On what two days of the year in latitude $52^{\circ} 14' 19 \cdot 6''$ is the sun's altitude $25^{\circ} 39' 32''$ at the middle of the time between sun-rise and noon, disregarding refraction, semi-diameter and parallax ?

SOLUTION, by Mr. BAINES, *the Proposer.*

Let 2θ represent the diurnal arc, l the latitude of the place, d the sun's declination, and $a =$ its altitude; then, by spherics and the nature of the question, we have

$$\cos 2\theta = \mp \tan l \tan d \dots (1),$$

$$\text{and } \cos \theta = \sin a \sec l \sec d - \tan l \tan d \dots (2).$$

Subtracting (1) from (2), and dividing, we have

$$\sec d = \frac{\cos \theta - \cos 2\theta}{\sin a \sec l} \dots (3).$$

$$\text{But } \sec^2 d - 1 = \tan^2 d = \frac{(\cos \theta - \cos 2\theta)^2}{\sin^2 a \sec^2 l} - 1 \dots (4).$$

$$\text{From (1), we have } \tan^2 d = \frac{\cos^2 2\theta}{\tan^2 l} \dots (5).$$

Equating (4) and (5),

$$\frac{\cos^2 2\theta}{\tan^2 l} = \frac{\cos^2 \theta - 2 \cos \theta \cos 2\theta + \cos^2 2\theta}{\sin^2 a \sec^2 l} \dots (6).$$

Or substituting the numerical values of a and l , and reducing,

$$14 \cos^4 \theta - 20 \cos^3 \theta - 9 \cos^2 \theta + 10 \cos \theta + 1 = 0;$$

this solved gives $\cos \theta = \sqrt{\frac{1}{2}}$, or $\theta = 45^{\circ}$; hence $d = 0^{\circ}$, and the times required are the vernal and autumnal equinoxes. Besides the above, the equation has another serviceable root, but this will be best obtained by substituting $\frac{1}{2}(1 + \cos 2\theta)$ for $\cos^2 \theta$, which reduces it to

$$(49 \cos^2 2\theta - 130 \cos 2\theta - 175) \cos^2 2\theta = 0,$$

of which the roots are $\cos 2\theta = 0$, giving the same

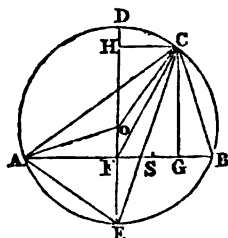
answer as above, and $\cos 2\theta = \frac{65 \pm 20\sqrt{11}}{7}$; the negative value gives $\cos 2\theta = -0.1903566$, or $2\theta = 100^\circ 58' 25''$; hence $\tan d = \frac{\mp \cos 2\theta}{\tan l} = \frac{13\sqrt{15} - 4\sqrt{165}}{7} = 0.1474496$, that is, $d = 7^\circ 58' 44''$ = the declin. N., answering to April 10, and September 2.

II. QUESTION 552, by Mr. W. GODWARD, London.

Given the vertical angle, the rectangle of the segments of the base by the contact of the inscribed circle, and the difference of the segments of the base by a perpendicular from the vertical angle, to determine the plane triangle.

SOLUTION, by Mr. GODWARD, the Proposer.

Let $\triangle ACB$ be the plane triangle circumscribed by a circle whose centre is O . Draw the diameter DE , perpendicular to AB , CH to DE , and CG to AB . Join AO , CO , AE , CF , CE , and let s be the contact of the inscribed circle.



Since the $\angle FAE = ECB = \frac{1}{2} \angle ACB$, and $\angle AOE = \angle ACB$, the triangles FAE and AOE are given in species; hence the ratios $AF : FE$ and $AF : FO$ are each given, and therefore the ratio $FE : FO$ or its equivalent $FE.FH : FO.FH$ becomes known. But $FE.FH = AS.SB$ is given, and consequently $FO.FH$ and $AF.FH$. Again $CF^2 - AF^2 = CH^2 + FH^2 - AF^2$, but $CH^2 = CO^2 - HO^2$, $FH^2 = HO^2 + FO^2 + 2HO.FO$ and $AF^2 = AO^2 - FO^2 = CO^2 - FO^2$; whence $CF^2 - AF^2 = 2FO^2 + 2HO.FO = 2FO(FO + OH) = 2FO.FH$ a given rectangle, from which deducting the given square CH^2 , the remainder $FH^2 - AF^2$ is known; but $AF.FH$ being also given, AF and FH are each given, and thence the required triangle.

III. QUESTION 553, by Mr. J. H. SWALE, Liverpool.

Theorem. The diameter DE cuts the chord AB perpendicularly at F , and O is the centre of the circle: describe semicircles

upon the segments FD, FE, FQ, and draw the tangents CT, TI, CV from any point C in the periphery of the segment AEB. Then

$$1^{mo} (AC + CB)^2 = 4CT^2$$

$$2^{do} (AC - CB)^2 = 4CI^2$$

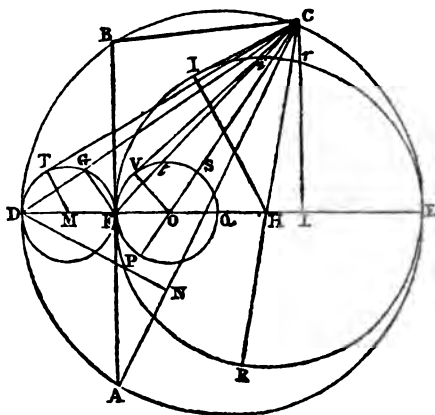
$$3^{ls} AC^2 + CB^2 = 4CV^2$$

$$4^{to} 4AC \cdot CB = 2CT^2 - 2CV^2.$$

SOLUTION, by Mr. JOHN BAINES, Thornhill.

Case 1. Join CD, cutting the circle DTF in G, then FG, being joined, is perpendicular to CD. Now by the property of the circle $CT^2 = CG \cdot CD = \frac{1}{4} (AC + BC)^2$, Student, Prop. VII, cor., which is the first case.

Case 2. Join CF, cutting the circle FIE in i, then $CI^2 = Ci \cdot CF = Cr \cdot CR = (CH - Hr) \cdot (CH + Hr) = CH^2 - Hr^2 = CH^2 - FH^2$. Demit the perpendicular CL, then $CH^2 = CL^2 + HL^2$; hence $CI^2 = CL^2 + HL^2 - FH^2 = DL \cdot LE - FL \cdot LE = LE (DL - FL) = LE \cdot DF = \frac{1}{4} (AC - BC)^2$, the second case.



Case 3. $CV^2 = Ct \cdot CF = Cs \cdot CP = (CO + OS) \cdot (CO - OS) = CO^2 - OS^2 = CO^2 - OQ^2 = CL^2 + OL^2 - OB^2 = DL \cdot LE + FL \cdot QL$. Mult. by 4, and $4CV^2 = 4DL \cdot LE + 4FL \cdot LQ = 4DL (FE - FL) + 2FL \cdot (2DL - DE) = 4DL \cdot FE - 2FL \cdot DE = AC^2 + BC^2$, Student, Prop. VII, and cor. to Prop. XIX.

Case 4. Because $(AC + BC)^2 = 4CT^2$, and $(AC - BC)^2 = 4CI^2$, we have, by subtr. $4CT^2 - 4CI^2 = 4AC \cdot BC$, or $AC \cdot BC = CT^2 - CI^2$; also $(AC + BC)^2 - (AC^2 + BC^2) = 4CT^2 - 4CV^2$, or $AC \cdot BC = 2CT^2 - 2CV^2 = CT^2 - CI^2$, therefore $CT^2 = 2CV^2 - CI^2$, or $CT^2 + CI^2 = 2CV^2$. Hence, we have determined $AC \cdot BC = 2CT^2 - 2CV^2$, which was probably intended for case 4.

Besides the property demonstrated above, viz. $CT^2 + CI^2 = 2CV^2$, a great number of others may be inferred from the above figure.

Thus $4Ci \cdot CF = (AC - BC)^2 = 4AN^2$, $4OG \cdot CD = (AC + BC)^2 = 4CN^2$, and $4Cl \cdot CF = AC^2 + BC^2 = 2(FL^2 + CL^2 + BF^2) = 2(CF^2 + BF^2)$. Moreover, $CT = CN$, and $CI = AN$, therefore $CT + CI = AC$, and since $AC \cdot BC = CT^2 - CI^2 = (CT + CI) \cdot (CT - CI)$, of which the former $= AC$, the latter, viz. $CT - CI = BC$. Also $\frac{1}{4}(CT^2 + CV^2 + CI^2) = \frac{3}{4}(AC^2 + BC^2) = 6(CF^2 + BF^2)$, and $2CV^2 = CF^2 + BF^2$.

IV. QUESTION 554, by Mr. JOHN BAINES.

Find the equation of a curve whose tangent varies directly as the abscissæ.

SOLUTION, by Mr. BAINES, the Proposer.

Subtangent $= \frac{ydx}{dy}$, and $\tan = \sqrt{(y^2 + \frac{y^4 dx^2}{dy^2})}$, $= rx$ (by the question, r being a given quantity.) We have therefore, $y^2 dy^2 + y^2 dx^2 = r^2 x^2 dy^2$, and consequently $ydx = dy \sqrt{(r^2 x^2 - y^2)}$. Assume $x = uy$, then $dx = udy + ydu$, and $uydy + y^2 dx = ydy \sqrt{(r^2 u^2 - 1)}$. Divide by y and reduce, then

$$\begin{aligned} \frac{dy}{y} &= \frac{du}{\sqrt{(r^2 u^2 - 1)} - u} = \frac{udu + du \sqrt{r^2 u^2 - 1}}{r^2 u^2 - u^2 - 1} \\ &= \frac{udu}{r^2 u^2 - u^2 - 1} + \frac{du \sqrt{r^2 u^2 - 1}}{r^2 u^2 - u^2 - 1} =, \end{aligned}$$

by multiplying and dividing the latter part by $\sqrt{r^2 u^2 - 1}$, then adding and subtracting u^2 in the numerator,

$$\begin{aligned} &\frac{udu}{r^2 u^2 - u^2 - 1} + \frac{du (r^2 u^2 - u^2 - 1 + u^2)}{(r^2 u^2 - u^2 - 1) \sqrt{r^2 u^2 - 1}} \\ &= \frac{udu}{r^2 u^2 - u^2 - 1} + \frac{du}{\sqrt{r^2 u^2 - 1}} + \frac{u^2 du}{(r^2 u^2 - u^2 - 1) \sqrt{r^2 u^2 - 1}} \\ &= \frac{udu}{r^2 u^2 - u^2 - 1} + \frac{r^2}{r^2 - 1} \cdot \frac{du}{\sqrt{r^2 u^2 - 1}} \\ &\quad + \frac{1}{r^2 - 1} \times \frac{du}{(r^2 u^2 - u^2 - 1) \sqrt{r^2 u^2 - 1}}. \end{aligned}$$

We have now arrived at known forms; hence taking the integrals, after multiplying by $r^2 - 1$, we have

$$(r^2 - 1) h. l. y. = \frac{1}{2} h. l. (r^2 u^2 - u^2 - 1) + r \times h. l.$$

$$(ru + \sqrt{r^2u^2 - 1}) + h. l. \frac{u - \sqrt{r^2u^2 - 1}}{\sqrt{r^2u^2 - u^2 - 1}},$$

$$\text{or } y^{r-1} = (ru + \sqrt{r^2u^2 - 1})^r \times (u - \sqrt{r^2u^2 - 1}),$$

and substituting the value of $u = \frac{x}{y}$, we have

$$y^{r-1+r} = (x - \sqrt{r^2x^2 - y^2}) \times (rx + \sqrt{r^2x^2 - y^2})^r$$

= the equation of the curve required.

V. QUESTION 555, by Mr. T. S. DAVIES, Bath.

Let M, N, S, R , be the centres of four circles about AED , EBC , ABF , DCF (Fig. to the solution of question 524) and let L be the point where they intersect: then M, N, S, R, L are in one circumference.

SOLUTION, by Mr. DAVIES, the Proposer.

Instead of confining myself to the solution of the proposition mentioned, I shall add a few others which I have had in my possession for some years. The solutions, however, which are here given are not the original ones, as the MS. which contained a number of similar researches has been mislaid, and I have been obliged to re-investigate them; and though I cannot recall the methods I before employed, I am quite sure they were altogether different from the present, and, I think, too, they were much superior to them in several respects.

The subject is by no means exhausted: and in order to again give an opportunity of attending to it by the correspondents of the Repository, I shall add the opening proposition of a new but connected series, to the list of new questions which accompany the present number.

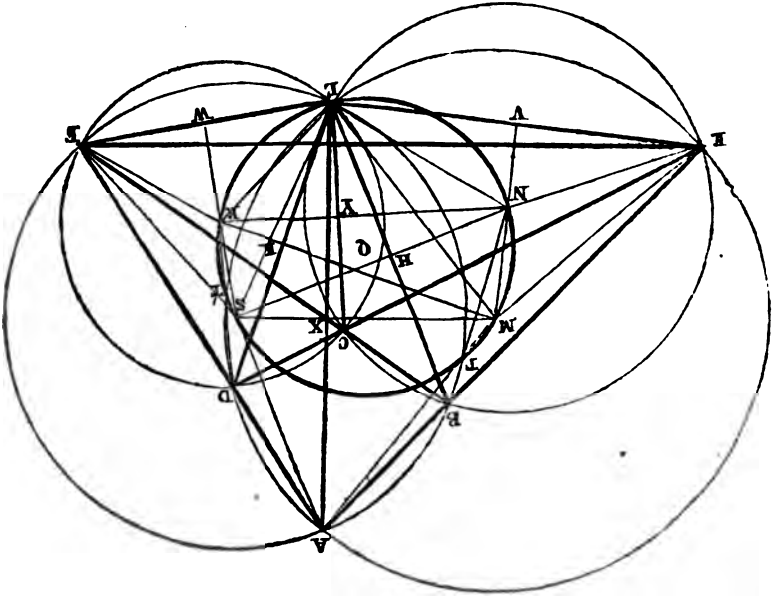
PROP. I. To prove that M, N, R, S are in the same circle.

Let MS, AL' intersect in X , and NC, CL' intersect in Y , and let MN, SR , meet $EL', L'F$ in V and W .

Then, because of the circles AEL', AFL' whose centres are M and S respectively intersect in A, L' , the line MS is perpendicular to AL' , or MXL' is a right angle. In like manner it is shewn that RYL' is also a right angle.

Again, because M, N are the centres of two circles which

intersect in E and L' , the line MN bisects EL' at right angles in v : and for a similar reason, SR bisects $L'F$ at right angles in w .



$$\begin{aligned}
 \text{Now } EL'A &= EL'B + BL'A \\
 &= ECB + CL'D \\
 &= DL'F + DL'C \\
 &= FL'C \\
 &= CL'W
 \end{aligned}$$

$$\text{Or } VL'X = YL'W \dots\dots (1)$$

But, because MVL' and MXL' are two right angles, the points M, V, L', X , are in a circle, and therefore, VMX and $VL'X$ are equal to two right angles. In like manner, because L', Y, R, U are in a circle, YRV and YLV are equal to two right angles. Hence,

$$VMX + VL'X = YRX + YL'W \dots\dots (2)$$

And from (1), (2) we get at once

$$VMX = YRW$$

that is,

$$NMS = NRW$$

From which it follows that the four points M, N, R, S are in a circle.

PROP. II. *To prove that the point L' is in the same circle with M, N, R, S .*

Draw the perpendiculars MU, NZ , from the centres M, N of the circles, upon the line AE . Let BL', NS intersect in H , and DL', MR in K . Then, N, S , being the centres of the circles whose common chord is BL' , the lines NS, BL' intersect at right angles in H . In like manner, MR, DL' intersect at right angles in K .

Then, since MS, SN are at right angles to $AL', L'B$ each to each, the angles $MSN, AL'B$ are equal.

Again,

$$\begin{aligned} AL'B &= EL'A - EL'B \\ &= EMU - ENZ \\ &= NEZ - MEU \\ &= NEM \\ &= NL'M \end{aligned}$$

Hence $MSN = AL'B = NL'M$: and the point L' is in the circle M, N, R, S .

PROP. III. *Let MA, BN intersect at T : then the point T is at the intersection of the circles $MNRS$ and AFB .*

$$\begin{aligned} \text{For } MTN &= BTA = NBA - MAB \\ &= NEB - MEB \\ &= MEN \\ &= ML'N. \end{aligned}$$

Hence, the angles $MTN, ML'N$ standing on the same base and being equal to one another, are in the same circle $MNRS$, and hence the point T is in that circle.

Again, the lines MS, SN being, as before shewn, perpendicular to AL', BL' , the angle $BL'A = MSN = MTN = BTA$; and hence T is also in the circle AFB . Hence the point T is at the intersection of the circles.

It may be shewn in a similar manner that AS, RD intersect at a point T' which is also the intersection of the circle $MNRS$ with ADE .

Rem. Several other intersections of similar kinds are easily assigned, and are proved in a manner very analogous. I omit them solely on account of the complexity which the necessary lines inevitably create in a figure of the size adapted to the Repository-page.

PROP. IV. *The following pairs of triangles are similar; viz.*

BEI' and DFI' , CBI' and DAL' , BAL' and CDL' .

For $DI'F = DCI' = BCE = BE'I'$.

$BEI' = L'CF = L'DF$

$DFI' = ECI' = EBI'$.

Hence the triangles EBI' , $L'DF$ are similar; and

$BE' : L'E : EB :: DF : FI' : L'D$.

Again, $L'BC = L'EC = L'ED = L'AD$

$BE'C = AL'D$ (Repos. quest. 524)

$\therefore BCI' = ADL'$, and

$BC : CI' : L'B :: AD : DI' : L'A$.

In precisely the same way we find,

$AB : BE' : L'A :: DC : CI' : L'D$.

Mem. It has been shewn in Repos. quest. 524, that

$CI'A = AFC - AEC$

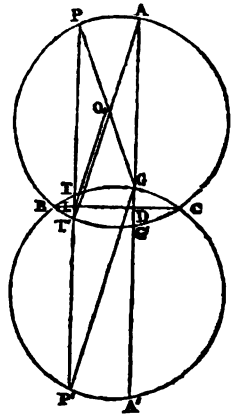
$BE'D = AFC + AEC$

Before we proceed to the other properties we must premise two or three simple lemmas.

LEMMA I. *Let APG' $A'P'G$ be two equal circles whose common chord (real or ideal*) is BC . Draw any two lines perpendicular to the chord, cutting the circles in $PT'P'$ and $AG'GA'$ and the chord itself in L and D respectively. Join AT' , GP' and these will be parallel.*

For the parallels AA' , PP' cut off equal segments from the equal circles, and the equal segments subtend equal angles. Hence $P'GA' = T'AG'$.

Conversely, if GP' be drawn parallel to AT' it will meet PP' at the point of intersection with the circle.



* This term is employed by Poncelet to designate the locus of the points from which tangents to the two circles are equal. *Real* when they cut, *ideal* when one lies wholly within or without the other. The term *radical axis* is applied to them both, by most of the Foreign Geometers: yet there are cases in which Poncelet's seems to be the best term.

LEMMA II. Draw LQ parallel to AT intersecting GP in Q : then GP is bisected in Q .

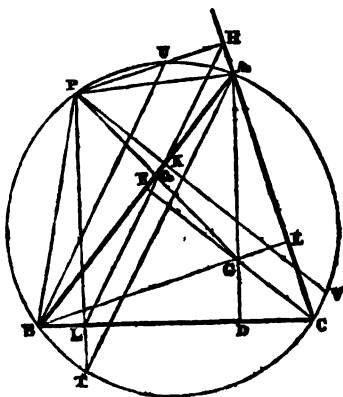
For, since LQ is parallel to AT , it is also (Lem. I.) parallel to GP' ; and since $PL = LP'$, and LQ is drawn parallel to the base OP' , it bisects the other side GP in Q .

LEMMA III. Let ABC be a triangle circumscribed by a circle, and from any point P in the circumference, draw perpendiculars PH, PK, PL , to the sides of the triangle, cutting them in H, K, L respectively: then, H, K, L are in one straight line.*

Because PHC, PLC are right angles the points P, H, C, L , are in a circle; and, therefore, $BPA = LPH$ each being the supplement of ACB . Hence, $BAT = BPT = APH = AKH$.

Again, because PKB, PLB are right angles, the points P, K, L, B , are in a circle; and therefore $BKT = BPT = BAT$. Hence $AKH = BKT$, or the lines LK, KH form one continued line.

Cor. The lines BU, AT , and vc being drawn, are all parallel to HKL .



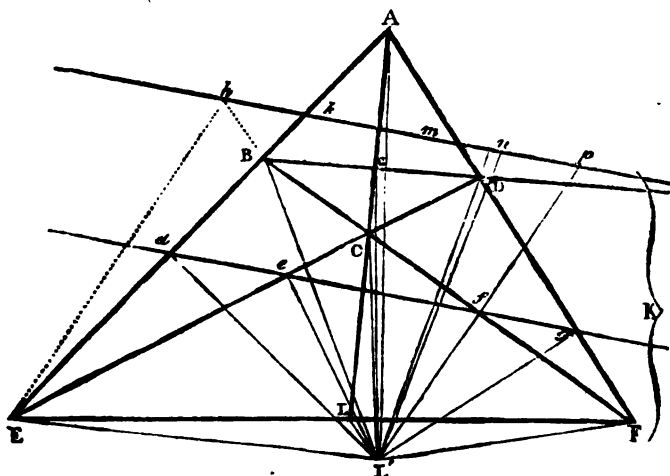
LEMMA IV. Let the perpendiculars from the angles upon the opposite sides intersect in G ; and join GP , cutting HKL in Q ; then GP is bisected in Q .

For, if a circle be described through cgb it will be equal to the circle ABC , and they will have BC for their common chord (which is *real* in the present case): and since LKH is parallel to AT , it bisects GP in Q , as is obvious from the preceding Lemmas.

We now return to the properties of the complete quadrilateral.

* This proposition is not new. It appears to have been first noticed in the Gentleman's Mathematical Companion, No. —, by —; but I am not aware that any consequences have before been drawn from it. It is very fertile: though we have here occasion for only one of its results.

PROP. V. From the point L' draw the perpendiculars $L'd, L'e, L'f, L'g$ upon the sides AE, ED, BF, FD : then d, e, f, g are in one straight line.



For since L' is a point in the circle ADE , the points d, e, g are in one line, (Lem. III): and since L' is a point also in the circle FCD , the points e, f, g are in one line also. Hence all four d, e, f, g , are in one straight line.

PROP. VI. Let the perpendiculars from the angles upon the sides of the triangles EBC meet in h ; those of EAD in m ; those of FBA in k , and those of FCD in n : then h, k, m, n are in one straight line, and that line is parallel to $defg$.

For draw nL' . Then since L' is a point in the circle CED , and n is the intersection of the perpendiculars; and since, moreover, efg is the line through the foot of the perpendiculars from L' upon the sides of the triangle CED : it follows from Lemma IV. that nL' is bisected by the line eg . The same takes place with the other three points h, k, m ; and therefore the four points h, k, m, n are in one line, parallel to dg , as was affirmed in the proposition.

PROP. VII. If circles be described upon the three diagonals AC, BD, EF ; these shall have the line hn for a common chord:—real, when EAF is acute, but ideal, when EAF is obtuse.

This proposition will form one in the series of questions issued with the present number of the Repository. Vide Question 600.

VI. QUESTION 556, by Mr. HUMPHREY CLINKER,
Caer Badon.

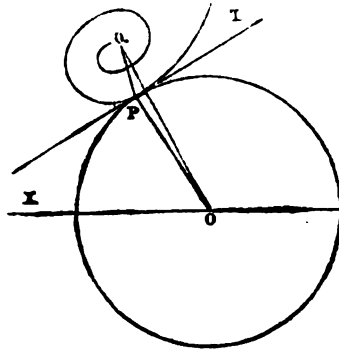
The logarithmic spiral rolls upon the circumference of a given circle, what is the locus of its pole?

SOLUTION, by Mr. T. S. DAVIES, Bath.

Let Q be the pole of the logarithmic spiral, O the centre of the circle and x the point of the circumference with which the pole was in contact.

Put $OQ = r$; $OP = a$;
 $rQ = \rho$; $xOQ = \theta$; QPO
 $= \frac{\pi}{2} + \alpha$; and $POQ = \psi$.

Then since the arcs which have been in contact are equal,



$$\psi = \frac{\rho}{a} \sec \alpha \dots (1)$$

$$\cos POQ = \cos (\theta - \psi) = \frac{a^2 + r^2 - \rho^2}{2ar} \dots (2)$$

$$\sin POQ = \sin (\theta - \psi) = \frac{r \cos \alpha}{\rho} \dots (3)$$

Squaring (2) and (3), adding the results, and reducing, we have

$$\rho^6 - 2(a^2 + r^2)\rho^4 + (a^2 - r^2)^2\rho^2 + 4a^2r^4\cos^2\alpha = 0 \dots (4)$$

From which we get $\rho = f(r) \dots (5)$

Equating, now, the equations (1) and (3) we have after transposition,

$$\begin{aligned} \theta &= \frac{\rho}{a} \sin \alpha + \cos^{-1} \frac{r \cos \alpha}{\rho} \\ &= \frac{f(r)}{a} \cdot \sin \alpha + \cos^{-1} \frac{r \cos \alpha}{f(r)} \dots (6) \end{aligned}$$

Now since the value of ρ can be actually exhibited in terms of r , by means of Cardan's formula, the question is solved, and the equation of the curve assigned. It will be found upon trial that

two of the values of ϵ^2 will be essentially imaginary; and therefore, that there are only two branches corresponding to $\pm \epsilon$ in the locus in question.

VII. QUESTION 557, by Mr. JAMES CUNLIFFE.

Let a, b and c denote the rational numerical values of the sides of a plane triangle, and let Λ denote its double area, then will $a^2b^2 - \Lambda^2, a^2c^2 - \Lambda^2$ and $b^2c^2 - \Lambda^2$ be all rational squares: required the proof.

SOLUTION, by Mr. JOHN BAINES, *Thornhill*.

The half sum of the sides, and the three remainders are

$$\frac{a+b+c}{2}, \frac{c-(a-b)}{2}, \frac{c+(a-b)}{2}, \text{ and } \frac{a+b-c}{2}; \text{ hence}$$

$$\Lambda^2 = \frac{1}{4} \{ (a+b) + c \} \cdot \{ (a+b) - c \} \cdot \{ c + (a-b) \} \cdot$$

$$\{ c - (a-b) \} = \frac{1}{4} \{ (a+b)^2 - c^2 \} \cdot \{ c^2 - (a-b)^2 \}$$

$$= \frac{2c^2(a^2 + b^2) - (a^2 - b^2)^2 - c^4}{4}.$$

Hence

$$\begin{aligned} a^2b^2 - \Lambda^2 &= a^2b^2 - \frac{1}{4} \{ 2c^2(a^2 + b^2) - (a^2 - b^2)^2 - c^4 \} \\ &= \frac{1}{4} \{ a^2 + b^2 \}^2 - 2c^2(a^2 + b^2) + c^4 \\ &= \frac{1}{4} (a^2 + b^2 - c^2)^2 = \square. \end{aligned}$$

In like manner we have

$$a^2c^2 - \Lambda^2 = \frac{1}{4} (a^2 - b^2 + c^2)^2 = \square, \text{ and}$$

$$b^2c^2 - \Lambda^2 = \frac{1}{4} (-a^2 + b^2 + c^2)^2 = \square. \quad \text{Q. E. D.}$$

Because a^2, b^2 and c^2 only enter the expressions for the roots of the three squares, the property holds good if a, b and c are of the forms $\sqrt{a}, \sqrt{b}$, and $\sqrt{c}$. Also, it is evident from this solution, that the sides of a plane triangle may all of them be irrational, and yet so related that the area may be rational. Thus, if the sides are $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$, the area is rational when

$$2c(a+b) - (a-b)^2 - c^2 = \square,$$

and this will always be a square when

$$a = gr^2 + s^2, b = r^2 + gs^2, \text{ and } c = 4(r^2 + s^2),$$

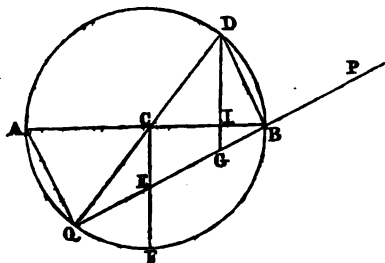
r and s being arbitrary, but such as give sides consistent for a triangle. Ex. If $r = 3$, and $s = 2$, $\sqrt{85}, \sqrt{45}$ and $\sqrt{52}$ are the sides of a triangle whose area is 24.

VIII. QUESTION 558, by Mr. J. H. SWALE, *Liverpool*

To determine (*geometrically*) the position of equilibrium, of a uniformly heavy, inflexible rod, of a given length, resting on the edge of the brim of a given hemispherical bowl, while the lower end rests against the concave surface of the bowl.

SOLUTION, by ———.

Let $AQFB$ be a vertical section of the hemispherical bowl, AB its horizontal diameter, PQ the rod resting on the edge of the bowl at B and supported by the concave surface at Q ; and let G be the centre of gravity or middle of the rod. Draw the diameter QD and join DB ; also draw DI and CF perpendicular to AB , the latter meeting PQ in E :



then, the re-action of the bowl against the end of the rod at Q is in the direction QD , perpendicular to the surface of the bowl; and the re-action at B is in the direction BD , perpendicular to the rod; and therefore, by a well-known principle in *Statics*, the perpendicular DI passes through the centre of gravity G when the rod is in equilibrium. Hence, because QC is $= CD$ and CE parallel to DC , QR is $= EC = \frac{1}{4}$ of the given length of the rod; therefore to determine the position of the rod we have only to draw BQ so that the part BQ , intercepted between the perpendicular CF and the circumference, may be equal to a fourth part of the length of the rod, which is a well known problem.

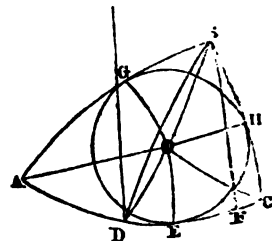
The construction, indeed, is evident; for the triangles CBE , BQA being similar $QB \times EB$ is $= AB \times BC = 2BC^2$; hence, if a line equal to $\frac{1}{4}$ of the length of the rod be taken, and produced till the rectangle contained by the whole line thus produced, and the part produced, be equal to twice the square of the radius of the bowl, the length of the part of the rod within the bowl, and consequently its position, will be determined.

IX. QUESTION 559, by Mr. T. S. DAVIES.

Given the base and radius of the inscribed circle, to find the locus of the vertex of the spherical triangle?

FIRST SOLUTION, by Mr. DAVIES, the Proposer.

Let AC be the base, D its middle point, B the vertex of the triangle, and O the centre of its inscribed circle. Draw the radii OE, OH, OG to the points of contact: and draw the perpendicular BF.



Put $DF = \theta$, $DE = \psi$, $FB = \phi$
 $AD = DC = \alpha$, and $EO = GO = HO = \rho$
 Then the arcs AO, OC bisect the angles A and C at the base of the triangle: and we have

$$\cot OAE = \cot \frac{1}{2} A = \sin (\alpha + \psi) \cot \rho \dots (1)$$

$$\cot OCE = \cot \frac{1}{2} C = \sin (\alpha - \psi) \cot \rho \dots (2)$$

$$\cot BAE = \cot A = \sin (\alpha + \theta) \cot \phi \dots (3)$$

$$\cot BCE = \cot C = \sin (\alpha - \theta) \cot \phi \dots (4)$$

$$\left. \begin{array}{l} \text{But } \cot A = \frac{1}{2} (\cot \frac{1}{2} A - \tan \frac{1}{2} A) \\ \text{and } \cot C = \frac{1}{2} (\cot \frac{1}{2} C - \tan \frac{1}{2} C) \end{array} \right\} \dots (5)$$

Inserting in the equations (5) the values furnished by (1, 2, 3, 4) we have

$$2 \sin (\alpha + \theta) \cot \phi = \sin (\alpha + \psi) \cot \rho - \operatorname{cosec} (\alpha + \psi) \tan \rho \dots (6)$$

$$2 \sin (\alpha - \theta) \cot \phi = \sin (\alpha - \psi) \cot \rho - \operatorname{cosec} (\alpha - \psi) \tan \rho \dots (7)$$

Between (6, 7) eliminate ψ , and the resulting equation will define the locus of the point B.

Resolving these questions we have

$$\sin \overline{\alpha + \psi} = \{ \sin (\alpha + \theta) \cot \phi \pm \sqrt{\sin^2 \overline{\alpha + \theta} \cot^2 \phi + 1} \} \tan \rho \dots (8)$$

$$\sin \overline{\alpha - \psi} = \{ \sin \overline{\alpha - \theta} \cot \phi \pm \sqrt{\sin^2 \overline{\alpha - \theta} \cot^2 \phi + 1} \} \tan \rho \dots (9)$$

Write the second sides K and L; and add and subtract the first sides: then

$$\left. \begin{array}{l} \cos \psi = \frac{K + L}{2 \sin \alpha} \\ \sin \psi = \frac{K - L}{2 \cos \alpha} \end{array} \right\} \dots (10)$$

Hence $\sin^2 \psi + \cos^2 \psi = 1$ becomes

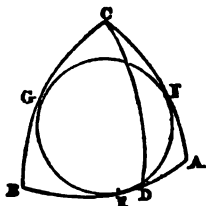
$$K^2 - 2 \cos 2\alpha \cdot KL + L^2 = \sin^2 \alpha \cos^2 \alpha \dots (11)$$

Where K and L are known functions of $\phi \theta$: and hence the equation is determined. It admits however, of some consider-

able simplifications by reduction ; but as in a work which I am composing on Spherical Geometry, the question is discussed at length, I shall waive all further consideration of it here.

SECOND SOLUTION, by Mr. BAINES.

Let ABC represent a spherical triangle, and EFG a circle inscribed in it, touching the sides AB , AC and BC at the points E , F and G ; then, by the properties of the inscribed circle $AE = AF$, $BE = BG$, and $CG = CF$. Let fall the perpendicular CD ; put $AD = x$, $CD = y$, $CF = CG = a$, $BE = BG = \beta$, $AE = AF = \gamma$, the base $AB = c$, and the radius of the inscribed circle $= r$; then $\cos x \cos y = \cos AC$, and $\cos(c-x) \cos y = \cos BC$. Hence, we have



$$\cos(a+y) = \cos x \cos y \dots (1),$$

$$\cos(a+\beta) = \cos(c-x) \cos y \dots (2),$$

$$\beta + \gamma = c \dots (3),$$

and, by Mr. Davies's Supplement to Young's Trigonometry, p. 223,

$$\tan^2 r = \frac{\sin a \sin \beta \sin \gamma}{\sin(a+\beta+\gamma)}, \text{ or } \tan^2 r \sin(a+c) = \sin a \sin \beta \sin \gamma \dots (4).$$

By common trigonometrical formulæ, from (1) and (2) we have

$$\cos(\beta-\gamma) = \cos x \cos(c-x) \cos^2 y + \sqrt{(1-\cos^2 x \cos^2 y) \cdot \{1-\cos^2(c-x) \cos^2 y\}} \dots (5).$$

Taking the difference between (5) and (3) we have

$$2 \sin \beta \sin \gamma = \cos x \cos(c-x) \cos^2 y + \sqrt{(1-\cos^2 x \cos^2 y) \cdot \{1-\cos^2(c-x) \cos^2 y\}} - \cos c \dots (6).$$

Also from (1) and (3),

$$\cos(a-\beta) = \cos c \cos x \cos y + \sin c \sqrt{1-\cos^2 x \cos^2 y} \dots (7).$$

Take the difference between (2) and (7), and

$$2 \sin a \sin \beta = \cos c \cos x \cos y + \sin c \sqrt{1-\cos^2 x \cos^2 y} - \cos(c-x) \cos y \dots (8).$$

But from (2) and (3) $\cos(a-\gamma) = \cos c \cos(c-x) \cos y$

$$\sqrt{1-\cos^2(c-x)^2 \cos^2 y} \dots (9).$$

The difference between (1) and (9) gives,

$$2 \sin \alpha \sin \gamma \approx \cos c \cos (c - x) \cos \gamma + \sin c \sqrt{1 - \cos^2 (c - x) \cos^2 \gamma} - \cos x \cos \gamma \dots (10)$$

Hence, dividing the continued product of (6), (8) and (10), by 8, and expressing the square root of the quotient, we have the value of $\sin \alpha \sin \beta \sin \gamma$ in terms of x and y ; also, dividing the value of $2 \sin \alpha \sin \beta \sin \gamma$ by the value of $2 \sin \beta \sin \gamma$ from (6), we have $\sin \alpha$, and consequently $\cos \alpha$, in terms of x and y ; therefore $\sin (\alpha + c) = \sin \alpha \cos c + \cos \alpha \sin c$ is known in the same terms. These values substituted in (4) give the equation of the required locus in terms of the spherical co-ordinates required.

X. QUESTION 560, by Mr. T. S. DAVIES.

Find the equation of the hour lines on the Horodietical Quadrant; see Hutton's Dictionary, art. Quadrant, or Ozanam's Recreations, vol. iii. p. 313.

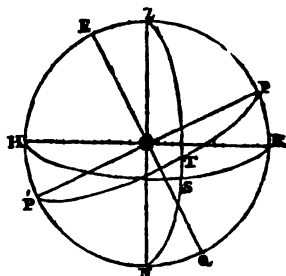
SOLUTION, by Mr. DAVIES, the Proposer.

The construction of this quadrant has not been completely defined so far as the hour-lines are concerned: inasmuch, as the function of the declination set off upon AB (Fig. 2), is arbitrary. The equation will hence be different according to the function we may select, or in other words it must involve an arbitrary characteristic, which we shall call $f(\)$. Hence if ϕ denote the polar distance of the sun (from the pole of an opposite name to the latitude,) we have

$$r = af(\phi) \dots (1)$$

when r is the radius-vector of the curve, and a the radius of the circle of the sphere to the scale of which the dial is constructed.

Let PF' be the axis of the sphere (Fig. 1), and the other letters have their usual meaning in this figure. Then ZT is the zenith distance when the sun has the polar distance PT and the hour angle is ZPT .



Put $ZP = \lambda$; $PT = \phi$; $ZPT = x$; and $ZT = \theta$.

Then $\cos \theta = \cos \lambda \cos \phi + \sin \lambda \sin \phi \cos \alpha \dots (2)$

Resolve this for $\sin \phi$, then

$$\sin \phi = \frac{\cos \theta \sin \lambda \pm \cos \lambda \{\sin^2 \theta - \sin^2 \lambda \sin^2 \alpha\}^{\frac{1}{2}}}{1 - \sin^2 \lambda \sin^2 \alpha} \dots (3)$$

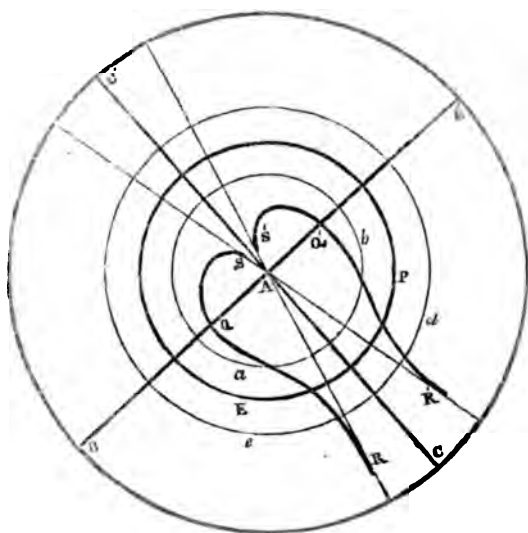
Insert this in (1) then the equation sought is

$$r = a f \sin^{-1} \left\{ \frac{\cos \theta \sin \lambda \pm \cos \lambda \{\sin^2 \theta - \sin^2 \lambda \sin^2 \alpha\}^{\frac{1}{2}}}{1 - \sin^2 \lambda \sin^2 \alpha} \right\} \dots (4)$$

In the method employed by Ozanam, Bion, &c. the function $f(\phi)$ designates $(\phi + \beta)$ and the equation of the curve then is,

$$r = a \sin^{-1} \left\{ \frac{\cos \theta \sin \lambda + \cos \lambda \{\sin^2 \theta - \sin^2 \lambda \sin^2 \alpha\}^{\frac{1}{2}}}{1 - \sin^2 \lambda \sin^2 \alpha} + \beta \right\} \dots (5)$$

A much neater construction is where $\beta = 0$: that is where the line AB is the meridian unrolled into a straight line, and the point A designates one of the poles. The figures in Ozanam, Bion and Hutton are constructed on the supposition of A being in the opposite hemisphere from the place for which the dial is constructed. The figure, however, exhibits only a part of the locus sought, viz, the intertropical portion of it: but the whole hour line when completed takes the form exhibited in the an-



nexed figure (2). It is terminated abruptly at both ends: and is composed of two similar and equal branches RQS and $R'Q'S'$.

The lines RAS' , $R'A's$ are tangents to the curves; and are inclined at an angle $\pm \theta$ such that $\sin^2 \theta = \sin^2 \lambda \sin^2 \alpha$.

They cross AB at Q , Q' such, that

$$r = a \sin^{-1} \sqrt{\frac{\pm \cos \lambda}{1 - \sin^2 \lambda \sin^2 \alpha}}$$

By giving to α its several values we get the corresponding hour-lines: and the tangents at the abrupt points of the curves make different angles in each case with AB or AC *

Had we taken A for the pole of the same name as the latitude, the curves would have been in all respects equal: but reversed in position, so that s would have been in the line AR , and R in the line AS , or in other words, the plane sector SAB would have been turned over, and the parts inverted.

It is unnecessary to pursue the simple inquiry further, as its character must be sufficiently apparent, and its processes are too simple to offer the least difficulty to the student.

XI. QUESTION 561, *by the Rev. Mr. BRICE BRONWIN, Denby, near Wakefield.*

The arch of a semicircle stands in a vertical plane, with its two extremities in the same horizontal line, and convexity downwards. Two material points connected by an inextensible rod without weight are laid on the inside of the arch, one of them at its extremity. In this situation they are left to the action of gravity. Find the difference of the times of their descent through any given arch in this case and that in which the two points form but one, namely, the uppermost, the length of the rod being small. Find also the pressures and the tension.

SOLUTION, *by Mr. BRONWIN, the Proposer.*

Let m, m' be the masses of the points; p, p' their pressures on the curve; τ the tension of the rod, l its length; r the radius of the semicircle, g gravity; x and y the co-ordinates of m ; x', y' those of m' ; x, x' being vertical, and origin the centre of the semicircle. Then

$$\frac{x}{r} \frac{p}{m}, \quad \frac{y}{r} \frac{p}{m}, \quad \frac{x'}{r} \frac{p'}{m'}, \quad \frac{y'}{r} \frac{p'}{m'}$$

are the resolved parts of p and p' in the directions of x, y and x', y' retardative. Also

* EP denotes the equator, ab the tropic of Capricorn, and cd the tropic of cancer.

$$\frac{x - x'}{l} \frac{\chi}{m}, \quad \frac{y - y'}{l} \frac{\chi}{m}, \quad \frac{x' - x}{l} \frac{\chi}{m'}, \quad \frac{y' - y}{l} \frac{\chi}{m'}$$

are the resolved parts of r in the same directions accelerative. Therefore

$$\frac{d^2x}{dt^2} = g - \frac{x}{r} \frac{p}{m} + \frac{x - x'}{l} \frac{\chi}{m}, \quad \frac{d^2y}{dt^2} = -\frac{y}{r} \frac{p}{m} + \frac{y - y'}{l} \frac{\chi}{m},$$

$$\frac{d^2x'}{dt^2} = g - \frac{x'}{r} \frac{p'}{m'} + \frac{x' - x}{l} \frac{\chi}{m'}, \quad \frac{d^2y'}{dt^2} = -\frac{y'}{r} \frac{p'}{m'} + \frac{y' - y}{l} \frac{\chi}{m'}.$$

$$r^2 = x^2 + y^2, \quad x dx + y dy = 0, \quad r'^2 = x'^2 + y'^2, \quad x' dx' + y' dy' = 0, \quad (x' - x)^2 + (y' - y)^2 = l^2, \quad (x' - x)(dx' - dx) + (y' - y)(dy' - dy) = 0.$$

Add the first four of these multiplied by $m dx$, $m dy$, $m' dx'$, $m' dy'$ respectively. The result by means of the last reduces to

$$m \left(dx \frac{d^2x}{dt^2} + dy \frac{d^2y}{dt^2} \right) + m' \left(dx' \frac{d^2x'}{dt^2} + dy' \frac{d^2y'}{dt^2} \right)$$

$$= g m dx + g m' dx'. \quad \text{Or}$$

$$m \left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} \right) + m' \left(\frac{dx'^2}{dt^2} + \frac{dy'^2}{dt^2} \right)$$

$$= 2gm x + 2gm' x' + C.$$

Let z = angle described by m , λ that subtended by l . Then $r^2 dz^2 = dx^2 + dy^2 = dx'^2 + dy'^2$, $x = r \sin z$, $x' = r \sin (z + \lambda)$, and the correction $C = -2gm'r \sin \lambda$. To abridge let

$$\frac{m}{m + m'} = n, \quad \frac{m'}{m + m'} = n', \quad \text{our equation becomes } r \frac{dz^2}{dt^2}$$

$$= 2gn \sin z + 2gn' \sin (z + \lambda) - 2gn' \sin \lambda; \quad \text{and}$$

$$dt = \sqrt{\frac{r}{2g}} \cdot \sqrt{\frac{dz}{n \sin z + n' \sin (z + \lambda) - n' \sin \lambda}}$$

$$= \sqrt{\frac{r}{2g}} \cdot \sqrt{\frac{dz}{\sin z}} + n' \lambda \sqrt{\frac{r}{2g}} \cdot \frac{\sin^2 \frac{z}{2} dz}{\sin \frac{1}{2} z}$$

nearly, λ being supposed small. But if greater accuracy be required, we must carry the expansion of the radical further, and include higher powers of λ .

$$\text{When the weights make but one, } dt = \sqrt{\frac{r}{2g}} \cdot \sqrt{\frac{dz}{\sin z}}.$$

Therefore $\Delta t = \sqrt{\frac{nl}{2gr}} \int \frac{\sin^2 \frac{z}{2} dz}{\sin^{\frac{1}{2}} \frac{z}{2}}$ nearly. Putting $\sin z = x$, this integral becomes $\int \frac{\frac{1}{2} dx}{x^{\frac{3}{2}} \sqrt{1-x^2}} + \frac{1}{x^{\frac{1}{2}}}$, the value of which may be found by series. Thus we obtain the difference Δs of the times required.

From the two first differential equations we find

$$y \frac{d^2 x}{dt^2} - x \frac{d^2 y}{dt^2} - gy = \frac{xy' - x'y}{lm} \tau; \text{ or}$$

$r^2 \frac{d^2 z}{dt^2} - gr \cos z = -\frac{r^3 \sin \lambda}{lm} \tau$. Differentiating the differential equation in z in order to find $\frac{d^2 z}{dt^2}$, we get after substituting its value

$$\begin{aligned} \tau &= \frac{2glmn' \sin \frac{\lambda}{2} \sin(z + \frac{\lambda}{2})}{r \sin \lambda} = \frac{glmn' \sin(z + \frac{\lambda}{2})}{r \cos \frac{\lambda}{2}} \\ &= \frac{glmn'}{r} \sin(z + \frac{\lambda}{2}) \text{ very nearly.} \end{aligned}$$

Again, we shall easily form the equation

$$\begin{aligned} (y - y') \frac{d^2 x}{dt^2} - (x - x') \frac{d^2 y}{dt^2} - (y - y') g &= \frac{xy' - x'y}{rm} p; \text{ or} \\ r \sin \frac{\lambda}{2} \frac{d^2 z}{dt^2} - r \cos \frac{\lambda}{2} \frac{dz^2}{dt^2} - g \sin(z + \frac{\lambda}{2}) \\ &= -\cos \frac{\lambda}{2} \frac{p}{m}; \text{ and } p = 3gm \sin z - 4gm n' \sin \lambda \sin^2 \frac{z}{2}, \end{aligned}$$

true to quantities of the order λ^2 .

Similarly,

$$\begin{aligned} (y' - y) \frac{d^2 x'}{dt^2} - (x' - x) \frac{d^2 y'}{dt^2} - (y' - y) g &= \frac{x'y - xy'}{rm'} p'; \\ \sin \frac{\lambda}{2} \frac{d^2 z}{dt^2} + r \cos \frac{\lambda}{2} \frac{dz^2}{dt^2} + g \sin(z + \frac{\lambda}{2}) &= \cos \frac{\lambda}{2} \frac{p'}{m'}; \end{aligned}$$

and $p' = 3gm' \sin z + 2gm' \sin \frac{\lambda}{2} \cos z - 4gm'n' \sin \lambda \sin^2 \frac{z}{2}$ nearly.

We might have solved the problem otherwise. Thus $\tau \cos \frac{\lambda}{2}$ = tension in direction of the curve; $gm \cos z$, and $gm' \cos (z + \lambda)$ = motive force of gravity of the weights in the same direction. Therefore $g \cos z - \frac{\tau}{m} \cos \frac{\lambda}{2}$ = accelerative force of m , and $g \cos (z + \lambda) + \frac{\tau}{m} \cos \frac{\lambda}{2}$ = that of m' . As these generate equal velocities, they are equal. Equalling them we find $\tau = \frac{gm'n'}{\cos \frac{\lambda}{2}} (\cos z - \cos (z + \lambda))$ which is easily reduced

to the former expression of τ . Hence the accelerating force of either weight is

$$gn \cos z + gn' \cos (z + \lambda)$$

This multiplied by $2r dz$, and integrated, gives the square of the velocity

$$2grz \sin z + 2grn' \sin (z + \lambda) - 2grn' \sin \lambda,$$

the integral being corrected. From hence we may find the time as before.

The velocity and time might have been found by regarding the system as a complex pendulum, length r , and centre of oscillation in the radius passing through the centre of gravity of the weights.

The pressures may be found by adding the resolved parts of the gravity and tension perpendicular to the curve to the centrifugal force multiplied by the respective weights.

XII. QUESTION 562, by the Rev. Mr. BRONWIN.

Suppose the arch of a semicircle placed in a vertical plane, extremities parallel to the horizon, and convexity downwards. A bar of heavy metal of uniform thickness is laid on the inside of the curve, one end at the top of the semicircle, and descends along the arch by its weight. What is the difference between the time of descent through a given arch, and that of a material point descending down the same, the length of the bar being small? required also the pressures at each end of the rod?

SOLUTION, by Mr. BRONWIN, the Proposer.

Let p, p' be the pressures, m the mass, r the rad. of the circle described by the middle point of the bar, z the angle r makes with the horizontal line, λ the angle subtended by half the bar, l half its length.

$$\frac{p}{m} \cos (z - \lambda), \quad \frac{p'}{m} \cos (z + \lambda)$$

are the horizontal accelerative pressures ;

$$\frac{p}{m} \sin (z - \lambda), \quad \frac{p'}{m} \sin (z + \lambda)$$

are those vertical. Therefore

$$\frac{d^2 (r \cos z)}{dt^2} = - \frac{p}{m} \cos (z - \lambda) - \frac{p'}{m} \cos (z + \lambda),$$

$$\frac{d^2 (r \sin z)}{dt^2} = g - \frac{p}{m} \sin (z - \lambda) - \frac{p'}{m} \sin (z + \lambda).$$

From these we easily obtain

$$\begin{aligned} rd \cos z \frac{d^2 \cos z}{dt^2} + rd \sin z \frac{d^2 \sin z}{dt^2} \\ = gd \sin z + \frac{p - p'}{m} \sin \lambda dz. \end{aligned}$$

Again, $p \cos \lambda, p' \cos \lambda, (p' - p) \cos \lambda$ are the pressures and their difference perpendicular to the bar. $\frac{\sqrt{l^2}}{3}$ is the distance of the centre of gyration from the middle point. Therefore $l^2 : \frac{l^2}{3} :: m : \frac{m}{3}$ the matter which placed at the end of the bar would resist the communication of angular motion the same as the whole bar. Consequently $\frac{p' - p}{m} 3 \cos \lambda$ is the accelerative force producing rotary motion. And therefore

$$\frac{d^2 (lz)}{dt^2} = \frac{p' - p}{m} 3 \cos \lambda.$$

Eliminating $p' - p$ from the former equation by this ;

$$rd \left(\frac{d \cos z}{dt} \right)^2 + rd \left(\frac{d \sin z}{dt} \right)^2 = 2gd \sin z - \frac{l}{3} \tan \lambda d \left(\frac{dz}{dt} \right)^2 ;$$

$$\text{or } r \left(\frac{d \cos z}{dt} \right)^2 + r \left(\frac{d \sin z}{dt} \right)^2$$

$$= 2g \sin z - \frac{l}{3} \tan \lambda \frac{dz^2}{dt^2} + c.$$

At the commencement of the motion, the velocity is zero, and $z = \tau$; therefore $c = -2g \sin \lambda$. Reducing the last equation, it becomes

$$r \left(1 + \frac{1}{3} \tan^2 \lambda \right) \frac{dz^2}{dt^2} = 2g (\sin z - \sin \lambda).$$

Or, as we shall neglect quantities of the order λ^2 ;

$$\begin{aligned} r \frac{dz^2}{dt^2} &= 2g (\sin z - \sin \lambda), \quad dt = \sqrt{\frac{r}{2g}} \sqrt{\sin z - \sin \lambda} \frac{dz}{\sqrt{\sin z - \sin \lambda}} \\ &= \sqrt{\frac{r}{2g}} \text{ into } \frac{dz}{(\sin z)^{\frac{1}{2}}} + \frac{1}{2} \sin \lambda \frac{dz}{(\sin z)^{\frac{3}{2}}} \end{aligned}$$

nearly. The first term is the time of the descent of a material point. Therefore the difference required, or

$$\Delta t = \sqrt{\frac{r}{2g}} \frac{\sin \lambda}{2} \int \frac{dz}{(\sin z)^{\frac{3}{2}}}$$

nearly. This integral may be found by series.

Eliminating p' from the two differential equations,

$$\begin{aligned} r \sin(z + \lambda) \frac{d^2 \cos z}{dt^2} - r \cos(z + \lambda) \frac{d^2 \sin z}{dt^2} + g \cos(z + \lambda) \\ = -\sin 2\lambda \frac{p}{m}; \text{ or } r \cos \lambda \frac{d^2 z}{dt^2} + r \sin \lambda \frac{d^2 z}{dt^2} - g \cos(z + \lambda) \\ = \sin 2\lambda \frac{p}{m}; \text{ and } p = \frac{3}{2} gm \sin z - \frac{1}{6} gm \tan \lambda \cos z \\ - gm \sin \lambda. \text{ Afterwards } p' = \frac{3}{2} gm \sin z + \frac{1}{6} gm \tan \lambda \cos z \\ - gm \sin \lambda. \end{aligned}$$

These are true to quantities of the order λ . They might have been given exact, but the expressions would have been complicated.

We might consider the bar as a complex pendulum. By the common formulæ its length would be $r + \frac{l^2}{3r}$. Now $g \cos z$ = the accelerative force of gravity. Therefore if v equal the velocity,

$$\begin{aligned} v dv &= \left(r + \frac{l^2}{3r} \right) g \cos z dz, \text{ and } v^2 = \left(r + \frac{l^2}{3r} \right) 2g \sin z + c \\ &= \left(r + \frac{l^2}{3r} \right) 2g (\sin z - \sin \lambda). \text{ If we diminish this in the} \end{aligned}$$

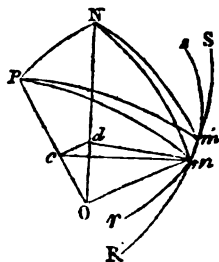
ratio of $(1 + \frac{r^2}{3r})^2$, or $(1 + \frac{1}{3} \tan^2 \lambda)^2$ to 1; we have the same expression as before. But we could not find the pressures without having recourse to the former method.

XIII. QUESTION 563, by Mr. W. S. B. WOOLHOUSE.

Find the equation to the evolute surface of any proposed loxodromic curve. And shew that if any number of loxodromes, having different meridional inclinations, be cut by a parallel of latitude, that the centres of absolute curvature, at the points where they are intersected, will all of them range in the surface of a sphere.

SOLUTION, by Mr. WOOLHOUSE, the Proposer.

Let m be any point in the loxodromic curve RS , and p the pole of a small circle rs , on the sphere, which osculates the curve at this point. As nm , nm' , make equal angles with the curve, and consequently also with rms , it is evident that they will both touch a small circle described through N round p , and therefore, that the spherical angle pnm is a right angle. Let λ , ϕ , be the longitude and latitude of m , λ' , ϕ' , those of p , the origin being taken where the curve intersects the equator. Then if m be the co-tangent of the inclination with the meridian, the equation of RS will be



$$\sin \phi = \frac{e^{2m\lambda} - 1}{e^{2m\lambda} + 1}.$$

But $\lambda = \lambda' + \frac{\pi}{2}$, and by spherics, $\sin nm \cdot \tan \angle nmp = \tan np$,

$$\text{or } m \cos \phi = \cot \phi' \therefore \sin \phi = \frac{1}{m} \sqrt{m^2 - \cot^2 \phi'}.$$

These substituted, we derive

$$\tan \phi' = \frac{1}{2m} \left\{ e^{m(\lambda' + \frac{\pi}{2})} + \frac{1}{e^{m(\lambda' + \frac{\pi}{2})}} \right\}$$

which determines the curve on the sphere to which p is limited.

This curve is evidently the spherical evolute of rs ; and the evolute surface is that of a cone through this evolute, having its vertex at the centre of the sphere. The value of $\tan \phi'$ will

obviously be the least possible when $e^{m(\lambda' + \frac{\pi}{2})} = 1$, or when $\lambda' + \frac{\pi}{2} = \lambda = 0$; and this value is therefore $\frac{1}{m}$: thus we see

that ϕ' commences with its least value, which is equal to the inclination of the loxodromic with the meridian. The circle rs is the osculating circle at the point m , and its plane is the osculating plane at that point. Let o be the centre of the sphere and op , ON , Om , radii to the points p , N , m . Draw md perpendicular to ON and dc to op ; then, as the spherical $\angle N$ is a right angle, the plane mon and consequently md is perpendicular to the plane nop ; and as dc is perpendicular to op , mc is thus also perpendicular to op , and therefore c is the centre of the circle rs , or the centre of absolute curvature. Now for all loxodromic curves which can be drawn round the pole N , the point d will be the same, when m is determined by the intersection of the curve with a given parallel of latitude, because this point is the centre of that parallel; and hence, as o , d , are thus fixed points, and the angle ocd always a right angle, all the centres c , of absolute curvature, will range in the surface of a sphere described on od as a diameter.

XIV. QUESTION 564, by Mr. WOOLHOUSE.

A curve whose equation is $f(x, y) = 0$, is successively modified in its figure by the simultaneous motion of all its points. At each instant the points move all in directions perpendicular to their curve and the motions resolved parallel to a given line are all the same. It is required to determine the general equation for the curve through all its changes, and also that of the curve which each point describes.

SOLUTION, by Mr. WOOLHOUSE, the Proposer.

Let the axis of x be taken parallel to the given line; and suppose

$$\phi(x, y, t) = 0$$

or

$$\phi = 0$$

to be the general equation to all the curves, t denoting the time,

which is to be fixed for each curve. By transferring the point xy to a consecutive position

$$0 = \left(\frac{d\phi}{dx}\right) \delta x + \left(\frac{d\phi}{dy}\right) \delta y + \left(\frac{d\phi}{dt}\right) \delta t \dots (1)$$

But, by the point varying along the curve at the time t ,

$$0 = \left(\frac{d\phi}{dx}\right) dx + \left(\frac{d\phi}{dy}\right) dy \dots \dots \dots (2)$$

From (1) multiplied by dy take away (2) multiplied by δy , and

$$0 = \left(\frac{d\phi}{dx}\right) (dy \delta x - dx \delta y) + \left(\frac{d\phi}{dt}\right) dy \delta t$$

Since δx has the same value for all the points we may assume it $= c \delta t$; and as δx , δy , relate to the normal, we have

$$\delta y = -\frac{dx}{dy} \delta x = -c \frac{dx}{dy} \delta t. \text{ These substituted give}$$

$$\begin{aligned} 0 &= c \left(\frac{d\phi}{dx}\right) \left(dy + \frac{dx^2}{dy}\right) + \left(\frac{d\phi}{dt}\right) dy \\ &= c \left(\frac{d\phi}{dx}\right) \cdot \frac{ds^2}{dy} + \left(\frac{d\phi}{dt}\right) dy \\ &= c \left(\frac{d\phi}{dx}\right) \cdot \frac{ds^2}{dy^2} + \left(\frac{d\phi}{dt}\right) \end{aligned}$$

But by (2)

$$\begin{aligned} \frac{dx}{dy} &= -\frac{\left(\frac{d\phi}{dy}\right)}{\left(\frac{d\phi}{dx}\right)} \\ \therefore \frac{ds^2}{dy^2} &= \frac{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2}{\left(\frac{d\phi}{dx}\right)^2} \end{aligned}$$

Hence, by substitution,

$$c \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 \right\} + \left(\frac{d\phi}{dx}\right) \left(\frac{d\phi}{dt}\right) = 0 \dots (3)$$

By this relation amongst the partial differential coefficients of the function ϕ , it may be shewn that the time t cannot be expressed by an algebraic function of the co-ordinates xy . For the purpose of integration suppose ϕ to be of the form

$$f(x, y) + t f_1(x, y) + t^2 f_2(x, y) + t^3 f_3(x, y) + \&c.$$

in which f characterises the equation of the given curve when $t = 0$, and f_1, f_2, f_3 , &c. determinate functions. Then,

$$\left(\frac{d\phi}{dx}\right) = \left(\frac{df}{dx}\right) + t \left(\frac{df_1}{dx}\right) + t^2 \left(\frac{df_2}{dx}\right) + \&c.$$

$$\left(\frac{d\phi}{dy}\right) = \left(\frac{df}{dy}\right) + t \left(\frac{df_1}{dy}\right) + t^2 \left(\frac{df_2}{dy}\right) + \&c.$$

$$\left(\frac{d\phi}{dt}\right) = f_1(x, y) + 2t f_2(x, y) + 3t^2 f_3(x, y) + \&c.$$

and (3) becomes

$$\begin{aligned} & c \left\{ \left(\frac{df}{dx}\right) + t \left(\frac{df_1}{dx}\right) + t^2 \left(\frac{df_2}{dx}\right) + \&c. \right\}^2 \\ & + c \left\{ \left(\frac{df}{dy}\right) + t \left(\frac{df_1}{dy}\right) + t^2 \left(\frac{df_2}{dy}\right) + \&c. \right\}^2 \\ & + \left\{ \left(\frac{df}{dx}\right) + t \left(\frac{df_1}{dx}\right) + t^2 \left(\frac{df_2}{dx}\right) + \&c. \right\} \times \{ f_1(x, y) + \\ & 2t f_2(x, y) + 3t^2 f_3(x, y) + \&c. \} = 0 \end{aligned}$$

which expanded, the coefficients of the several powers of t give

$$\begin{aligned} & c \left(\frac{df}{dx}\right)^2 + c \left(\frac{df}{dy}\right)^2 + \left(\frac{df}{dx}\right) f_1(x, y) = 0 \\ & 2c \left(\frac{df}{dx}\right) \left(\frac{df_1}{dx}\right) + 2c \left(\frac{df}{dy}\right) \left(\frac{df_1}{dy}\right) \\ & + \left(\frac{df_1}{dx}\right) f_1(x, y) + 2 \left(\frac{df}{dx}\right) f_2(x, y) = 0 \\ & c \left(\frac{df_1}{dx}\right)^2 + c \left(\frac{df_1}{dy}\right)^2 + 2c \left(\frac{df}{dx}\right) \left(\frac{df_2}{dx}\right) + 2c \left(\frac{df}{dy}\right) \left(\frac{df_2}{dy}\right) \\ & + \left(\frac{df_2}{dx}\right) f_1(x, y) + 2 \left(\frac{df_1}{dx}\right) f_2(x, y) + 3 \left(\frac{df}{dx}\right) f_3(x, y) = 0 \\ & \qquad \qquad \qquad \&c. \qquad \qquad \qquad \&c. \end{aligned}$$

which will serve for the determination of the functions $f_1(x, y)$, $f_2(x, y)$, $f_3(x, y)$, &c.

But this integration may be more easily effected as follows:

By (3)

$$\left(\frac{d\phi}{dt}\right) = c \cdot \frac{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2}{-\left(\frac{d\phi}{dx}\right)}$$

Let ψ characterise the second term of this equation, as it arises from ϕ ; that is, let

$$c. \frac{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2}{-\left(\frac{d\phi}{dx}\right)} = \psi(\phi)$$

Then

$$\left(\frac{d\phi}{dt}\right) = \psi(\phi), \text{ which gives}$$

$$\phi(x, y, t) = f(x, y) + \int dt \cdot \psi\{\phi(x, y, t)\} \dots (4)$$

From this equation we may determine successively the several terms of $\phi(x, y, t)$ by repeating the last determination for the derivation of a new one as under :

$$\begin{aligned} \phi &= f(x, y) \\ \phi_1 &= f(x, y) + t \psi(\phi) \\ \phi_2 &= f(x, y) + \int dt \psi(\phi_1) \\ \phi_3 &= f(x, y) + \int dt \psi(\phi_2) \\ &\quad \&c. \qquad \&c. \end{aligned}$$

Each of these will successively include a new term of the expression.

For the path of each point, take the equation (2)

$$\left(\frac{d\phi}{dx}\right) \frac{dx}{dy} + \left(\frac{d\phi}{dy}\right) = 0$$

and put $-\frac{\partial y}{\partial x}$ instead of $\frac{dx}{dy}$; then

$$\left(\frac{d\phi}{dx}\right) \delta y - \left(\frac{d\phi}{dy}\right) \delta x = 0 \dots (5)$$

As $\delta y, \delta x$, appertain to the track of the point, this equation, if it can be expressed in terms of the ordinates, will be the differential of the equation to the curve described by the point. The

coefficients $\left(\frac{d\phi}{dx}\right), \left(\frac{d\phi}{dy}\right)$, will involve the time t . If t cannot be eliminated by means of the equation $\phi(x, y, t) = 0$, it must be considered as an implicit function of xy determined by its involution in this equation, which is the result of (4).

Suppose the relation between x and y to be involved in the coexistence of two equations $\phi(x, y, t) = 0, \phi'(x, y, t) = 0$, the former being the determined equation of the curve at different times; then, in addition to (1), we shall have

$$\left(\frac{d\phi'}{dx}\right) \delta x + \left(\frac{d\phi'}{dy}\right) \delta y + \left(\frac{d\phi'}{dt}\right) \delta t = 0$$

and, eliminating dt with (1),

$$\left\{ \left(\frac{d\phi}{dx} \right) \left(\frac{d\phi'}{dt} \right) - \left(\frac{d\phi'}{dx} \right) \left(\frac{d\phi}{dt} \right) \right\} dx + \left\{ \left(\frac{d\phi}{dy} \right) \left(\frac{d\phi'}{dt} \right) - \left(\frac{d\phi'}{dy} \right) \left(\frac{d\phi}{dt} \right) \right\} dy = 0$$

But, in order that the path of the point may be perpendicular to the curve $\phi(x, y, t) = 0$, we must have by (5)

$$\frac{\delta y}{\delta x} = \frac{\left(\frac{d\phi}{dy} \right)}{\left(\frac{d\phi}{dx} \right)}$$

$$\begin{aligned} & \therefore \left(\frac{d\phi}{dx} \right) \left\{ \left(\frac{d\phi}{dx} \right) \left(\frac{d\phi'}{dt} \right) - \left(\frac{d\phi'}{dx} \right) \left(\frac{d\phi}{dt} \right) \right\} \\ & + \left(\frac{d\phi}{dy} \right) \left\{ \left(\frac{d\phi}{dy} \right) \left(\frac{d\phi'}{dt} \right) - \left(\frac{d\phi'}{dy} \right) \left(\frac{d\phi}{dt} \right) \right\} = 0, \text{ or} \\ & \left\{ \left(\frac{d\phi}{dx} \right)^2 + \left(\frac{d\phi}{dy} \right)^2 \right\} \left(\frac{d\phi'}{dt} \right) - \left\{ \left(\frac{d\phi}{dx} \right) \left(\frac{d\phi'}{dx} \right) + \left(\frac{d\phi}{dy} \right) \left(\frac{d\phi'}{dy} \right) \right\} \left(\frac{d\phi}{dt} \right) = 0 \end{aligned}$$

Multiply this by $\left(\frac{d\phi}{dx} \right)$, observing that by (2)

$$\left(\frac{d\phi}{dx} \right) \left(\frac{d\phi}{dt} \right) = -c \left\{ \left(\frac{d\phi}{dx} \right)^2 + \left(\frac{d\phi}{dy} \right)^2 \right\}$$

and we get

$$\left(\frac{d\phi}{dx} \right) \left(\frac{d\phi'}{dt} \right) + c \left\{ \left(\frac{d\phi}{dx} \right) \left(\frac{d\phi'}{dx} \right) + \left(\frac{d\phi}{dy} \right) \left(\frac{d\phi'}{dy} \right) \right\} = 0 \dots \dots (6)$$

A particular integration of this equation is $\phi' = a\phi + b$, since it will then give (3); but this solution does not apply to the present case as the functions ϕ, ϕ' , must be independent. If, in reference to the function ϕ' , we put

$$c \frac{\left(\frac{d\phi}{dx} \right) \left(\frac{d\phi'}{dx} \right) + \left(\frac{d\phi}{dy} \right) \left(\frac{d\phi'}{dy} \right)}{\left(\frac{d\phi}{dx} \right)} = \psi_1(\phi')$$

we shall have, similarly to (4),

$$\phi'(x, y, t) = f'(x, y) + \int dt \cdot \psi_1 \{ \phi'(x, y, t) \} \dots (7)$$

in which the form of f' is arbitrary: and we may thus find various integrations of (6) by different assumptions of f' , but all these integrations must necessarily determine the same curve for the track of the point. By taking $f'(x, y) = ax + b$, and using this for ϕ' , in the first instance, we find

$$\psi_1(\varphi') = -ac;$$

and as this is constant, we obtain for a complete integral of (6)

$$\varphi'(x, y, t) = ax + b - act = 0$$

Let x_1 be the value of x at the commencement, when $t = 0$, and

$$(x - x_1) - ct = 0 \dots \dots (8)$$

a result which we might have inferred from the assumption $\delta x = c\delta t$. This combined with $\varphi(x, y, t) = 0$ determines the curve described by the point; and hence by substituting $\frac{x-x_1}{c}$ instead of t , the equation of its track will be

$$\varphi\left(x, y, \frac{x-x_1}{c}\right) = 0$$

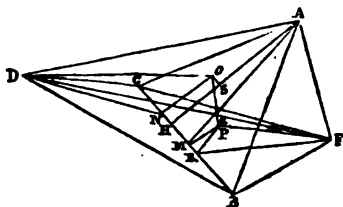
in which x_1 is the arbitrary constant that modifies the curve for the different points.

XV. QUESTION 565, by Mr. WILLIAM GODWARD.

The six edges of any irregular tetrahedron are opposed two by two, and the nearest distance of two opposite edges is called *breadth*; so that the tetrahedron has *three breadths* and *four heights*. It is required to demonstrate, independently of the geometry of co-ordinates, that in every tetrahedron, the sum of the reciprocals of the squares of the *breadths*, is equal to the sum of the reciprocals of the squares of the heights. (See Ladies' Diary, Question 1523, and Young's Differential Calculus, page 235).

FIRST SOLUTION, by Mr. GODWARD, the Proposer.

Let ABCD be any irregular tetrahedron, and draw planes through the edges AB, BC, AC parallel to the opposite edges, forming the tetrahedron ABCE; then it is obvious that the *breadths* of the original tetrahedron are equal to the ls drawn from A, B, C, to the opposite faces BCF, ACF and ABF. Draw DO, FP $\perp$ the plane ABC, and ON, AH, PR $\perp$ BC, also OS $\parallel$ BC cutting AH in S, and join DN, FR, which are $\perp$ BC. Join DF intersecting the plane ABC in G, and through G draw AM cutting BC in M. Also in order to



prevent obscurity in the diagram let us conceive AO, BO, CO, BG, CG to be joined. Now the pyramids BCFA, BCFD, having the same base BCF and equal altitudes, are equal. Similarly pyr. BCFA = CAFD = ABFD = $\therefore$ to BCFD; hence each of these solids is = $\frac{1}{3}$ of the whole solid DABCF, or pyr. ABCD = 2ABCF, and $\therefore$ DO = 2FP. Whence the pyr. BGCD = 2BGCF = $\frac{2}{3}$ BCDF, AGCD = 2AGCF = $\frac{2}{3}$ CAFD, and AGBD = 2AGBF = $\frac{2}{3}$ ABFD; but BCDF = CAFD = ABFD; hence BGCD = AGCD = AGBD, or the Δ BGC = AGC = AGB, evidently implying that G is the centre of gravity of the triangle ABC, and $\therefore$ AG = 2GM. Now it is manifest that the points P, G, O, are in a straight line, and the right-angled Δ s DOG, FPG similar; hence, since DO = 2FP, GO = 2GP. Whence the Δ s AOG, MPG, having the sides about the equal angles at G proportional, are similar; $\therefore$ AO is = and $\parallel$ to 2MP; hence the right-angled Δ s ASO, PRM are similar, and AS = 2PR, or PR = $\frac{1}{2}$ AS = $\frac{1}{2}$ (AH - ON). Let A = area BDC, B = Δ CD, C = Δ BD, D = Δ BC, α = BCF, β = ACF, γ = ABF, α' = BCO, β' = AOC, γ' = ABO; also let h, h', h'', h''' denote the heights, and b, b', b'' the breadths of Δ BCD, and put ABCD

= $\frac{8}{3}$. Then, since Δ s upon the same base are as their altitudes, DN : AH :: A : D, and DN : ON :: A : α' ; $\therefore$ DN² : AH.ON :: A² : $\alpha'D$ = $\frac{1}{DN^2} \times$ AH.ON.A², also DN² : AH² ::

$$A^2 : D^2 = \frac{AH^2 \cdot A^2}{DN^2}, \text{ and } DN^2 : FR^2 = PR^2 + PF^2 = PR^2 + \frac{1}{4} DO^2$$

$$\therefore A^2 : \alpha^2 = \frac{1}{4} (AH^2 - 2AH \cdot ON + ON^2 + OD^2) \times A^2 \div DN^2$$

$$= \frac{1}{4} \left(\frac{AH^2 \cdot A^2}{DN^2} - \frac{2AH \cdot ON}{DN^2} A^2 + A^2 \right) \text{ or}$$

$$\alpha^2 = \frac{1}{4} (A^2 + D^2 - 2\alpha'D)$$

$$\text{Similarly } \beta^2 = \frac{1}{4} (B^2 + D^2 - 2\beta'D)$$

$$\gamma^2 = \frac{1}{4} (C^2 + D^2 - 2\gamma'D)$$

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = \frac{1}{4} \{A^2 + B^2 + C^2 + 3D^2 - 2D(\alpha' + \beta' + \gamma')\}$$

$$= \frac{1}{4} (A^2 + B^2 + C^2 + D^2).$$

$$\text{But } \frac{1}{h^2} + \frac{1}{h'^2} + \frac{1}{h''^2} + \frac{1}{h'''^2} = \frac{1}{8^2} (A^2 + B^2 + C^2 + D^2)$$

$$\text{and } \frac{1}{b^2} + \frac{1}{b'^2} + \frac{1}{b''^2} = \frac{4}{8^2} (\alpha^2 + \beta^2 + \gamma^2)$$

$$= \frac{1}{8^2} (A^2 + B^2 + C^2 + D^2).$$

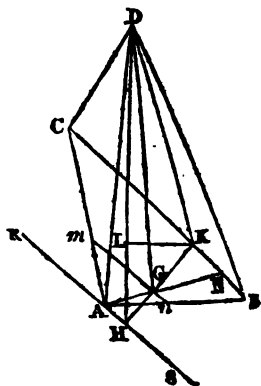
$$\text{Therefore } \frac{1}{b^2} + \frac{1}{b'^2} + \frac{1}{b''^2} = \frac{1}{h^2} + \frac{1}{h'^2} + \frac{1}{h''^2} + \frac{1}{h'''^2}.$$

Q. E. D.

SECOND SOLUTION, by Mr. WOOLHOUSE.

The following is the Solution I sent to the Diary :

Let ABCD be any tetrahedron; DC, perpendicular to the face ABC, one of the heights. Draw RAS, AGN, parallel and HGK perpendicular to CB; draw AGN meeting CB at N, and join DH, DK; and let GQ, KL, be perpendiculars to DH. To avoid the drawing of more lines, which would complicate the figure, denote the heights from the corners A, B, C, D, by the letters h, h', h'', h''' ; and the breadths between DA, DB, DC, and their opposite sides by β, β', β'' .



First consider β : As this line is perpendicular to both DA and CB, it must evidently be perpendicular to the plane of DA and RS; and as CB is parallel to this plane, such perpendicular will have the same value from all points in CB, and will be equal to KL. That is, $\beta = KL$.

$$\therefore \frac{1}{\beta^2} = \frac{DH^2}{(2 \Delta DHK)^2} = \frac{GH^2 + h'''^2}{HK^2 \cdot h'''^2} = \frac{GH^2}{HK^2} \cdot \frac{1}{h'''^2} + \frac{1}{HK^2}$$

$$\text{Also, as } h \cdot \Delta DBC = h''' \cdot \Delta ABC \text{ we have } \frac{1}{h^2} = \left(\frac{\Delta DBC}{\Delta ABC} \right)^2 \cdot \frac{1}{h'''^2} \\ = \left(\frac{DK}{HK} \right)^2 \cdot \frac{1}{h'''^2} = \frac{GK^2 + h'''^2}{HK^2} \cdot \frac{1}{h'''^2} = \frac{GK^2}{HK^2} \cdot \frac{1}{h'''^2} + \frac{1}{HK^2}$$

Hence as $HK = GH + GK$, we derive

$$\frac{1}{\beta^2} - \frac{1}{h^2} = \frac{GH - GK}{HK} \cdot \frac{1}{h'''^2} = \left(1 - 2 \frac{GK}{HK} \right) \cdot \frac{1}{h'''^2} \quad \text{That is,} \\ \frac{1}{\beta^2} - \frac{1}{h^2} = \left(1 - 2 \frac{GN}{AN} \right) \frac{1}{h'''^2}$$

If we now conceive BGN' , CGN'' , to be drawn to the opposite sides, we shall similarly have

$$\frac{1}{\beta'^2} - \frac{1}{h'^2} = \left(1 - 2 \frac{GN'}{BN'} \right) \frac{1}{h'''^2} \\ \frac{1}{\beta''^2} - \frac{1}{h''^2} = \left(1 - 2 \frac{GN''}{CN''} \right) \frac{1}{h'''^2}$$

These three equations combined, observing that

$$\frac{GN}{AN} + \frac{GN'}{BN'} + \frac{GN''}{CN''} = \frac{GN}{AN} + \frac{Gm + Gn}{BC} = \frac{GN}{AN} + \frac{GA}{AN} = 1, \text{ we derive}$$

$$\frac{1}{\beta^2} + \frac{1}{\beta'^2} + \frac{1}{\beta''^2} = \frac{1}{h^2} + \frac{1}{h'^2} + \frac{1}{h''^2} = \frac{1}{h'''^2}$$

$$\therefore \frac{1}{\beta^2} + \frac{1}{\beta'^2} + \frac{1}{\beta''^2} = \frac{1}{h^2} + \frac{1}{h'^2} + \frac{1}{h''^2} + \frac{1}{h'''^2}$$

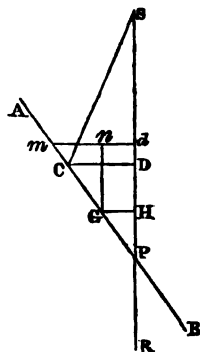
XVI. QUESTION 566, by Mr. W. S. B. WOOLHOUSE.

A heavy metal rod AB is suspended at c , by a string SC connected to a fixed point s , and is made to describe a double conic surface round an unknown point P in the vertical right line SR . There are given $SC = s$, $AG = GB = a$, $GC = b$, and the angular velocity round $SR = \omega$; to find the position of P and the inclination of the rod. Also $AG = GB$ and GC being given as above, there are particular determinate values of SC and GP which are required, being such that, if the angular velocity be varied at pleasure, the point P will be found in the vertical right line SR ; that is, P will always be the centre of rotation, or common vertex, to the two conic surfaces described by the rod.

SOLUTION, by Mr. WOOLHOUSE, the Proposer.

Since the appearance of the above question in the Repository I have observed one of the same kind proposed by Mr. Davies, which forms Question (12) of the Gentleman's Diary, for 1831. The solution of Mr. Jones, printed in the following Diary, is widely different however from the question as proposed, so that I have not been anticipated in the following solution to the above, which leads to very interesting results:

Let m be any point in the rod and draw CD , md , GH , perpendicular to SP , and gn to md . Denote GP by β , the angle csp by θ , cps by i , and md , the distance of any particle m of the rod from SP , by y . Then the centrifugal force of the particle in the direction dm , arising from the angular motion, $= my\omega^2$. The moment round G of the



centrifugal forces of all the particles = $\Sigma m (y\omega^2) \cdot Gn$, which is readily found = $\frac{a^2}{3} \omega^2 \cos i \sin i$. Also since the vertical

resolvent of the tension of the string supports the rod, and $\therefore = g$, assuming *unity* for the mass of the rod, we have the tension itself = $g \sec \theta$ and its moment round G in opposition to the tendency of centrifugal force, = $(g \sec \theta) b \sin (i + \theta)$

$$= gb \frac{(\sin i + \theta)}{\cos \theta} = gb (\sin i + \cos i \tan \theta). \text{ Hence, as the}$$

rod at any instant has no tendency to turn round G in the plane SPA, we have

$$0 = gb (\sin i + \cos i \tan \theta) - \frac{a^2}{3} \omega^2 \cos i \sin i \dots\dots\dots(1)$$

Again, the moments round P of the centrifugal forces of all the particles which tend to increase the angle SPA, = $\Sigma m (y\omega^2)$.

Pd, which is found = $\left(\frac{a^2}{3} + \beta^2\right) \omega^2 \cos i \sin i$; and the similar

tendency of gravity = $g\beta \sin i$. The moment round P of the tension which destroys this tendency = $g(\beta + b)(\sin + \cos i \tan \theta)$.

$$\therefore 0 = g(\beta + b)(\sin i + \cos i \tan \theta) - \left(\frac{a^2}{3} + \beta^2\right) \omega^2 \cos i \sin i - g\beta \sin i \dots\dots\dots(2)$$

From this deduct (1) and

$$\begin{aligned} 0 &= g\beta (\sin i + \cos i \tan \theta) - \beta^2 \omega^2 \cos i \sin i - g\beta \sin i \\ &= g\beta \cos i \tan \theta - \beta^2 \omega^2 \cos i \sin i \\ &= g \tan \theta - \beta \omega^2 \sin i \dots\dots\dots(3) \end{aligned}$$

Here $g \tan \theta$ is the resolvent of the tension of the string in the direction CD, and $\beta \omega^2 \sin i$ is the centrifugal force of all the particles which compose the rod, in the contrary direction, being the same as if concentrated in the centre of gravity, G. Substitute $\beta \omega^2 \sin i$ instead of $g \tan \theta$ in (1), and

$$0 = gb \sin i + \beta b \omega^2 \cos i \sin i - \frac{a^2}{3} \omega^2 \cos i \sin i$$

$$= gb - \left(\frac{a^2}{3} - \beta b\right) \omega^2 \cos i$$

$$\therefore \cos i = \frac{gb}{\left(\frac{a^2}{3} - \beta b\right) \omega^2} \dots\dots\dots(4)$$

By the relations of the figure

$$s \sin \theta = (\beta + b) \sin i \dots\dots(5)$$

Divide this by $g \tan \theta = \beta \omega^2 \sin i$, and $\frac{s}{g} \cos \theta = \frac{\beta + b}{\beta \omega^2}$

$$\therefore \cos \theta = \left(\frac{\beta + b}{s} \right) \frac{g}{\beta \omega^2} \dots\dots(6)$$

But (5) gives $s^2 (1 - \cos^2 \theta) = (\beta + b)^2 (1 - \cos^2 i)$; in this, substitute the values of $\cos i$, $\cos \theta$ by (4), (6), and

$$s^2 \left\{ 1 - \left(\frac{\beta + b}{s} \right)^2 \frac{g^2}{\beta^2 \omega^4} \right\} = (\beta + b)^2 \left\{ 1 - \frac{g^2 b^2}{\left(\frac{a^2}{3} - \beta b \right)^2 \omega^4} \right\}$$

which reduces to

$$0 = g^2 (\beta + b)^2 \left\{ \left(\frac{a^2}{3} - \beta b \right)^2 - \beta^2 b^2 \right\} + \beta^2 \left(\frac{a^2}{3} - \beta b \right)^2 \{ (\beta + b)^2 - s^2 \} \omega^4 \dots\dots(7)$$

an equation for determining the value of β or GP, and hence i and θ by (4) and (6). It gives, for the angular velocity,

$$\begin{aligned} \left(\frac{\omega^2}{g} \right)^2 &= \frac{(\beta + b)^2 \left\{ \left(\frac{a^2}{3} - \beta b \right)^2 - \beta^2 b^2 \right\}}{\beta^2 \left(\frac{a^2}{3} - \beta b \right)^2 \{ s^2 - (\beta + b)^2 \}} \\ \therefore \frac{\omega^2}{g} &= \frac{\beta + b}{\beta \left(\frac{a^2}{3} - \beta b \right)} \sqrt{\frac{\left(\frac{a^2}{3} - \beta b \right)^2 - \beta^2 b^2}{s^2 - (\beta + b)^2}} \\ &= \frac{\beta + b}{\beta \left(\frac{a^2}{3} - \beta b \right)} \sqrt{\frac{\frac{a^2}{3}}{s + (\beta + b)}} \sqrt{\frac{\frac{a^2}{3} - 2\beta b}{s - (\beta + b)}} \dots\dots(8) \end{aligned}$$

Now, as long as the motion is possible, this value must be posi-

tive and real, and therefore $\sqrt{\frac{\frac{a^2}{3} - 2\beta b}{s - (\beta + b)}}$ must be positive for positive values of β ; and consequently

$$\left(\frac{a^2}{3} - 2\beta b \right) \{ s - (\beta + b) \} > 0$$

$$\therefore \left(\frac{a^2}{6b} - \beta \right) \{ (s - b) - \beta \} > 0$$

This shows, that whenever the motion is possible, the positive value of β cannot be between the values of $\frac{a^2}{\delta b}$ and $s - b$,

being always less than the less and greater than the greater of these two limiting values. The former value answers to $\omega = 0$; the latter to $\omega = \infty$. Many other particulars may be inferred from the equation (8); and it will be seen that this investigation is not restricted to the particular motion specified in the question: β may assume negative values, and then the motion may be similar to the common conical pendulum..... (9)
We have also

$$\begin{aligned} \text{SP} &= s \cos \theta + (\beta + b) \cos i = (\beta + b) \left\{ \frac{g}{\beta \omega^2} + \frac{gb}{\left(\frac{a^2}{3} - \beta b\right) \omega^2} \right\} \\ &= \left\{ \frac{\beta + b}{\beta} \cdot \frac{\frac{a^2}{3}}{\left(\frac{a^2}{3} - \beta b\right)} \right\} \frac{g}{\omega^2} = \frac{a^2}{3} \sqrt{\frac{s^2 - (\beta + b)^2}{\left(\frac{a^2}{3} - \beta b\right)^2 - \beta^2 b^2}} \\ &= \sqrt{\frac{a^2}{3}} \cdot \sqrt{\frac{s^2 - (\beta + b)^2}{\frac{a^2}{3} - 2\beta b}} \dots\dots\dots (10) \end{aligned}$$

When β is independent of the angular velocity, the coefficient of ω^4 must disappear in (7) and $\therefore$ either $\frac{a^2}{3} - \beta b = 0$, or $(\beta + b)^2 - s^2 = 0$. The former is inadmissible, for then the value of β could not fulfil (7). The latter gives $(\beta + b)^2 - s^2 = 0$ and $\left(\frac{a^2}{3} - \beta b\right)^2 - \beta^2 b^2 = 0$; or $(\beta + b) - s = 0$ and $\left(\frac{a^2}{3} - \beta b\right) - \beta b = 0$; or, which is the same thing,

$$\left. \begin{aligned} 2\beta b &= \frac{a^2}{3} \\ \beta + b &= s \end{aligned} \right\} \dots\dots\dots (11)$$

The conditions of (11) will cause the rod to always revolve round the same point P of itself, whatever be the angular motion; these conditions being symmetrical with respect to β and b shew that the points c and P are reciprocal to each other; that

is, if the rod be suspended from P with the same string, it will always turn round C in the same manner in which it always turns round

P . By using the value of $\frac{\omega^2}{g}$ by (8), in (4) and (6) we have

$$\left. \begin{aligned} \cos i &= \frac{\beta b}{\frac{a}{\sqrt{3}}(\beta + b)} \sqrt{\frac{s^2 - (\beta + b)^2}{\frac{a^2}{3} - 2\beta b}} \\ \cos \theta &= \frac{\frac{a^2}{3} - \beta b}{\frac{a}{\sqrt{3}} \cdot s} \sqrt{\frac{s^2 - (\beta + b)^2}{\frac{a^2}{3} - 2\beta b}} \end{aligned} \right\} \dots\dots\dots (12)$$

which determine the geometrical relations when β is known.

XVII. QUESTION 567, by Mr. W. S. B. WOOLHOUSE.

Given $(a^2 - x^2) \left(y \cdot \frac{d^2 y}{dx^2} + \frac{dy^2}{dx^2} \right) - \frac{y}{x} (y^2 - 2x^2) \frac{dy}{dx} + \frac{y^4}{4x^2} - \frac{9x^2}{4} + 2y^2 - 3a^2 = 0$, to find y in terms of x without series ;
supposing $y = 0$ when $x = 0$.

SOLUTION, by Mr. WOOLHOUSE, the Proposer.

Assume $z = \frac{y^2 - 3x^2}{2x}$; then, $y^2 = 3x^2 + 2xz$, and

$$\begin{aligned} y \frac{dy}{dx} &= x \frac{dz}{dx} + z + 3x \\ y \frac{d^2 y}{dx^2} + \frac{dy^2}{dx^2} &= x \frac{d^2 z}{dx^2} + 2 \frac{dz}{dx} + 3 \\ \frac{y^2 - 2x^2}{x} &= z + 2z \end{aligned}$$

$$\frac{y^4}{4x^2} - \frac{9x^2}{4} + 2y^2 - 3a^2 = z^2 + 7xz + 6x^2 - 3a^2$$

and consequently the equation becomes, after a little reduction

$$x(a^2 - x^2) \frac{d^2 z}{dx^2} + (2a^2 - 3x^2 - 2xz) \frac{dz}{dx} - z^2 = 0$$

$$\therefore \{x(a^2 - x^2) \frac{d^2 z}{dx^2} + (a^2 - 3x^2) dz\} + a^2 dz - (2xz dz + z^2 dx) = 0$$

which integrated gives

$$x(a^2 - x^2) \frac{dz}{dx} + a^2 z - xz^2 = 0$$

Let now $z = p \frac{\sqrt{(a^2 - x^2)}}{x}$; and

$$\frac{dz}{dx} = \frac{dp}{dx} \cdot \frac{\sqrt{(a^2 - x^2)}}{x} - \frac{a^2 p}{x^2 \sqrt{(a^2 - x^2)}}$$

$$a^2 z - xz^2 = a^2 p \frac{\sqrt{(a^2 - x^2)}}{x} - \frac{p^2 (a^2 - x^2)}{x}$$

Therefore by substitution

$$\begin{aligned} 0 &= \frac{dp}{dx} (a^2 - x^2)^{\frac{3}{2}} - a^2 p \frac{\sqrt{(a^2 - x^2)}}{x} + a^2 p \frac{\sqrt{(a^2 - x^2)}}{x} - \frac{p^2 (a^2 - x^2)}{x} \\ &= \frac{dp}{dx} (a^2 - x^2)^{\frac{3}{2}} - \frac{p^2 (a^2 - x^2)}{x} \end{aligned}$$

which gives

$$\frac{dp}{p^2} = \frac{dx}{x \sqrt{(a^2 - x^2)}}$$

and hence by integration

$$\frac{1}{p} = \frac{1}{a} \left\{ c + \text{hyp. log.} \frac{a + \sqrt{(a^2 - x^2)}}{x} \right\}$$

$$\therefore p = \frac{a}{c + \text{hyp. log.} \frac{a + \sqrt{(a^2 - x^2)}}{x}}$$

$$z = p \frac{\sqrt{(a^2 - x^2)}}{x} = \frac{a \sqrt{(a^2 - x^2)}}{x \left\{ c + \text{hyp. log.} \frac{a + \sqrt{(a^2 - x^2)}}{x} \right\}}$$

$$y^2 = 3x^2 + 2xz$$

$$= 3x^2 + \frac{2a \sqrt{(a^2 - x^2)}}{c + \text{hyp. log.} \frac{a + \sqrt{(a^2 - x^2)}}{x}}$$

$$\therefore y = \sqrt{\left\{ 3x^2 + \frac{2a \sqrt{(a^2 - x^2)}}{c + \text{hyp. log.} \frac{a + \sqrt{(a^2 - x^2)}}{x}} \right\}}$$

in which c is the arbitrary constant.

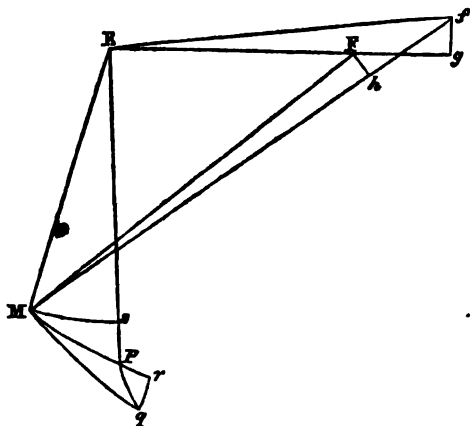
XVIII. QUESTION 568, by the Rev. Mr. BRICE
BRONWIN, Denby, near Wakefield.

Investigate the expressions of La Grange for the variation of the elements a, e, π of a disturbed orbit, considered as an ellipse; and make use of the same principle and method as those by which Newton determined the regression and variation of the inclination.

FIRST SOLUTION, by Mr. BRONWIN, the Proposer.

Let mp denote the velocity and direction of the moon at m , pq the velocity communicated by the disturbing forces at the same point: then mq will represent the new velocity and direction.

If E and F be the foci; EM, FM must make equal angles with the curve mp : and f being the new focus; EM, FM must make equal angles with the curve mq . Consequently the $\angle FMh = \angle pmq$. And if fh, fg be perpendicular to MF, Eg ; fh will be the momentary variation of the axis major, Fg that of the



eccentricity, and $\angle feg$ the progression of the apside.

Let μ the absolute force or gravity of M to E be unity; and D be the disturbing force along the rad. vect. D that perp. there-

to. Then by a well known property of elliptic movement we have

$$Mq^2 = \frac{2 f M}{EM (EM + fM)}, Mp^2 = \frac{2 F M}{EM (EM + FM)}$$

And taking the difference of these, neglecting the powers of hf above the first, we have

$$Mq^2 - Mp^2 = Mr^2 - Mp^2 = 2 Mp. rp;$$

$$\frac{2 f M}{EM (EM + fM)} - \frac{2 F M}{EM (EM + FM)} = \frac{2 fh}{(EM + FM)^2} = \frac{fh}{2a^2}.$$

$$\text{Therefore } fh = 4 a^2 Mp. rp = 4 a^2. Mp. pq. \cos rpq. = 4 a^2. Mp. pq \cos Epq. \cos Epr + \sin Epq \sin Epr =$$

$$4 a^2 Mp \left(Epr. D + \sin Epr. D \right) = 4 a^2 (- D. ps + D. Ms);$$

$$\text{or } \delta a = 2 a^2 \left(D r dv - D dr \right).$$

$$\text{Again } fh = FM. \angle FMh = 2 MF. \angle rMq =$$

$$2 MF. \frac{rq}{Mq} = 2 MF. \frac{rq}{Mp} = \frac{2 MF}{Mp}. pq. \sin rpq =$$

$$\frac{2 MF}{Mp}. pq. (\sin Epq \cos Epr - \cos Epq \sin Epr) =$$

$$\frac{2 MF}{Mp} \left(- \sin Epr D + \cos Epr. D \right).$$

$$\text{But } fg = fh \cos F \times fh \sin E; \text{ and } fg = \delta (2ae) = 2e\delta a + 2a\delta e. \text{ Consequently}$$

$$\delta e = \frac{1}{2a} \left\{ (\cos F - e) fh + FH. \sin F \right\} =$$

$$\frac{MF - A}{2a. MF} \cdot \frac{1 - e^2}{e} fh + \frac{ME \sin E}{2A. MF} \cdot fh =$$

$$- a. \frac{1 - e^2}{e} D. ps + \left(a \frac{1 - e^2}{e} - \frac{ME^2}{ae} \right) D. Ms,$$

denoting by E and F the angles at E and F . Or adopting the usual symbols

$$\delta e = -a \frac{1 - e^2}{e} D dr + \left(a \frac{1 - e^2}{e} - \frac{r^2}{ae} \right) D r dv.$$

$$\text{Again } \angle feg = \frac{fg}{eg} = \frac{fg}{ef} = \frac{fh \sin F - fh \cos F}{ef}$$

$$= \frac{ME \sin E}{MF. EF} \cdot fh - \frac{MF - a(1 - e^2)}{MF. EF. e} \cdot fh =$$

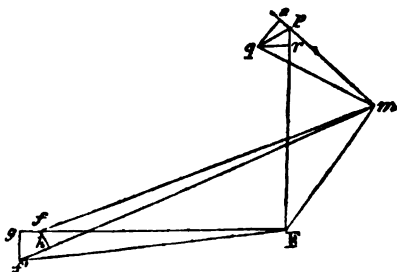
$$a \frac{1}{2} \cdot \frac{\sqrt{1-e^2}}{e} \cos E. D + \frac{a(1-e^2) + ME}{a \frac{1}{2} e \sqrt{1-e^2}} \sin E. D^1;$$

$$\text{or } \delta\pi = a \frac{1}{2} \cdot \frac{\sqrt{1-e^2}}{e} \cos cv. D dt + \frac{a(1-e^2) + r}{a \frac{1}{2} e \sqrt{1-e^2}} \sin cv. D dt^1.$$

These are the very expressions of La Grange for the variation of these elements.

SECOND SOLUTION, by Mr. WOOLHOUSE, London.

Let F be the focus to which the body (m) is attracted and let us inquire into the variations produced in the elements of the orbit by the disturbing forces, during the preceding instant dt . Let mp represent the velocity of m , supposing the disturbing forces not to have acted during the preceding instant dt ; and let f be corresponding position of the secondary focus; also let mq be the velocity of the body under the action of those forces, and f' the new position of f in consequence. Draw the perpendiculars qr to pf , qs to mp , fh to mf' and $f'g$ to ff . Since $mf + mf' = 2(a + \epsilon a)$, it is evident that $f'h = 2\delta a$; it is also obvious that $gf = \delta(2ae)$ and the $\angle fff' = \delta\pi$. Let D be the disturbing force along the radius vector, and D' that perpendicular to the radius vector, and we shall also have $pr = Ddt$, $qr = D'dt$.



Supposing the attractive force towards the centre F to be unity at the distance unity, the common theory of the undisturbed elliptic motion, gives

$$v^2 = mp^2 = \frac{ds^2}{dt^2} = \frac{2}{r} - \frac{1}{a} \dots \dots \dots (1)$$

The effect produced on this equation by the disturbing forces will be $2v\delta v = \frac{\delta a}{a^2}$; but ϵv the effect along the curve arises

from the resolvents of pr , qr , thus, $\epsilon v = ps = D'dt \frac{rdv}{ds} -$

$\frac{dr}{ds}$. Hence using $\frac{ds}{dt}$ for v , we find

$$\delta a = 2a^2 v dv = 2a^2 (v' r dv - v dr) \dots \dots (A)$$

Again, as mr , mf , and mf' , make equal angles with their respective curves, mp , mq , we have $\angle fmh = 2 \angle pmq$, and therefore $fh = mf \cdot \frac{fh}{mf} = 2mf \cdot \frac{qs}{mq} = 2mf \cdot \frac{qs}{mp} = 2(2a-r)$.

$\frac{qs \cdot v}{v^2}$. Also $qs = \frac{dr}{ds} + v' \frac{dr}{ds}$ and $qs \cdot v = qs \cdot \frac{ds}{dt}$
 $= v' r dv + v' dr$; this and the value of v^2 by (1), give fh
 $= 2ar (v' r dv + v' dr)$. But the equation of the ellipse,
 $r(1 + e \cos v) = a(1 - e^2)$, gives

$$r dv = \frac{a}{r \sin v} \cdot \frac{1 - e^2}{e} dr, \quad dr = \frac{er \sin v}{a(1 - e^2)} r dv$$

$$\therefore fh = \frac{2a^2(1 - e^2)}{e \sin v} v' dr + \frac{2er^2 \sin v}{1 - e^2} v' r dv \dots \dots (2)$$

We hence find gf , gf' , from the expressions

$$gf = f'h \cdot \cos f - fh \cdot \sin f = 2\delta a \cos f - fh \cdot \sin f$$

$$gf' = f'h \cdot \sin f + fh \cdot \cos f = 2\delta a \sin f + fh \cdot \cos f$$

in which f denotes the $\angle mfr$. By first using

$$\cos f = \frac{1}{e} - \frac{1 - e^2}{e} \cdot \frac{a}{2a - r}, \quad \sin f = \frac{r}{2a - r} \sin v, \text{ and after}$$

$$\text{terwards } \sin^2 v = \frac{1 - e^2}{e^2} \cdot \frac{r(2a - r) - a^2(1 - e^2)}{r^2} \text{ we thus find}$$

$$gf = \delta(2ae) = -2a^2 \frac{1 + e^2}{e} v' dr + 2(a^2 \frac{1 + e^2}{e} - \frac{r^2}{e}) v' r dv$$

$$gf' = \frac{2a^2(1 - e^2)}{e \sin v} \cdot \frac{a(1 - e^2) - r}{er} v' dr + 2r \frac{a(1 - e^2) + r}{1 - e^2} \sin v v' r dv$$

From the value of gf we derive

$$\delta e = \frac{\delta(2ae)}{2a} - e \frac{\delta a}{a} = -a \frac{1 - e^2}{e} v' dr + (a \frac{1 - e^2}{e} - \frac{r^2}{ae}) v' r dv \dots (B)$$

$$\text{And from the value of } gf', \text{ observing that } \frac{a(1 - e^2) - r}{er} =$$

as v , we deduce

$$\delta e = \frac{gf'}{2ae} = \frac{a(1 - e^2)}{e^2 \sin v} \cos v v' dr + \frac{r}{ae} \cdot \frac{a(1 - e^2) + r}{1 - e^2} \sin v v' r dv$$

But the equations of the ellipse and (1) give

$$dr = \frac{e \sin v}{a^3 \sqrt{(1-e^2)}} dt, \quad r dv = \frac{a^3 \sqrt{(1-e^2)}}{r} dt$$

$$\therefore \delta \pi = \frac{a^3 \sqrt{(1-e^2)}}{e} \cos v D dt + \frac{a(1-e^2)+r}{ae^3 \sqrt{(1-e^2)}} \sin v D' dt \dots (c)$$

(A), (B), (C), are the expressions required.

XIX. QUESTION 569, by Mr. T. S. DAVIES.

Take a globe in any oblique position with respect to the horizon, and π -sect all its semi-diurnal arcs. Now if any parallels of equal declination D, D' be drawn, and great-circle chords be drawn through the corresponding sections of D, D' : Then as π takes consecutive values, the intersection of the resulting consecutive great-circle chords, will trace out upon the surface of the sphere a curve which it is required to determine.

FIRST SOLUTION, by the Proposer.

By my Treatise on Spherical Loci, and referring the system to the elevated pole instead of the equator (which changes D into $\frac{\pi}{2} - D$, or $\tan D$ into $\cot D$) we have the following values of L , and $L_{//}$:

$$L = \frac{\frac{\pi}{2} - \sin^{-1}(\cot D \cot I)}{\pi} \dots \dots (1)$$

$$L_{//} = \frac{\frac{\pi}{2} - \sin^{-1}(-\cot c \cot I)}{\pi}$$

$$\frac{\frac{\pi}{2} + \sin^{-1}(\cot D \cot I)}{\pi} \dots \dots (2)$$

Now $\frac{\pi}{2} - D$ and $\frac{\pi}{2} + D$ are the polar distances of the points through which the circles, given by their dependence upon π , are to pass, and which correspond therefore to α , and $\alpha_{//}$ of that Treatise: and $L, L_{//}$ are the corresponding values of the data, which are marked β , and $\beta_{//}$. But the general form of the equation of a circle through α, β , and $\alpha_{//}, \beta_{//}$ is

$$\cot \phi = -\operatorname{cosec} \{ \beta_{//} - \beta_{\perp} \} \left\{ \frac{\cos \theta (\cot \alpha_{//} \sin \beta_{\perp} - \cot \alpha_{\perp} \sin \beta_{//})}{\sin \theta (\cot \alpha_{//} \cos \beta_{\perp} - \cot \alpha_{\perp} \cos \beta_{//})} \right\} \dots (3)$$

In (3) inserting the above values of the constants, we shall have

$$\begin{aligned} \cot \phi &= -\operatorname{cosec} \frac{2 \sin^{-1} (\cot D \cot I)}{n} \times \\ &\{ \cos \theta (\sin L_{\perp} + \sin L_{//}) - \sin \theta (\cos L_{\perp} + \cos L_{//}) \} \times \tan D \\ &= -\operatorname{cosec} \frac{2 \sin^{-1} (\cot D \cot I)}{n} \times \\ &\left\{ \begin{aligned} &\cos \theta \left\{ 2 \sin \frac{\pi}{2n} \cdot \cos \left(\frac{2 \sin^{-1} (\cot D \cot I)}{n} \right) \right\} \\ &-\sin \theta \left\{ 2 \cos \frac{\pi}{2n} \cdot \cos \left(\frac{2 \sin^{-1} (\cot D \cot I)}{n} \right) \right\} \end{aligned} \right\} \times \tan D \\ &= -\operatorname{cosec} \frac{2 \sin^{-1} (\cot D \cot I)}{n} \times \cos \frac{2 \sin^{-1} (\cot D \cot I)}{n} \\ &\quad \times \tan D \times 2 \left\{ \cos \theta \frac{\pi}{2n} - \sin \theta \cos \frac{\pi}{2n} \right\} \\ &= -\cot \frac{2 \sin^{-1} (\cot D \cot I)}{n} \times \tan D \times 2 \sin \left(\frac{\pi}{2n} - \theta \right) \dots (4) \end{aligned}$$

Put $\sin^{-1} (\cot D \cot I) = \gamma$: then (4) is changed into

$$\cot \phi = -2 \cot \frac{2\gamma}{n} \tan D \sin \left(\frac{\pi}{2n} - \theta \right) \dots \dots \dots (5)$$

and which is the equation of the circle through L, D , and $L_{//}, D_{//}$.

But the problem requires us to assign the locus of the points of the intersections of the curve (5) in its consecutive states at infinitesimal distances: and hence for the reasons usually given in such cases, we must eliminate n between (5) and its first differential co-efficient taken as a function of n . But the differential is

$$\frac{d \frac{2\gamma}{n}}{\cos^2 \frac{2\gamma}{n}} = -2 \cot \phi \tan D \cos \left(\frac{\pi}{2n} - \theta \right) d \frac{\pi}{2n},$$

and actually performing the indicated operation, it becomes

$$\frac{-2\gamma dn}{n^2 \cos^2 \frac{2\gamma}{n}} = +2 \cot \phi \tan D \cos \left(\frac{\pi}{2n} - \theta \right) \frac{2\pi dn}{4n^2},$$

or dividing out, it is

$$- \frac{2\gamma}{\pi} \tan \phi \cot D = \cos^2 \frac{2\gamma}{n} \cos \left(\frac{\pi}{2n} - \theta \right) \dots\dots(6)$$

Again, from (5), (6) we have

$$\sin \left(\frac{\pi}{2n} - \theta \right) = - \frac{\cot \phi \cot D}{2 \cot \frac{2\gamma}{n}} \dots\dots(7)$$

$$\cos \left(\frac{\pi}{2n} - \theta \right) = - \frac{2\gamma}{\pi} \cdot \frac{\tan \phi \cot D}{\cos^2 \frac{2\gamma}{n}} \dots\dots(8)$$

But writing (7), (8) in $\cos^2 + \sin^2 = 1$ we have

$$\frac{4\gamma^2 \tan^2 \phi}{\pi^2 \cos^4 \frac{2\gamma}{n}} + \frac{\cot^2 \phi}{4 \cot^2 \frac{2\gamma}{n}} = \tan^2 D \dots\dots(9)$$

or putting for $\frac{2\gamma}{n}$ its values in terms of $\cot \frac{2\gamma}{n}$, it becomes after simple reduction

$$\cot^4 \frac{2\gamma}{n} + \cot^2 \frac{2\gamma}{n} \times \frac{\pi^2 \cot^2 \phi - 8\gamma^2 \tan^2 \phi}{4\pi^2 \tan^2 D - 4\gamma^2 \tan^2 \phi} = \frac{\gamma^2 \tan^2 \phi}{\pi^2 \tan^2 D - \gamma^2 \tan^2 \phi} \dots\dots(10)$$

From which double quadratic we get $\cot \frac{2\gamma}{n}$ in terms of ϕ and constants; and we may write

$$\cot \frac{2\gamma}{n} = f(\phi) \text{ or } n = \frac{2\gamma}{\cot^{-1} f(\phi)} \dots\dots(11)$$

Insert this in (5) and the solution is obtained: viz.

$$\cot \phi = - 2 \tan D \cdot f(\phi) \cdot \sin \left\{ \frac{\pi \cot^{-1} f(\phi)}{4\gamma} - \theta \right\} \dots\dots(12)$$

Schol. Had we inserted the value of $\cot \frac{2\gamma}{n}$ in terms of $\cos$

$\frac{2\gamma}{n}$ we should have obtained an equation having different coefficients from those of (10), and in a form better adapted to discussion: but as the inquiry itself is involved in the subject of the "*Antique Hour-lines*," I shall not discuss it here, it being intended to form a paragraph in my concluding paper on that subject, in the Transactions of the Royal Society of Edinburgh.

SECOND SOLUTION, *by Mr. WOOLHOUSE.*

The two semi-diurnal arcs for the equal parallels of declination $\mathfrak{p}$, $\mathfrak{p}'$, must evidently together, make 180° . Let their values be $90^\circ(1 + \pi)$, $90^\circ(1 - \pi)$; and denote the declination by δ ; then, the distances of the points of division from the meridian will always be, of the forms $\alpha(1 + \pi)$, $\alpha(1 - \pi)$. Let x be estimated along the equator from the meridian, and y in declination. It is evident that the great circle chord will cut the equator at the distance α from the meridian; and therefore, if i denote the inclination of the circle with the equator, we shall have $\sin n\alpha \tan i = \tan \delta$, and its equation will be

$$\tan y = \sin(x - \alpha) \tan i = \frac{\sin(x - \alpha)}{\sin n\alpha} \tan \delta \dots\dots\dots(1)$$

in which xy is any point in the great circle chord. At the point of consecutive intersection (xy) the ordinates xy will not change by slightly varying α ; therefore, differentiating with α , we find

$$\sin n\alpha \cos(x - \alpha) + n \cos n\alpha \sin(x - \alpha) = 0 \dots\dots\dots(2)$$

From these two equations we learn that the required curve will be determined by

$$\tan(x - \alpha) = -\frac{\tan n\alpha}{n} \quad \tan y = \frac{+ \tan \delta}{\sqrt{(n^2 \cos^2 n\alpha + \sin^2 n\alpha)}}$$

the arc $x - \alpha$ must be either in the second or fourth quadrant; the former will relate to the hemisphere containing the greater, and the latter to that containing the less of the two semi-diurnal arcs. The curve in each hemisphere will evidently be the same, and they will be wholly without the two parallels of declination.

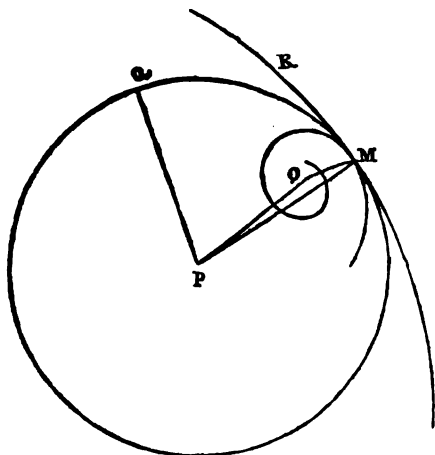
XX. PRIZE QUESTION 570, *by Mr. T. S. DAVIES.*

If the loxodrome were detached from its fixed position and made to roll upon one of the circles of the sphere, its pole would trace out a particular kind of epicycloid. It is required to find the equations of this epicycloid—to ascertain whether in any case it can be algebraical, quadrable or rectifiable—and to assign the conditions which must be fulfilled by any class of loci, which employed as directrices instead of

the circle, shall confer upon the epicycloid the properties above-mentioned either singly, or simultaneously.

FIRST SOLUTION, by Mr. DAVIES.

Let P the pole of the circular direction, be taken as origin of the radii-vectores, and Q , the point with which the pole



of the loxodrome was in contact, as the origin of the polar angle. Let o be the pole of the loxodrome, and m the current point of contact. Join PM , MO , OP by the great circle arcs.

Then PM is perpendicular to the great-circle tangent MR (MR being the common great-circle tangent to the loxodrome and the directrix), and $OMR = \alpha$ is the generating angle of the loxodrome whose pole o traces out the locus sought.

Put $OM = \chi$, $PQ = \epsilon$, $PO = \phi$, and $QPO = \theta$. Then the arcs OM , QM are equal by the genesis: that is,

$$QM = \chi \sec \alpha \operatorname{cosec} \rho \dots\dots\dots(1)$$

Also by the triangle OPM we have

$$\sin OPM = \frac{\cos \alpha \sin \chi}{\sin \phi} \dots\dots\dots(2)$$

Again,

$$\theta = QPM - MPO = \chi \sec \alpha \operatorname{cosec} \epsilon - \sin^{-1} \frac{\cos \alpha \sin \chi}{\sin \phi} \dots\dots\dots(3)$$

$$\text{and } \cos \phi = \cos \epsilon \cos \chi + \sin \epsilon \sin \chi \sin \alpha \dots\dots\dots(4)$$

Between (3) and (4) eliminate χ , and we shall have the locus sought.

From (4) we have

$$\left. \begin{aligned} \cos \chi &= \frac{\cos \varphi \cos \varrho \pm \sin \alpha \sin \varrho \sqrt{\sin^2 \varphi - \sin^2 \varrho \cos^2 \alpha}}{1 - \sin^2 \varphi \cos^2 \alpha} \\ \sin \chi &= \frac{\sin \varrho \sin \alpha \cos \varphi \mp \cos \varrho \sqrt{\sin^2 \varphi - \sin^2 \varrho \cos^2 \alpha}}{1 - \sin^2 \varphi \cos^2 \alpha} \end{aligned} \right\} \dots (5)$$

Hence the equation of the curve traced out by the pole is

$$\begin{aligned} \theta &= \sec \alpha \operatorname{cosec} \varrho \sin^{-1} \frac{\sin \varrho \sin \alpha \cos \varphi \pm \cos \varrho \sqrt{\sin^2 \varphi - \cos^2 \alpha \sin^2 \varrho}}{1 - \cos^2 \alpha \sin^2 \varrho} \\ &- \sin^{-1} \frac{\sin \varrho \cos \alpha \sin \alpha \cos \varphi \pm \cos \varrho \cos \alpha \sqrt{\sin^2 \varphi - \cos^2 \alpha \sin^2 \varrho}}{\sin \varphi (1 - \cos^2 \alpha \sin^2 \varrho)} \dots (6) \end{aligned}$$

This will be algebraical when $\sec \alpha \operatorname{cosec} \varrho$ is an integer, and under no other conditions.

We have next to ascertain whether the curve is capable of rectification or quadrature. There are two general cases to be considered in relation to each of these problems : viz. where $\sec \alpha \operatorname{cosec} \varrho$ is integral, and where fractional or irrational.

1. When $\sec \alpha \operatorname{cosec} \varrho$ is fractional or irrational, the quantities $\sin \varphi$ and $\cos \varphi$ can only be found in an infinite series of θ : and in this case therefore, the expressions for the differential of the arc and the area will also be composed of an infinite number of terms ; and hence in this case the integral will not be finite when φ is put in terms θ . These expressions are respectively

$$d \text{ arc} = \sqrt{\sec^2 \varphi d\theta^2 + d\varphi^2} \dots \dots \dots (7)$$

$$d \text{ area} = (1 - \cos \varphi) d\theta \dots \dots \dots (8)$$

(*Vide Ed. Trans.* vol. xii.)

the slightest contemplation of which in conjunction with the conditions we have been considering is sufficient to verify our conclusion, without more minute and tedious statements being added here.

Let $\sec \alpha \operatorname{cosec} \varrho =$ be an integer : then (6) will involve $\sin \varphi$ or $\cos \varphi$ (according to which is inserted for the other) of the $4m$ th degree. For the sine or cosine of the multiple arc will involve those of the simple arc of the m th degree : and the elimination of the radical involved in the result will double this, or raise it to the $2m$ th degree. There is yet in this result both $\sin \varphi$ and $\cos \varphi$, the substitution of one of which in terms of the other, will introduce a new radical of the second degree, the elimination of which will again double the degree of the equation obtained in our last result, or raise it to the $4m$ th power.

The expression for the length and area will be integrable in

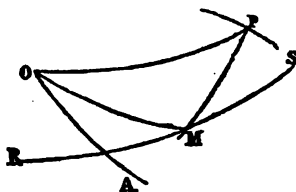
finite terms between limits, when the coefficient of the arcs which enter into it is zero: and not otherwise. But as there are only two arbitrary quantities α and φ involved in the data we shall be unable to assign any relations between them so as to effect this purpose when the equation is of a higher order than the second,—which it always is. As, moreover, the equation itself even in the most simple case $\operatorname{cosec} \varphi \sec \alpha = 1$, is of a form which does not admit of integration in finite terms, we cannot expect to find it so in more complex cases: but it must be confessed that a direct examination of the state of the equation is next to impossible in consequence of the complexity of the expressions involved.

3. The expressions for the arcs and areas, when we put the value of θ in terms of φ , are still of a kind to offer, on the face of them, little hope of success in our attempts at integration in finite terms. I have, however, gone into the inquiry so far as to see that the integral is essentially incapable of a finite form, though the length of the reasoning and the incommensurate utility of the inquiry scarcely, upon re-consideration, justifies my occupying the pages of the Repository with its details. I shall therefore drop the inquiry at its present stage.

Though in my work on "Spherical Geometry," I may find it desirable to discuss the remaining part of the question, in illustration of a principle which it involves, yet as that principle is not now before us, I shall defer further detail respecting it, till I am able to publish that volume.

SECOND SOLUTION, by Mr. WOOLHOUSE.

Consider the question generally, and let RS be the directrix on which the curve is supposed to roll, the commencement being at R. Let M be the point of contact at any position and P the pole; and let the curve of P as well as the directrix be referred to a point O and the fixed great circle OA. Assume thus $OM = y$, $\angle AOM = \theta$, $RM = s$, $OP = y'$, $\angle AOP = \theta'$, $\angle OMR = \epsilon$, const. $\angle PMS = i$ and $\cos i = \pi$. Then, by the loxodromic, $MP =$



$\frac{1}{2} \pi - \pi s$, and we have also, $\cos s = \frac{dy}{ds}$, $\sin s = \frac{d\theta}{ds} \sin y$, and

hence $\cos \angle OMP = -\cos (\epsilon + i) = \sin \epsilon \sin i - \cos \epsilon \cos i$

$= \frac{d\theta}{ds} \sin i \sin y - n \frac{dy}{ds}$. In relation to the triangle OMP, we have therefore, by spherics,

$$\cos y' = \cos y \sin ns + \sin y \cos ns \left(\frac{d\theta}{ds} \sin i \sin y - n \frac{dy}{ds} \right) \dots (1)$$

$$\sin ns = \cos y \cos y' + \sin y \sin y' \cos (\theta' - \theta) \dots (2)$$

The curve described by P will be algebraical when its equation is of the form

$$f(\cos y', \cos \theta') = 0 \dots (3)$$

f denoting any algebraical function. This function may be assumed at pleasure, and then we shall have three equations to eliminate y' , θ' , and thence find the equation of the directrix. In the same manner we may assume, instead of (3), the equation of any quadrable or rectifiable curve and determine the directrix.

When the directrix is a great circle, take one of its poles for the origin o : then $y = \frac{1}{2}\pi$, $dy = 0$, $d\theta = ds$, and hence (1), (2), become

$$\cos y' = \sin i \cos ns \dots (a)$$

$$\sin ns = \sin y' \cos (\theta' - \theta) \dots (b)$$

Eliminate y' , and we find

$$\tan (\theta' - \theta) = \cos i \cot ns \dots (c)$$

From (a) and (c) we deduce

$$\sin^2 y' = \cos^2 i + \sin^2 i \sin^2 ns$$

$$dy' = \sin i (ds \sin ns) \frac{\cos i}{\sin y'} \quad d\theta' = \sin i (ds \sin ns) \frac{\sin i \sin ns}{\sin^2 y'}$$

And hence, s' denoting the length of the curve traced by P , we have

$$ds' = \sqrt{(dy'^2 + d\theta'^2 \sin^2 y')} = \sin i (ds \sin ns)$$

$$\therefore s' = c - \tan i \cos ns \dots (d)$$

Instead of $\cos ns$ put its value by (a), and

$$s' = c - \frac{\cos y'}{\cos i} \dots (e)$$

for the equation of the curve described by P . The plane of the great circle directrix being supposed horizontal, this equation determines the curve through a very neat property, viz: that at each point it makes the same constant angle (i) with the vertical which the loxodrome makes with the meridians.

We have also the sectorial surface round the pole

$$= \int d\theta' (1 - \cos y') = \int ds \sin^2 i \frac{\sin^2 \pi s}{1 + \cos y'} = \sin^2 i \int ds \frac{\sin^2 \pi s}{1 + \sin i \cos \pi s}$$

$$= (s - \frac{\pi}{2}) - \tan i \sin \pi s + \tan^{-1} \frac{\sin i + \cos \pi s}{\cos i \sin \pi s} \dots \dots (f)$$

which is estimated from the position $s = 0$, $y' = \frac{\pi}{2} - i$,

$$\theta' = \left(\frac{\pi}{2} + \theta \right).$$

TWO SOLUTIONS of QUESTION 564, by Mr. DAVIES.

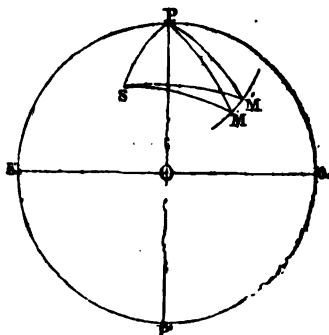
[These Solutions were accidentally omitted in their proper places in the preceding pages.—Ed.]

FIRST SOLUTION.

Let MM' be an element of the loxodrome, and its equation be referred to the pole P and the meridian of the point of intersection with the equator. Then the differential equation of the curve is

$$\frac{\sin \phi' d\theta'}{d\phi'} = \tan \alpha \dots \dots (1)$$

where $\phi' = PM$, θ , is the angle made between PM and the first meridian, and α is the angle of inclination the loxodrome to the meridian.



Let $\phi\theta$ denote the co-ordinates of the spherical centre s of the circle of osculation with the element MM' . Then this point R traces out the spherical evolute of the loxodrome: and we are required to find the equation of this evolute. Now this same point may be considered as the locus of the intersection of the consecutive great circles, which are normal to the loxodrome: and by means of this property the evolute may be found.

The general equation of the normal to a spherical curve at the point $\phi'\theta'$ is

$$\sin (\theta' - \theta) \sin \phi' \frac{d\theta'}{d\phi'} - \cos (\theta' - \theta) \cos \phi' + \sin \phi' \cot \phi = 0 \dots (2)$$

And by virtue of (1), (2) we have

$$\sin (\theta' - \theta) \tan \alpha - \cos (\theta' - \theta) \cos \phi' + \sin \phi' \cot \phi = 0 \dots (3)$$

Differentiate this on the hypothesis of φ, θ' being the variables, and comparing with equation (1) we get

$$\frac{d\theta'}{d\varphi} = - \frac{\sin \varphi' \cos (\theta' - \theta) + \cos \varphi' \cot \varphi}{\cos (\theta' - \theta) \tan \alpha + \sin (\theta' - \theta) \cos \varphi'} = \frac{\tan \alpha}{\sin \varphi'} \dots \dots (4)$$

Multiply out equation (4), and multiply (3) by $\cos \varphi'$: then we obtain the result

$$\sin^2 \varphi' \cos (\theta' - \theta) + \sin \varphi' \cos \varphi' \cot \varphi + \tan^2 \alpha \cos (\theta' - \theta) + \tan \alpha \cos \varphi' \sin (\theta' - \theta) = 0 \dots \dots (5)$$

$$-\cos^2 \varphi' \cos (\theta' - \theta) + \sin \varphi' \cos \varphi' \cot \varphi + \tan \alpha \cos \varphi' \sin (\theta' - \theta) = 0 \dots (6)$$

Subtracting (6) from (5) we get

$$\cos (\theta' - \theta) \sec^2 \alpha = 0 \dots \dots (7)$$

Now, as $\sec^2 \alpha$ cannot be 0, this equation can only be fulfilled by

$$\cos (\theta' - \theta) = 0, \text{ or}$$

$$\theta' - \theta = \pm \frac{\pi}{2} \dots \dots (8)$$

From this we learn that φ, φ' are at right angles to one another—a conclusion that may be readily verified by geometrical considerations. The above general method is retained, however, both for preserving the unmixed character of the spherical methods, as well as to illustrate the *general system* by means of equation (2). Further details respecting these methods, however, will be found in my work on Spherical Geometry, which is now nearly ready for press.

By the insertion of (7) in (3) we get

$$\cot \varphi = \mp \frac{\tan \alpha}{\sin \varphi'} \dots \dots (9)$$

But by the properties of the loxodrome, first given in my paper in the Edinburgh Transactions, we have

$$\sin \varphi' = \frac{2\kappa^{\theta'}}{1 + \kappa^{2\theta'}} \dots \dots (10)$$

(where $\kappa = e^{\cot \alpha}$)

And the combination of (8) (9) (10) at once furnish the equation sought: viz.

$$\cot \varphi = \pm \frac{\kappa^{\left(\theta \pm \frac{\pi}{2}\right)} + \kappa^{-\left(\theta \pm \frac{\pi}{2}\right)}}{2 \cot \alpha} \dots \dots (11)$$

If a cone whose vertex is the centre of the sphere, and directrix

the curve whose equation (11) we have just found, be described, it will be the *evolute surface* sought by the problem: and its equation may be readily found by transforming the polar into rectangular co-ordinates. We should then thus have at once

$$\frac{z}{\sqrt{x^2+y^2}} = \frac{\tan \alpha}{z} \left\{ \kappa \sin^{-1} \frac{\pm x}{\sqrt{x^2+y^2}} + \kappa \sin^{-1} \frac{\pm x}{\sqrt{x^2+y^2}} \right\} \dots (12)$$

$$= \frac{1}{2} \tan \alpha \left\{ \tan^{-1} \left(-\frac{x}{y} \right) + \kappa \tan^{-1} \left(-\frac{x}{y} \right) \right\} \dots (13)$$

Again, for the locus of the centre of *absolute curvature*, we have the cone upon which it is situated already found: and the *distance from the centre* of the sphere (or vertex of the cone) of the point whose *direction* is already known by means of (11). We have therefore only to use this general theorem:—

$$r = a \{ \cos \phi \cos \phi' + \sin \phi \sin \phi' \cos (\theta' - \theta) \} \dots (14)$$

Where a = radius of the sphere, and $\phi, \phi', \theta, \theta'$ denote the same quantities as before. This is shown readily: and is explained in my Work.

In the present case we have, then,

$$r = a \cos \phi \cos \phi' \dots \dots \dots (15)$$

But as ϕ' is given in this equation, we have, for the locus sought, $r = a \cos \phi' \times \cos \phi$ an equation between r and ϕ , which obviously designates a spherical surface. Its diameter is coincident with the polar axis; it passes through the centre of the loxodromic sphere; and finally, its diameter is $a \cos \phi'$.

Moreover, for different values of a whilst ϕ' and θ' remain the same, (that is, for all the loxodromes which pass through ϕ', θ') the quantity $\theta' - \theta$ will be constant: and hence the position of ϕ will be the same, and its length only variable. The locus of the centres of absolute curvature in this case, then, is a meridional circle upon the spherical surface of absolute centres just determined.

SECOND SOLUTION.

The solution above given is purely spherical and by means of spherical polar co-ordinates: the following is by the method of rectilinear and rectangular co-ordinates. It has been my object to keep the two methods of solution perfectly distinct.

That the logarithmic spiral is the stereographic projection of the loxodrome, may be readily proved by means of rectilinear

co-ordinates; and hence we may at once assume the equation,

$$r, = ak' \dots\dots\dots (1)$$

and this, converted into rectilinear co-ordinates, is

$$\sqrt{x,^2 + y,^2} = ak \cos^{-1} \frac{x,}{\sqrt{x,^2 + y,^2}} \dots\dots\dots (2)$$

$$\text{or } \sqrt{x,^2 + y,^2} = ak \sin^{-1} \frac{y,}{\sqrt{x,^2 + y,^2}} \dots\dots\dots (3)$$

Making this the directrix of a cone whose vertex has for co-ordinates 0, 0, and $-a$, we find

$$\frac{a}{z, + a} \{x,^2 + y,^2\}^{\frac{1}{2}} = ak \cos^{-1} \frac{x,}{\sqrt{x,^2 + y,^2}} \dots\dots\dots (4)$$

$$\frac{a}{z, + a} \{x,^2 + y,^2\}^{\frac{1}{2}} = ak \sin^{-1} \frac{y,}{\sqrt{x,^2 + y,^2}} \dots\dots\dots (5)$$

But the equation of the sphere concentric with the logarithmic spiral, and having radius a , is

$$x,^2 + y,^2 + z,^2 = a^2 \dots\dots\dots (6)$$

Or for $x,^2$ and $y,^2$ in (4) and (5) write their values from (6) then

$$\left\{ \frac{a - z,}{a + z,} \right\}^{\frac{1}{2}} = k \cos^{-1} \frac{x,}{\sqrt{a^2 - z,^2}} \dots\dots\dots (7)$$

$$\left\{ \frac{a - z,}{a + z,} \right\}^{\frac{1}{2}} = k \sin^{-1} \frac{y,}{\sqrt{a^2 - z,^2}} \dots\dots\dots (8)$$

These, (7) (8), are the projections of the loxodrome, on the planes of xz and yz respectively. To render the differentiation more simple and convenient, we may write (7) and (8) in the following forms:

$$x, = \sqrt{a^2 - z,^2} \cos \frac{l \left(\frac{a - z,}{a + z,} \right)}{2lk} \sqrt{a^2 - z,^2} \cos \chi \dots\dots\dots (9)$$

$$y, = \sqrt{a^2 - z,^2} \sin \frac{l \left(\frac{a - z,}{a + z,} \right)}{2lk} \sqrt{a^2 - z,^2} \sin \chi \dots\dots\dots (10)$$

$$\begin{aligned} \text{Now } d \sin \chi &= -\cos \chi \frac{adz,}{(a^2 - z,^2)lk} \left\{ \right. \\ &\text{and } d \cos \chi = \sin \chi \frac{adz,}{(a^2 - z,^2)lk} \left. \right\} \dots\dots\dots (11) \end{aligned}$$

Hence the differential co-efficients of (9) and (10) are found at once to be

$$\frac{dy}{dz} = - \frac{\frac{a}{lk} \cos \chi + z \sin \chi}{\sqrt{a^2 - z^2}} \dots\dots\dots (12)$$

$$\frac{dx}{dz} = + \frac{\frac{a}{lk} \sin \chi - z \cos \chi}{\sqrt{a^2 - z^2}} \dots\dots\dots (13)$$

The second differential co-efficients taken on the same hypothesis, are

$$\frac{d^2x}{dz^2} = - \frac{a^2 (1 + lk^2)}{(a^2 - z^2)^{\frac{3}{2}} lk^2} \cos \chi \dots\dots (14)$$

$$\frac{d^2y}{dz^2} = - \frac{a^2 (1 + lk^2)}{(a^2 - z^2)^{\frac{3}{2}} lk^2} \sin \chi \dots\dots (15)$$

Let xyz be a point in the normal plane which is drawn to the loxodrome at the point x, y, z ; then the normal plane itself, in consequence of equations (9), (10), (14), (15) is found to be

$$\left. \begin{aligned} (z-z) - (y - \sqrt{a^2 - z^2} \sin \chi) \frac{\frac{a}{lk} \cos \chi + z \sin \chi}{\sqrt{a^2 - z^2}} \\ + (x - \sqrt{a^2 - z^2} \cos \chi) \frac{\frac{a}{lk} \sin \chi - z \cos \chi}{\sqrt{a^2 - z^2}} \end{aligned} \right\} = 0 \dots (16)$$

or, breaking up the vincula and otherwise reducing, it becomes

$$z\sqrt{a^2 - z^2} = \frac{a}{lk} (y \cos \chi - x \sin \chi) + (x \cos \chi + y \sin \chi) z \dots (17)$$

Also, taking the differential of the equation of the normal plane (16) by means of (14), (15) we have, z' being the principal variable, and xyz constant,

$$\begin{aligned} & - (y - \sqrt{a^2 - z^2} \sin \chi) \frac{1 + lk^2}{lk^2} \cdot \frac{a^2}{(a^2 - z^2)^{\frac{3}{2}}} \cdot \sin \chi \\ & - (x - \sqrt{a^2 - z^2} \cos \chi) \frac{1 + lk^2}{lk^2} \cdot \frac{a^2}{(a^2 - z^2)^{\frac{3}{2}}} \cdot \cos \chi \\ & - \frac{1 + lk^2}{lk^2} \cdot \frac{a^2}{a^2 - z^2} = 0 \dots\dots\dots (18) \end{aligned}$$

or multiplying out and cancelling the common factor,* it becomes

$$y \sin \chi + x \cos \chi = 0 \dots\dots (19)$$

It now remains to eliminate z , between (17) and (19), which may be readily effected as follows:—

From (19) we have

$$\left. \begin{aligned} \cos \chi &= \frac{+y}{\sqrt{x^2 + y^2}} \\ \sin \chi &= \frac{-x}{\sqrt{x^2 + y^2}} \end{aligned} \right\} \dots\dots (20)$$

Insert (20) in (17), and we shall have

$$a^2 - z'^2 = \frac{a^2}{lk^2} \cdot \frac{x^2 + y^2}{z^2} \dots\dots\dots (21)$$

Again from (19) we have

$$\tan \chi = -\frac{x}{y} \text{ or } \chi = \tan^{-1} -\frac{x}{y} \dots\dots (22)$$

But $\chi = \frac{l(a - z')}{2lk}$; or $2\chi \cdot lk = l(a - z')$; or, taking l^{-1} of both sides, it becomes

$$\kappa^2 \chi = \frac{a - z'}{a + z'},$$

or, by reduction,

$$z' = a \cdot \frac{1 - \kappa^2 \chi}{1 + \kappa^2 \chi}$$

from which

$$a^2 - z'^2 = \frac{4a^2 \kappa^2 \chi}{(1 + \kappa^2 \chi)^2} \dots\dots\dots (23)$$

Equating (21), and (23) we have

* This factor $\frac{1 + lk^2}{lk} \cdot \frac{a^2}{(a^2 - z'^2)^{\frac{3}{2}}}$ it is true, involves z' ; but a little consideration would shew that the second member of it, which involves z' , is a foreign factor, introduced by elimination. Our limits do not allow us to enter into detail, and we can therefore only here mention the fact.

$$\frac{a}{lk} \left\{ \frac{x^2 + y^2}{z^2} \right\}^{\frac{1}{2}} = \frac{a \cdot 2k \chi}{1 + k^2 \chi^2} \text{ or}$$

$$\sqrt{\frac{z}{x^2 + y^2}} = \frac{1}{2l} \left\{ k \chi + k^{-1} \chi \right\} \dots\dots (24)$$

$$= \frac{1}{2lk} \left\{ k^{\sin^{-1} \frac{\pm x}{\sqrt{x^2 + y^2} + k}} \sin^{-1} \frac{\pm x}{\sqrt{x^2 + y^2}} \right\} \dots\dots (25)$$

$$= \frac{1}{2lk} \left\{ k^{\cos^{-1} \frac{\pm y}{\sqrt{x^2 + y^2} + k}} \cos^{-1} \frac{\pm y}{\sqrt{x^2 + y^2}} \right\} \dots\dots (26)$$

Bearing in mind, that $lk = \cot \alpha$ we have precisely the same results as in (19), (13) of the former solution. The sign and the value of $\tan \chi$ given in (25) also confirm our inference, that zz , was a right angle; vide (8) of the former solution.

It only remains to find the locus of the centre absolute curvature of the loxodrome, and apply the resulting formula to the case of $z = \text{constant}$ and lk variable.

The general equation of the osculating plane through x, y, z , is $y''(x - x_1) - x''(y - y_1) = (x''y' - x'y'')(z - z_1) \dots\dots (26)$

From (14) (15) we have x'' and y'' , and from (12,) (13,) (14,) (15) we have $x''y' - y''x'$. These, inserted in (26) give, after putting for lk its value $\cot \alpha$,

$$(y - y_1) \cos \chi - (x - x_1) \sin \chi = \frac{a(z - z_1) \tan \alpha}{\sqrt{a^2 - z_1^2}} \dots\dots (27)$$

For x, y , put their values from (9, 10); and then for $\sin \chi$ and $\cos \chi$ their values from (20): then there will result, after squaring both sides of the equation,

$$(x^2 + y^2)(a^2 - z_1^2) = a^2(z' - z_1)^2 \tan^2 \alpha \dots\dots (28)$$

But by (23) we have

$$z_1 = a \cdot \frac{1 - k^2 \chi}{1 + k^2 \chi}, \text{ and } a^2 - z_1^2 = \frac{4a^2 \cdot k^2}{(1 + k^2 \chi^2)}$$

which inserted in (28) gives

$$(x^2 + y^2) \frac{4k^2 \chi}{1 + k^2 \chi} = \left(z - a \frac{1 - k^2 \chi}{1 + k^2 \chi} \right)^2 \tan^2 \alpha \dots\dots (29)$$

By a slight transformation of (29) we obtain, keeping in view our equation (22),

$$\pm 2 (x^2 + y^2)^{\frac{1}{2}} \cot \alpha = z \left(k^{-\tan^{-1} \frac{x}{y}} + k^{\tan^{-1} \frac{x}{y}} \right) \\ - a \left(k^{-\tan^{-1} \frac{x}{y}} - k^{\tan^{-1} \frac{x}{y}} \right) \dots \dots (30)$$

which is the second equation of the locus of the centre of absolute curvature; and which with the former, (25), serves to determine that curve.

But in the case proposed in the question, z , is constant, and the angle α of the loxodrome as well as xy , is arbitrary. To find the locus of the centre of absolute curvature in this case we have only to put z , = constant, and eliminate $\tan \alpha$ between (21) and (28). This gives

$$\frac{(x^2 + y^2)(a^2 - z^2)}{a^2(z - z_0)^2} = \tan^2 \alpha = \frac{(a^2 - z^2)z^2}{a^2(x^2 + y^2)} \dots \dots (31)$$

From this we get

$$(x^2 + y^2)^2 = z^2(z - z_0)^2 \text{ or extracting the square root} \\ x^2 + y^2 = \pm z(z - z_0) \dots \dots (32)$$

That is $x^2 + y^2 \pm z^2 = \mp zz_0, \dots \dots (33)$

Of these two roots one is foreign to our present object, viz. the $+ z(z - z_0)$: and the other gives the result we have been seeking: viz.

$$x^2 + y^2 + z^2 = zz_0, \dots \dots (34)$$

which designates a spherical surface, whose axis is coincident with the axis of z , whose diameter is equal to z_0 , and which passes through the origin of co-ordinates.

The question is therefore now completely resolved. The properties of the loxodrome will, however, be found much more amply treated in my Work on Spherical Geometry.

THE MATHEMATICAL REPOSITORY.

VOL. VI. PART II.

ARTICLE I.

Two Indeterminate Problems.

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PROBLEM I.

To determine such a relation of the quantities a, b, c, d as will make the expression $abcd \times (a^2 - b^2) \times (c^2 - d^2)$ a rational square.

It is evident that the expression $abcd \times (a^2 - b^2) \times (c^2 - d^2)$ will be a square when $ab \times (a^2 - b^2) = cd \times (c^2 - d^2)$.

Take $a + b = c + d$, and dividing the former expression by this we shall have $ab \times (a - b) = cd \times (c - d)$: in this latter expression writing $c + d - b$ for a , it becomes

$$b \times (c + d - b) \times (c + d - 2b) = c^2 d - cd^2;$$

and this expression when properly arranged becomes

$$(b - d) \times c^2 - c \times (3b^2 - 2bd - d^2) = -b \times \{2b^2 - 3bd + d^2\};$$

and dividing by $b - d$, $c^2 - c \times (3b + d) = -b \times (2b - d)$,

$$\text{whence } d = \frac{c^2 - 3bc + 2b^2}{b + c}; \text{ and hence } a = c + d - b =$$

$$\frac{ac^2 - 3bc + b^2}{b + c}.$$

And because we have taken $ab \times (a^2 - b^2) = cd \times (c^2 - d^2)$ therefore

$$\frac{e}{f} = \frac{(ad + bc) \times (ac - bd) + 2\sqrt{abcd \times (a^2 - b^2) \times (c^2 - d^2)}}{(ac + bd)^2}$$

$$= \frac{(ad + bc) \times (ac - bd) + 2ab \times (a^2 - b^2)}{(ac + bd)^2},$$

$$\text{or } \frac{e}{f} = \frac{(ad + bc) \times (ac - bd) - 2\sqrt{abcd \times (a^2 - b^2) \times (c^2 - d^2)}}{(ad - bc)^2}$$

$$= \frac{(ad + bc) \times (ac - bd) + 2ab \times (a^2 - b^2)}{(ad - bc)^2}.$$

In these ex-

pressions, b and c may be taken at pleasure, and a and d determined from thence, by the expressions for these quantities found above.

Example 1. Take $c = 6$ and $b = 2$; then $a = 5$ and $d = 1$, whence $\frac{e}{f} = \frac{(ad + bc) \times (ac - bd) + 2ab \times (a^2 - b^2)}{(ac + bd)^2} = \frac{7}{8}$. therefore we may take, $e = 7$ and $f = 8$; whence $v = 7 \times 32 - 8 \times 7 = 168$, $x = 7 \times 32 + 8 \times 7 = 280$, $y = 7 \times 17 - 8 \times 28 = -105$, $z = 7 \times 28 - 8 \times 17 = 60$, which numbers are the roots of four squares that will answer the question.

Example 2. Take $c = 20$ and $b = 5$; then $a = 21$ and $d = 6$; $\frac{e}{f} = \frac{226 \times 390 + 210 \times 416}{450 \times 450} = \frac{13}{15}$: therefore we may take $e = 13$ and $f = 15$, whence $v = 13 \times 450 + 15 \times 26 = 6240$, $x = 13 \times 450 - 15 \times 26 = 5460$, $y = 13 \times 226 - 15 \times 390 = -2912$, $z = 13 \times 390 - 15 \times 226 = 1680$, which are the roots of four squares that will answer.

But each of the preceding numbers are divisible by 4; therefore 1560, 1365, 728, and 420 are the roots of four squares that will answer.

Example 3. Take $c = 20$ and $b = 12$; then $a = 7$ and $d = -1$, whence $\frac{e}{f} = \frac{233 \times 152 - 168 \times 195}{128^2} = \frac{19}{16}$; therefore we may take $e = 19$ and $f = 16$, whence $v = 19 \times 128 - 16 \times 247 = -1520$, $x = 19 \times 128 + 16 \times 247 = 6384$, $y = 19 \times 233 - 16 \times 152 = 1995$, $z = 19 \times 152 - 16 \times 233 = -840$, and these are the roots of four squares that will answer.

PROBLEM II.

To find four square numbers, the sum of any three of which shall be a rational square number.

Solution. Let v^2 , x^2 , y^2 , and z^2 , denote the four required squares, and put $v^2 + x^2 + y^2 + z^2 = N$. Then by the quest. we shall have $v^2 + x^2 + y^2 = N - z^2 = p^2$,
 $v^2 + x^2 + z^2 = N - y^2 = q^2$,
 $v^2 + y^2 + z^2 = N - x^2 = r^2$,
 $x^2 + y^2 + z^2 = N - v^2 = s^2$,
 and therefore $N = p^2 + z^2 = q^2 + y^2 = r^2 + x^2 = s^2 + v^2$.

Hence it appears, that x must be a number capable of being divided into the sum of two squares four different ways. Now it is plain enough that $(a^2 + b^2) \times (c^2 + d^2) = (ad + bc)^2 + (ac - bd)^2 = (ac + bd)^2 + (ad - bc)^2$. Put $u = ad + bc$, $w = ac - bd$; $u' = ac + bd$, $w' = ad - bc$. It will be evident by inspecting the foregoing expression, that $(a^2 + b^2) \times (c^2 + d^2) \times (e^2 + f^2) = (eu + fw)^2 + (ew - fu)^2 = (ew + fu)^2 + (eu - fw)^2 = (ew' + fu')^2 + (eu' - fw')^2 = (ew' - fu')^2 + (eu' + fw')^2$. And by comparing these expressions with those above, for the value of x it is obvious that we may take

$$p = eu + fw = e \times (ad + bc) + f \times (ac - bd).$$

$$z = ew - fu = e \times (ac - bd) - f \times (ad + bc).$$

$$q = ew + fu = e \times (ac - bd) + f \times (ad + bc).$$

$$y = eu - fw = e \times (ad + bc) - f \times (ac - bd).$$

$$r = ew' + fu' = e \times (ad - bc) + f \times (ac + bd).$$

$$x = eu' + fw' = e \times (ac + bd) - f \times (ad - bc).$$

$$s = ew' + fu' = e \times (ad - bc) - f \times (ac + bd).$$

$$v = eu' + fw' = e \times (ac + bd) + f \times (ad - bc).$$

From the equation $v^2 + x^2 + y^2 = p^2$, $v^2 + x^2 = p^2 - y^2$; and this by means of the foregoing values becomes

$$2e^2 \times (ac + bd)^2 + 2f^2 \times (ad - bc)^2 = 4ef \times (ad + bc)(ac - bd), \text{ or}$$

$$e^2 \times (ac + bd)^2 + f^2 \times (ad - bc)^2 = 2ef \times (ad + bc) \times (ac - bd);$$

completing the square on the first side by adding $2ef \times (ac + bd) \times (ad - bc)$ to the last equation we shall have

$$e^2 \times (ac + bd)^2 + 2ef \times (ac + bd) \times (ad - bc) + f^2 \times (ad - bc)^2$$

$$= \{e \times (ac + bd) + f \times (ad - bc)\}^2$$

$$= 2ef \times \{(ad + bc) \times (ac - bd) + (ac + bd) \times (ad - bc)\}$$

$$= 4efdc \times (a^2 - b^2).$$

Taking the square roots

$$e \times (ac + bd) + f \times (ad - bc) = 2\sqrt{efcd \times (a^2 - b^2)}.$$

Again completing the square of the first side of the expression $e^2 \times (ac + bd)^2 + f^2 \times (ad - bc)^2 = 2ef \times (ad + bc) \times (ac - bd)$, by subtracting $2ef \times (ac + bd) \times (ad - bc)$, from each side, and we shall have

$$e^2 \times (ac + bd)^2 - 2ef \times (ac + bd) \times (ad - bc) + f^2 \times (ad - bc)^2$$

$$= \{e \times (ac + bd) - f \times (ad - bc)\}^2$$

$$= 2ef \times \{(ad + bc) \times (ac - bd) - (ac + bd) \times (ad - bc)\}$$

$$= 4efab \times (c^2 - d^2)$$

and taking the square roots

$$e \times (ac + bd) - f \times (ad - bc) = 2\sqrt{efab \times (c^2 - d^2)}.$$

Multiplying this by the expression

$e \times (ac + bd) + f \times (ad - bc) = 2\sqrt{efcd \times (a^2 - b^2)}$, the product is

$$e^2 \times (ac + bd)^2 + f^2 \times (ad - bc)^2 = 4ef\sqrt{abcd \times (a^2 - b^2) \times (c^2 - d^2)}.$$

We have found

$$e^2 \times (ac + bd)^2 + f^2 \times (ad - bc)^2 = 2ef \times (ad + bc) \times (ac - bd),$$

and taking half the sum of these two expressions

$$e^2 \times (ac + bd)^2 + f^2 \times (ad - bc)^2 = ef \times (ad + bc) \times (ac - bd) + 2ef\sqrt{abcd \times (a^2 - b^2) \times (c^2 - d^2)}$$

whence

$$\frac{e}{f} = \frac{(ad + bc) \times (ac - bd) + 2\sqrt{abcd \times (a^2 - b^2) \times (c^2 - d^2)}}{(ac + bd)^2},$$

and by taking half the difference of the same expressions, after proper and obvious reduction,

$$\frac{e}{f} = \frac{(ad - bc)^2}{(ad + bc) \times (ac - bd) - 2\sqrt{abcd \times (a^2 - b^2) \times (c^2 - d^2)}}.$$

From whence it appears that in order to have e , and f , rational, $abcd \times (a^2 - b^2) \times (c^2 - d^2)$ must be a rational square. It also appears that either of the preceding expressions will give the relation of e , and f .

ARTICLE II.

A complete Solution of Problem IX. Gent. Diary, 1829.

By C. F. BARNWELL, Esq. of the British Museum.

(Extracted from a Letter from Mr. DAVIES to the Editor of the Mathematical Repository.)

* * * * *

I must also beg you to insert the following complete solution of Mr. Barnwell's problem, an abstract only of which I sent to the Editor of the Gentleman's Diary, well knowing that on account of its length it could not be inserted, unabridged, in that valuable little periodical. It is too little the fashion of our day to examine a proposition in all its cases, so that a complete solution, in the true manner of the ancients, from a modern pen, is become quite a rarity.

Mr. Barnwell desires me to state that the method was sug-

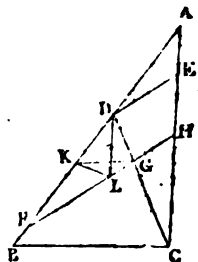
genced to his mind by considering the solution of a problem very analogous to it, given among Bland's Geometrical Problems (Section 9, Pr. 25.): and that he considers the analysis and distinction into cases, together with the modification of the method to suit these cases, as the whole that belongs to himself. This he remarked in his letter to Dr. Robertson (dated Aug. 9, 1824.) which contained the geometrical analysis and solution: but as I had not at the moment of writing my solutions for the *Diary*, the means of referring to that letter, I did not call to mind the circumstance, nor did I remark another (which oversight I now take the opportunity of correcting), that it is *only in one particular case that the problem admits of two solutions*,* as will appear from Mr. Barnwell's discussion of the several cases. This is one in addition to the many instances that might be adduced to shew the great advantages resulting from a complete examination of the various cases into which a problem naturally divides itself by the relative changes that may take place in the magnitudes which constitute the data of the proposition.

T. S. D.

PROBLEM. *Given the vertical angle, and the difference between the base and each of the sides, to construct the plane triangle.*

This problem naturally divides itself into three cases, *viz.* when the base (so calling the side opposite to the given angle) is *less* than either side; *greater* than either side; or *less* than one side, and *greater* than the other.

CASE 1st. Let A be the given $\angle$ (which, since the base is supposed $<$ than either of the sides, will be $< \frac{1}{2}$ of two right angles) and AD , AE the given differences, of which let AD be the greater. Join DE , and produce AD , AE . Draw $FH \parallel DB$, and $DL \parallel EH$, take $DK = EH$, in the same direction as DF , and join KL . Then, since $DK = EH = DL$, the $\triangle DKL$ is isosceles, and $\angle KDL = \angle DAE < \frac{1}{2}$ of two right angles; consequently $KL < DK$. If, therefore, a circle be described with centre K and radius DK , it will meet LH , or LH produced. Let the point of concourse be G , and join KG , DG .



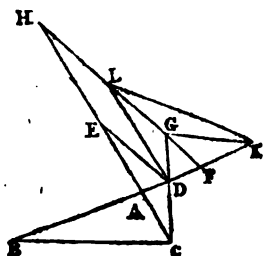
* The second solution which I hastily inferred to belong to the same question was the solution to the question in Bland already referred to, *viz.* that in which the *sums* of the base and each side were given.

Then, since $BL \parallel BH$, DG will meet AE produced. Let them meet in C , and draw $CB \parallel GK$; then will ABC be the triangle required.

For, by sim. triangles $DB : DK :: BC : KG (:: DC : DG) :: EC : EH$: and since $DK = KG = EH$ by construction, $DB = BC = CE$.

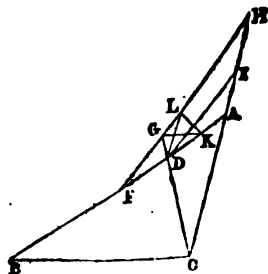
CASE 2nd. Here, since the base is supposed $>$ either side, the angle $A > \frac{1}{2}$ of two right angles.

Suppose AE the greater of the differences, and construct as before. Then LDK will be an isosceles Δ of which the $\angle KDL > \frac{1}{2}$ of two right angles, and conseq. $KL > KD$. But $KD > KF$; for since $DK = DL$, $DK : DF :: DL : DF :: AE : AD$; therefore F will be between D and K . Consequently, since $KD < KL > KF$, a circle with centre K and radius KD will cut LF between L and F . Let the point of concurrence be G ; then GD being joined and produced will meet EA produced, and the rest of the construction and demonstration will be as before.



CASE 3rd. This divides itself into three subordinate cases; for the given $\angle$ may be $>$, $=$, or $< \frac{1}{2}$ of two right angles: though it will always be acute.

Subcase (A). Suppose it less. Let AD be the excess of the greater side above the base, and AE the excess of the base above the lesser side; and let the $\angle DAB$ be the supplement of the given angle. Construct as before, except that DK is to be taken in the direction *opposite* to DF : then KL may be proved $< KD$, as in the first case, and $KF > KD$ by construction; therefore G will fall between L and F , and GD produced will meet EA produced; and the rest as before.



Subcase (B). Let the given $\angle = \frac{1}{2}$ of two right angles. In this case there is a limitation necessary to render the construction possible: *viz.* that AE the difference between the base and the lesser side, must be greater than AD the difference between the greater side and the base. The same construction being made as in

ARTICLE III.

ON RULE SURFACES.

By Mr. T. S. DAVIES, Bath.*

DEFINITIONS.

1. A RULE SURFACE (Surface Replée, Fr.) is that generated by the motion of a straight line whose movement in space is subject to *three conditions*.

2. When those conditions are such as to cause *its consecutive positions taken two and two to be in the same plane*, the surface is called a DEVELOPABLE SURFACE (Surface Dévelopable, Fr.)

3. When the consecutive positions have *not* the property of Def. 2. the Surface is called a TWISTED SURFACE (Surface Gauche, Fr.)

SECTION I.—PRELIMINARY.

The ancients in their methods of investigating the properties of figures commonly chose a defining property divested of all considerations respecting the manner of generating that figure. This was the natural result of that metaphysical system with which Plato is said to have infected the science in general; and which is mainly built upon an opinion that the rigour and purity of geometrical reasoning is totally lost by the introduction of any idea of motion. How far the moderns have drank at the same fountain it would be easy to shew did our limits permit us to dilate.

It is true that in some cases the defining properties employed in the writings of Euclid (or considered as his) do involve the idea of motion: for instance those of the sphere, cone, and cylinder in the Eleventh Book of the "Elements." It will

* These notes towards a history of the investigations of the properties of surfaces traced by the motion of a straight line, are intended as a Scholium to Question 506 of the present number of the Repository. I call them "*notes towards a history*" rather than a history itself, because it is possible that other works may exist than those I have yet consulted, and which it would be necessary to analyse in order to present any thing like a dissertation pretending to the character of a complete history. However, one thing is certain, that nearly the whole that composes the present part of these notes has escaped the notice of the modern French Mathematicians, as they consider the investigation to have been altogether effected in the Polytechnic School.

be remarked, however, that these definitions are not used in the course of the work ; and the fair inference from their want of conformity to the reigning principle of antient definition, and their total uselessness in the subsequent investigations,—is, that they are the explanatory interpolations of some of the later Greek scholiasts of Alexandria. Similar inconsistencies may be pointed out in other parts of the “ Elements.”

The properties of solids, however, never made any considerable portion of the study of the antient geometers; almost the whole of these investigations being confined to the properties of figures traced upon a single plane. The chief properties of solids which attracted their attention were those relating to the *equivalence*, or to the *ratio*, between different *cones*, *spheres*, and *pyramids*, *prisms* and *cylinders*, together with the fundamental properties of *solid angles*. Whether the 13th, 14th, and 15th Books of Euclid be genuine or not, is a question that does not affect our present enquiry, as, if they be, they do not indicate any material advance beyond the point I have stated.

In the case of the sections of the cone, which were amply considered by Appolonius, he yet, after obtaining some one property of lines situated within the cone and in the plane of section, which he thereby transfers from the cone to the plane figure itself, proceeds to conduct the whole of the subsequent investigation *in plano*. Nor do the speculations of Archimedes respecting the conoids, formed by the revolution of the conic sections about their principal axes, take a bolder course: and one of these, (the figure hereafter to be mentioned under the name of the *hyperbolic cylindroid*) that formed by the revolution of the hyperbola about its second axis he has not even alluded to. Archimedes, however, adopted the method of generating his conoids by revolution, and partially employed the principle in the derivation of the few inherent properties of the figures which he investigated.

DR. WALLIS, (1662.)

The other curve surfaces which were considered, in the slightest degree, before the application of algebra to geometrical investigations by Descartes, were few and unimportant. Indeed prior to the time of Dr. Wallis, no other method of generating curve surfaces than by the revolution of curve lines about an axis which divided them symmetrically, appears to have been imagined by mathematicians. Others, however, from time immemorial had been employed in the arts, the ostensible properties of which had been obtained by experiment, and verified by long and varied use. Amongst these was the “ shipwright’s circular

wedge," or as Wallis denominated it from its mode of generation, the cono-cuneus,—as partaking the character both of the cone and the wedge. The figure was generated by a straight line moving upon two directrices, one of which was a semicircle, the other a straight line parallel to the plane of the circle, such that the orthogonal projection of the line upon the plane of the circle coincided with its diameter: at the same time that the generating line was always parallel to the plane which cut the linear directrix at right angles. Dr. Wallis turned his attention to this in 1661 or 1662; and the occasion is thus explained in his dedication of the Tract to Sir Robert Moray. "Since I came home," (he writes from Oxford) "from London, I have taken some time to consider of those solids and lines made by the sections thereof; proposed to consideration (to my Lord Brouncker, and yourself, at your lodgings, where I was also present) by Mr. Pett, one of his Majesty's Commissioners for the Navy, and an excellent Shipwright.

"The bodies proposed to consideration were all of this form. On a plane Base, which was the Quadrant of a Circle, (like that of a Quadrantal Cone or Cylinder) stood an erect Solid, whose Altitude (being arbitrary) was there double to the Radius of that Quadrant, and from every Point of its Perimeter, straight lines drawn to the Vertex, met there, not in a Point (as in the Apex of a Line), nor in a parallel Quadrant (as in a Quadrantal Cylinder), but in a straight Line or sharp Edge, like that of a wedge or *Cuneus*. On which consideration I thought fit to give it the name of *Cono-Cuneus*, as having the Base of a *Cone*, and the Vertex of a *Cuneus*.

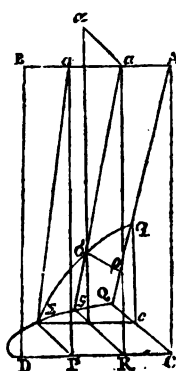
"By the various sections of this Solid, in several Positions, he did (rightly) conceive, that divers new Lines must arise, in great variety, different from those arising from the Section of a Cone. Some of which he supposed must be of great use in the Building of Ships:* in order to which it was, that he proposed them to consideration."

The first edition of the solution was printed (according to Watt) in 1662; the second, now before me, was printed in

* In the 19th century, a period of great commercial activity, after a war wherein England annihilated the European powers upon the high seas, we may blush to think that so much is due to the intrepidity of our naval heroes, and so little to our scientific prowess, as to render the fanciful surmises of Mr. Pett, after a lapse of nearly two centuries, as good a rule as we yet possess for "drawing the curve" to "lay down a ship" upon!

1684, and is annexed to his Algebra. It occupies 16½ folio pages, closely printed; and from the subsidiary calculations necessary for the determination of the numerical values of the ordinates, must have been a work of considerable labour. It is not however remarkable, otherwise than as being the first "locus of right lines" beyond the cone and cylinder whose character had been examined by the help of geometry. As it was undertaken solely with a view to the examination of the curves to which it gives rise by its intersection with a plane in given positions, it was not likely to suggest any new methods of investigating surfaces generally. It may not, however, be uninteresting to give a single specimen of his method.

"*Thirdly.* If it be cut by $c\Sigma\sigma q$, a Plain parallel to the Rectangle $CDBA$, the Curve Line will have this property. Drawing the Triangles as in the Scheme, it is, At, sR (the Ordinate from any Point s in the Arch ΣQ) to σq or ΣP (the Ordinate from Σ where the Plain $c\Sigma q$ cuts the Quadrantal Arch): $S\sigma$ is aR or AC (the whole height), to aq or $a\sigma$ (the Distance of the Point σ from the Plain Aax parallel to the Quadrant CDR). Because aRs , $a\sigma q$ are like Triangles.



"The Quantities I thus design in *Species*.

$$CQ = CD = R.$$

$$CP = C.$$

$$CR = c.$$

$$P\Sigma = q\sigma = \sqrt{R^2 - C^2}.$$

$$Rs = \sqrt{R^2 - c^2}.$$

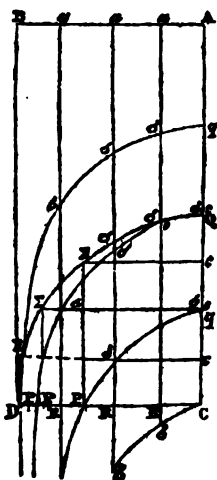
$$Ra = CA = A.$$

$$Rs \cdot Ra :: q\sigma (= P\Sigma) \cdot \sigma a = qa.$$

$$\sqrt{R^2 - c^2} \cdot A :: \sqrt{R^2 - C^2} \cdot A \quad \sqrt{\frac{R^2 - C^2}{R^2 - c^2}}.$$

$$\sigma a = qa = A \sqrt{\frac{R^2 - C^2}{R^2 - c^2}}.$$

"These Curve Lines (if made by Plains at equal distances) projected on the Rectangle CDBA (to which they are parallel), will appear as in the annexed Projection, which is thus made: Having drawn a Rectangle CDBA (like and equal to that in the Solid), and the Quadrant CDQ; divide CD into any number of equal parts at the Points c, and draw the Sines or Ordinates cΣ, with the Co-sines ΣP; Then supposing from the several points R in the line CD, the Lines Rσa parallel to CA, (cutting the Quadrant QD at s and AB at a:) And therein, As RS to PΣ; so AC (= aR) to aσ: The Curve Lines qσP in this Plain represent their Respectives qσs in the Solids. Where note, That as the Lines sσσs in the former Projection so are the Lines aσσs in this cut into equal Parts.



"As the Solid may be continued downwards at pleasure, beyond its Quadrantal Base CDQ, so may these Curve Lines qσP, in like manner, be so continued infinitely: And they will then be *Asymptotes* to each other; and to the straight Line BD so continued. And if the solid be continued upwards after a decussation in AB, the same Plains will cut off in the opposite Solid opposite Sections like to these.

"The Quantities in this Projection I thus design in *species*.

$$CD = R.$$

$$CP = C.$$

$$CR = c.$$

$$\Sigma P = S = \sqrt{R^2 - C^2}.$$

$$SR = s = \sqrt{R^2 - c^2}.$$

$$AC = A.$$

$$SR \cdot \Sigma P :: AC (= aR) \cdot a\sigma.$$

$$s \cdot S :: A \cdot \frac{S}{s} A.$$

$$a\sigma = \frac{S}{s} A = A \sqrt{\frac{R^2 - C^2}{R^2 - c^2}} = \frac{\sqrt{R^2 - C^2}}{\sqrt{R^2 - c^2}} A.$$

"In numbers thus; (putting $R = 1$, $A = 2$.)

	Aq	aσ	aσ	aσ	aσ
I.	0.5	0.516 +	0.577 +	0.756 —	Infin.
II.	1.	1.038 —	1.155 —	1.512 —	Infin.
III.	1.5	1.549 +	1.732 +	2.268 —	Infin.
IV.	2.	2.064 —	2.310 —	3.024 —	Infin."

We see here the same method of transferring the whole investigation of properties from the solid to the plane, as was usually employed with sections of the cone by all preceding writers. He goes on towards the end of the *Tractate*, to discourse upon the surface when any of the conic sections are substituted for the circular directrix; and also to suggest the oblique positions of the directrices. He however does not enter into particulars: nor would it have been a very hopeful undertaking either as to elegance in the form of the results, or as to their practical utility—nor yet in prospect of bringing the enquiry to any definite conclusion, even though conducted with an unwearied assiduity like his own, in reasonable time, by the methods of enquiry of which he was in possession,*

SIR CHRISTOPHER WREN, 1669.

The next instance of a "Rule surface," or "locus of straight lines" is the *hyperbolic cylindroid* of Sir Christopher Wren. This was described in a paper, brief indeed, but very elegant, which that illustrious Genius read before the Royal Society, in 1669; that is, seven years after the publication of Wallis's *Treatise on the Cono-Cuneus*.

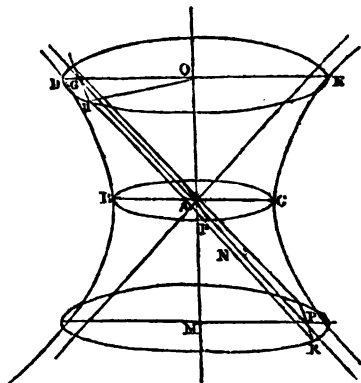
Wallis's method of considering the generations of curve surfaces by the motion of a line parallel to a given plane whilst at rest upon given directrices, may obviously be varied without end: yet the results to which he had arrived by employing data of great simplicity, were not such as to excite the most ardent enquirer to further efforts. He had, however, strangely omitted to consider the case when both directrices were rectilinear—the most simple of all cases, and which has been since so completely examined by modern writers. The method of Wren, was similar to this in one single particular, in as much as the surface was generated by the motion of a straight line: the regulation of the motion was altogether different, being produced by the revolution of the generatrix about an axis fixed in space, and invariable in its connection with the moving line. The state of optical science at that time held out, as desiderata, the art of producing by methods capable of easy practice, the different conoidal figures. The method employed by the builder of St. Paul's was characteristic of its inventor—its recommendations being simplicity, elegance, and perfect adaptation to the purpose

* Perhaps it may be interesting to remark that one direction of the cutting plane gives *elliptical sections*: viz. when it is parallel to the circular base. One also, from the characteristic generating property, is obviously a triangle.

in view. The discoveries of Newton in respecting light, however, causing the project, *for physical reasons alone*, to be deemed hopeless, all the methods of forming those bodies, mathematical and mechanical, were alike forgotten.

The following is the substance of Sir Christopher's paper on the solid in question, taken from the Philosophical Transactions of 1669.

Let the opposite hyperbolas, DB, CE revolve about the second axis OA generating thereby the hyperbolic cylindroid DBCBH; whilst the asymptote generates the asymptotic cone. If any plane, tangential to the asymptotic cone, cut the hyperbolic cylindroid, the section HA will be a parallelogram.



For let it be cut by a circular section BNC through the transverse axis; and through O and M in equal circles, which will be equal distant from the centre; and also through the axis, into the generating figure whose half is BDEC, the asymptote of which will be GP, so that the plane BDE be cut at right angles by the plane FBP, and join OH.

Then because the triangle OGH, is rectangular therefore $OH^2 = OG^2$, (or $OD^2 - OG^2 = GH^2$; and because DO is parallel to BA and cuts the asymptote in G, it will be (from the known properties of conics)

$$OG^2 + AB^2 = OD^2; \text{ that is, } OD^2 - OG^2 = AB^2 = AN^2.$$

$$\text{Hence } GH^2 = AN^2, \text{ or } GH = AN;$$

and GA, AN are at right angles to GA.

Also the same is demonstrated of all the other sections parallel to the base: therefore an hyperbolic cylindroid being cut by a tangential plane to the asymptotic cone, the section will be a parallelogram.

Q. E. D.

Cor. Hence it appears, that on the surface of a cylindroid, though consisting of a double flexure, innumerable right lines may be drawn. It appears, also, that the body may be otherwise generated, viz. by the revolution of a parallelogram about the axis, the angle at the axis remaining the same, or the generating line HR continuing revolving, and either generating or cutting the body.

"And if a sharp and straight edged tool have the same situation to the axis with the generating line, while the mandrel turns round; it is plain that hyperbolas may be as accurately wrought on the lathe, as circles; since nothing is more required for the formation of a cylindroid than for that of a cylinder, except that in cylinders the edge of the tool is parallel to the axis, but here inclined.

"Therefore it is to be observed, that the species of the hyperbola is varied according to the inclination of the angle $\angle GOA$; consequently it may be so easily fitted to a given hyperbola that there is no need of further demonstration. But if the angle continuing the same, the generating line approach nearer the centre, there will arise a less hyperbola, but quite similar to the former."

The ingenious contrivance by which this principle is applied to the formation of hyperboloids of two sheets not belonging to our present object, need not be further noticed here, than simply to state, that it is found in Phil. Trans. No. 53. p. 1059: or Hutton's Abridgment, Vol. I. p. 396.

M. PARENT, 1709.

The next notice we find of this body is by M. Parent, in the *Histoire de l'Académie*, 1709, p. 57: but this referring merely to its cubature, and the relation between its volume and that of an ellipsoid,—adds nothing to our knowledge respecting the interesting properties at present under our consideration. It is mentioned here, from its being the first time, after Wren, that the figure seems to have been noticed; and the dissertation refers expressly to the Paper in the Transactions, for the name by which the body is designated. It may also be remarked as something singular that in the *Hist. de l'Acad.* the name is said to be given "après M. Wallis," whilst D'Alembert referring to it in the *Encyclopædîe*, says that M. Parent named it "après M. Wren." I cannot take upon me to say that Dr. Wallis has not named and mentioned this figure; but I have never met with it, if he has. D'Alembert, however, only notices the dissertation of Parent, and makes no reference to the above quoted paper of Sir Christopher Wren: in consequence of which, we may presume he was unacquainted with it.

SIR ISAAC NEWTON.

Sir Isaac Newton, also, in his *Arithmetica Universalis* investigates the nature of the sections made generally in this solid. Part I. Problem 33. He conceives it to be generated according to Sir Christopher Wren's *Corollary*, by the revolution of a

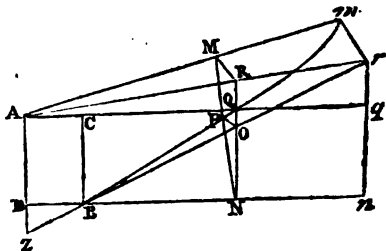
straight line about an axis, which it does not cut. He then investigates the nature of the section made by a plane in any position, and finds it to be generally of the same order, descending, however, under particular cases of relation amongst the data, to that of a straight line. We may thus consider that the general characteristic of the hyperboloid of revolution of one sheet have been determined by these illustrious men*.

DR. BRAKENRIDGE, 1759.

We come now to a more important document, the Paper of Dr. Brakenridge, in the Philosophical Transactions of 1759, p. 445. How little the nature of this class of enquiries, has been generally understood, will appear from the manner in which the venerable and learned Dr. Hutton characterizes this very memoir. In his Abridgment of the Transactions, we find this laconic account of it. "*A Letter to the Right Honourable Hugh Earl of Marchmont, F. R. S. concerning the Sections of a Solid, hitherto not considered by Geometry, from William Brakenridge, D. D.—An imaginary solid that never can occur, or be of any real use.*" Ab^l. XI. p. 425. Dr. Young, in his very imperfect catalogue (imperfect on this subject in particular) simply notices it as—"A locus of right lines." Such is the amount of attention which this valuable Paper has obtained!

"You have here," says he to Lord Marchmont, "new method of considering some geometrical curves, from the sections of a solid hitherto not taken notice of, and by which in particular you will see, that the two infinite curve lines from the sections of the cone are also sections of this; which may be of some use, as it seems to extend our views of their nature and properties. The description of it is very easy and obvious, and it has something remarkable in its form, that though in the most simple case, it is generated and bounded by right lines, the surface is incurvated. The solid is thus described.

"Let DN be a right line drawn on a plane, and a point A at any given distance from the line, from which having raised, above the plane, the right line AM in any given angle, and drawing from the same point A the line AD, meeting the line DN on the plane in D, make another plane MNR to move parallel to itself and to that line AD, in a given angle to the first



* There being no difficulty in consulting Sir Isaac Newton's work, I have thought it unnecessary to give any extract from it.

plane; and then suppose the intersections of it M, N, m, n , &c. with the lines AM and DN to be continually joined with the right lines MN, mn , &c. And there will be generated an incurvated surface by them, and bounded by the lines AM, MN, ND, DA .

"It is evident this incurvated surface may be infinitely extended on all sides beyond the lines AM, MN, ND, DA , and as well below the given plane ADN as above it; and therefore the various sections of it, if continued, will be infinite lines.

"The line AM raised above the given plane ADN may be called the vertical Directrix.

"And if through this directrix AM there be drawn a plane AMR in any given angle to the plane ADN , intersecting the moving plane NMR in the line MR , and meeting the given plane ADN in R ; then the points A, R and N, R being joined, there will be formed a trapezium $ARND$, that may be named the base, and in which the line DN may be called the Directrix of the Base.

"And the plane AMR passing through the vertical directrix, and the moving plane MNR together with the base $ARND$, and the curve surface will make a solid $AMRNDA$, something in the form of a wedge.

"In this solid there may be made five sections in a given angle with the base or parallel to it; viz. one parallel to the moved plane [section, a right line]; one parallel to the directrix of the base [section, an hyperbola]; one parallel to the vertical directrix [section, an hyperbola]; a fourth parallel to the base [section, also an hyperbola]; and a fifth intersecting both directrices [which gives also the same curve]*.

He then goes on to shew that "if the vertical directrix AM be made parallel to the plane of the base, but the plane passing through it not parallel to the other directrix, then the section meeting the two directrices will also be an hyperbola:" and he remarks that "in all those sections where the common intersection of the plane passing through the vertical with the base, is not parallel to the other directrix, the curve is an hyperbola."

"But now," he continues, "if we suppose the common intersection AR of the plane passing through the vertical with the base, to be parallel to the directrix DN of the base, and both directrices to be cut by the section, the curve will be a parabola."

One specimen of his method of demonstration will suffice on

* I have inserted the bracketed phrases to prevent a new enumeration further on.

this occasion, (p. 451. Art. 5), "If the section is so made as to meet the two directrices; the curve will be an hyperbola.

"Let the section BPO meet the directrices in B, m , and intersect the plane of the base in the line BO ; and make the moving plane NMR to be cut by the section in PO , and to meet the vertical directrix in M . Then from the point M draw MR parallel to PO ; and through the lines AM, MR imagine a plane to pass intersecting the base in the line Ar , and meeting the line BO in r and the section in mr . And from A draw AN parallel to NR , and AQ to DN ; and through r make nr parallel to NR and to meet AQ in q , and DN in n . Draw also from the point B the line BC parallel to AP and meeting AQ in c . The lines, then, $Aq, Ar, mr, rq, Br, Bn, rn$, will be given. Make, therefore, $AD = a = BC, AQ : QR :: a : q$; and $AQ : RM :: a : m$; $Br : Bn :: a : b$; $Br : rn :: a : n$; $AC = DB = c$; the abscissa $BO = x$, and ordinate $OP = y$. From which we have

$$BN = \frac{bx}{a}, ON = \frac{nx}{a}, AQ = c + \frac{bx}{a}, RQ = \frac{qca + qbx}{a^2},$$

$$MR = \frac{mca + mbx}{a^2}. \text{ And then, in the similar triangles } NPO,$$

MRN , having $NO : PO :: NR : MR$, the equation will be

$$ya^2 + yca^2q + yxbqa = nmcax + nmbx^2.$$

And the curve is an hyperbola; and in this case it will be convex towards the plane of the base. But when BN is negative the equation is

$$ya^2 + yqca^2 - yxbqa = nmcax - nmbx^2;$$

and the hyperbolic curve will be concave towards the base."

This surface, it is unnecessary to say is that to which I referred in the former part of my paper as a very important one, which had been overlooked (as Dr. Brakenridge quaintly remarks) by Dr. Wallis, "because often when searching after new discoveries, we are often like those, who groping in the dark, miss the things that are nearest them." It is, in short, the "*hyperbolic paraboloid*" of the modern French Geometers.*

Before dismissing my account of the labours of Englishmen in this branch of enquiry—all which, be it remarked, were antecedent to those of our Gallic neighbours,—it may be worth while to

* Mr. Nicholson has considered a particular case of this figure, (where the plane to which the generatrix is parallel is perpendicular to one of the linear directrices) in the *Repository*, Vol. IV. Part II. p. 27-30. His method in principle is very analogous to Brakenridge's: but it is neater in its details, and more uniform in its manner. This will atone for its rather less general grasp of idea.

mention that Brakenridge attempted to take a more general view of the subject, and to determine the highest order of the curve that could result from the substitution of any curvilinear directrix for one of the lines AM or BN. This he found to be $2n$, where n is the order of the directrix. The *cono-cuneus* of Wallis gives, therefore, as he remarks, lines generally of the fourth order. The possibility, however, of determining *from any generating property of a surface*, the highest order of lines which can result from its intersection with a plane is a problem, the legitimacy of every solution of which, by methods independent of finding the equation of the surface in terms of three co-ordinates, may reasonably be doubted. Certain it is, that no general method has yet been devised for that purpose: nor is it necessary to seek for such, now that we are in possession of a process so universal and so simple in its application, and at the same time so consonant to the general principle upon which the Analytical Geometry is founded and built up.

That beautiful system, the product of the fertile and precocious* mind of Clairault, was, however, unknown, or misunderstood in England for nearly a century, whilst in the hands of Euler, D'Alembert, Lagrange, Laplace, and a host even of minor mathematicians, it was giving birth to the most brilliant discoveries in the pure and mixed mathematics. The utmost point to which Englishmen attempted to extend their investigations respecting solids was confined to regular polyhedrons (in which however little was done more than to find the volumes) and to surfaces of revolution and these generally the simplest cases. The consequences were, that we fell behind the rest of cultivated Europe in our knowledge upon these subjects, and all power of extending the boundaries of science seemed to be lost in a universal torpor of English mind. Maclaurin† in-

* When Clairault wrote his *Treatise on Curves of Double Curvature*, in 1729, he was but 16 years of age.

† A claim has lately been set up for Maclaurin's being the author of the present system of Analytical Geometry of Three Dimensions, it may be well not to engage with that fastidious people, the French, in the discussion of any claims which we have not well ascertained to belong to ourselves, for complaisant as they are in their daily intercourse with their neighbours, it is confined to cases where discussions of "*meum et tuum*" cannot worm themselves between! In the present case, their sensitiveness may justly be alarmed: for I believe that since the days of Dr. Wallis no one ever dreamt of depriving Descartes of the honour of this fine device. His plan was to consider a curve of double curvature as the intersection of two cylinders at right angles to one another, that is, from its orthographic projections on two planes at right angles to one another;—in other words, to intro-

deed, the last of the bright spirits that shot up into mental vigour under the cheering eye of Newton, it is true, made some efforts to point his countrymen the road to glory; but in vain! Even to the general principles of integration—the interpretation of differential expressions—and the numerous geometrical and physical problems depending upon these,—did this unhappy spirit of *patriotic Anti-gallicism* extend itself.* The consequence

duce into Analytical Geometry the method employed by architects, of delineating a building by means of its *orthography* and *icnography*. The method has been illustrated by several cases of the curve of intersection of surfaces of revolution of the second order by the Jesuit Rahuel, (p. 398-409) in his Commentary on the Geometry of Descartes, 1731.

The more important contrivance, however,—*if representing a surface by one equation between three variables, and a curve of double curvature by two equations between three variables,—is due solely to CLAIRAUT*. It was made known by him to the *Académie* in 1728, and published in a thin 4to volume, under the title of “*Recherches sur les Courbes a Double Courbure*,” in 1731.—The publication having taken place *eleven years* prior to that of Maclaurin's *Fluxions*, in which it has been understood the system first appeared, I trust the general reference now given will be sufficient to put an end to this claim for Maclaurin. It is, however, sufficient honor to him that he was, probably, the first who *resolved forces* in the direction of three rectangular co-ordinates, without having also been the first to *resolve geometrical space* in that manner. I believe his claim to the resolution of motion will bear the strictest scrutiny: and such is what I understand Mr. Barlow to mean in the statement of his claims. *Encycl. Metrop. Art. Mech.* p. 5. Mr. Young has, I think, misunderstood him, in the preface of his very elegant and perspicuous “*Treatise on Analytical Geometry*,” just published. What Mr. Wright's authority was (Pref. to *Self Ex. in Euclid*) I cannot tell—I should suppose the same. Mathematicians will, I persuade myself, excuse me if I hint to them, that general statements should always be made in strict reference to authorities, and those authorities intelligibly stated. By neglecting the first part of this caution we are very liable to express ourselves with a vagueness often bordering on error, and in neglecting the latter we destroy to others the advantages which reference is primarily designed to afford. In matters relating to the history of disputed claims, the rule should be followed with the most rigorous minuteness:—and, indeed, authors who neglect this precaution can scarcely hope to gain ultimate authority with the world.

* Let any one compare the Philosophical Transactions of our own Royal Society, (the acknowledged focus of English science) with the *Mémoires* of the French Academy, with that of Berlin, or with that of Petersburg—then if he do not blush for England, I envy him neither his feelings nor his science. Even the paper of Brakenridge simple as it was, was not understood.

I am happy, however, to make an exception in favour of our plane geometry of that period, the only branch of mathematical science which was materially improved in our hands. I need not particularize authors, as every English Geometer is familiar with their writings.

was, that instead of teachers of science to the whole of our race, we found ourselves at the beginning of the present century, ignorant almost of its very elements! May the records of the present century in the history of English science make atonement for the disgraceful apathy of the last!

MAUDUIT, 1763.

We come now to an enumeration of the labours of the French Geometers with respect to this class of surfaces. To them it is, that we owe all the other properties of the Rule Surfaces, as well as the very elegant method which is now employed in their investigation. When the directrices were two straight lines, and the third condition that the generatrix divided the directrices in the same ratio, the surface itself, had been long known to the French Artisans and Architects; but it does not seem to have attracted the attention of the French mathematicians till about 1763, when M. Mauduit* gave in the *Mémoires Etrangeres*, tome IVeme. p. 629—33, a short paper on the subject. It had borne amongst Architectural writers the cognomen of '*surface gauche*,' or the *twisted surface*.† The professed object of M. Mauduit, is to give the method of calculating the volume contained be-

The problems resolved by this method have never been equalled either in number or difficulty, nor the methods conducted with so much elegance by any writers as the English. In this, however, our Universities had little share: it fell upon the shoulders of those men whom with supercilious ignorance it was the custom of the titled scholars of those days—perhaps scarcely less so now—to affect to treat as unworthy even of a passing notice. A geometrical work which is constantly employed at Cambridge is wholly composed of materials collected from the Diaries and other periodicals produced by Diarian writers—and yet there is not the slightest allusion made by the author to the sources of his *matériel*. Is this justice?

* Known to the English reader by Crakelt's Translation of his Spherical Trigonometry.

† *Diderot* thus notices them in the *Encyclopædie*, 1757.

"GAUCHE, (*coup de pierres*.) Il se dit de toute surface que n'a pas quatre angles dans une même plan; ensorte qu'étant regardée en profil, les côtes opposés se croisent. Telle est une portion de la surface d'un vis, et de la plupart des arrière-voussures. Ce terme est de tous les arts, tant de maçonnerie que de charpenterie et menuiserie d'où l'on a fait *gauchir*.

"GAUCHIR (vn). *Menuiserie*. Il se dit des faces ou paremens de quelque piece de bois ou ouvrage, lorsque toutes les parties n'en sont pas dans une même plan; ce qui se connoît en présentant une regle d'angle en angle: si l'angle ne touche point par-tout on la promenant sur la face de l'ouvrage, l'on dit que cette face a *gauchi*. Une porte

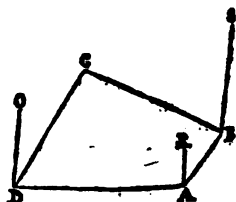
tween a portion of this surface and specified planes, and in spite of the great number of terms which the resulting differential expression contains, he has deduced from it a practical rule which is tolerably simple." He then goes on to find the equation referred to three co-ordinates, which is the more interesting portion of the *Mémoire*. He, however, does not take the question in all its generality—"as," says he, "we may readily convince ourselves that the nature of the surface is always the same, that is that it does not exceed an equation of the second degree of three variables; and as the computation becomes extremely long and complicated in the case where the base is a quadrilateral whose sides are not parallel, we shall satisfy ourselves with examining that only where two of the sides are parallel."

est gauche au milieu, si quand on la présente dans ses feuilures qui sont bien à-plomb elle, ne porte point par-tout également."

M. Mauduit gives the following "*Définition et Génération des Corps Gauches*" in the early part of his *Mémoire*: "où," says he, "l'on explique leur formation, la manière de les toiser sans être obligé de les décomposer; et les différentes propriétés de ces Corps par rapport aux courbes que l'on peut y trouver par l'intersection d'un plan."

"ABCD est un quadrilatère irrégulier quelconque, aux angles duquel s'élèvent trois hauteurs inégales AR, BS, DO, perpendiculaires à ce plan (on verroit aisément que le cas de quatre hauteurs se réduiroit à celui que nous examinons ici). Si après avoir tiré les lignes CO, CS, RO, RS, on imagine une règle qui par court dans le même temps les lignes opposées CS, RO, il en résultera un corps terminée par le quadrilatère ABCD, par plusieurs surfaces verticales qui seront toute des triangles ou des trapèzes, et enfin par une surface courbe ROCS, décrite dans le mouvement de la règle sur les lignes CS & RO; c'est à ce corps que l'on donné le nom de *Corps gauche*, à cause des formes singulières qu'il affecte, selon l'inégalité plus ou moins grande des lignes verticales AR, BS, DO." p. 624.

Savérien does not give the word in his *Dictionnaire des Mathématiques*.

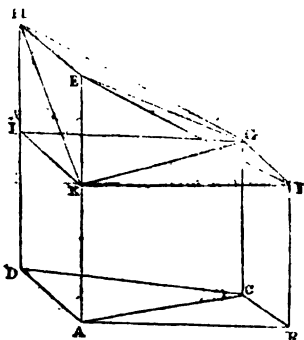


For the Volume.

Lemma. If any quadrilateral $ABCD$ composed of two triangles ACB, ADB whose areas are ϵ^2 and δ^2 , and if at the angles B, C, D, A , be drawn four perpendiculars, so that $BF = CG = \phi$ and $AE = DH = \gamma$; then the solid

$$ABCDHEFG = \frac{\epsilon^2\phi + 2\delta^2\gamma + \epsilon^2\gamma + \delta^2\phi}{3}$$

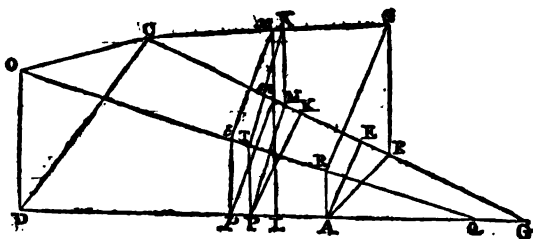
For, upon the verticals AE DH take the lines $AK \pm DI$ and $= BF$; and through the points ϵ, κ, F, C conceive a plane parallel to the base: then it is obvious that the solid between these parallel planes is equal to $(\epsilon^2 + \delta^2)\phi$. The remainder of the solid $HEKFG$ is a kind of wedge which contains three pyramids, one of which $EKFG$ has the triangle EFG for base and EK for altitude. It, therefore, is equal to $\frac{\gamma - \phi}{3} \times \epsilon^2$. The two others



$GHEK, GHFK$, and are together equal to $\frac{\gamma - \phi}{3} \times \delta^2$. By addition, we have, &c. as stated above.

The foregoing lemmatical proposition being established, we proceed to find the solidity of any *Corps gauche*, whatever.

Prolong the opposite sides AD, BC of the quadrilateral base till they meet at G . Prolong also OR till it meets AD in Q ; and demit from the point A the perpendicular AE upon the opposite side BC .



$$\text{Let } \left\{ \begin{array}{ll} AG = a, & BC = g \\ AE = b, & AR = m \\ AD = c, & BC = n \\ BG = f, & DO = q \end{array} \right\} \text{ be the given parts, and}$$

$$\left\{ \begin{array}{l} AP = x \\ FP = dx \\ BM = \frac{gx}{c} \\ Mm = \frac{gdx}{c} \end{array} \right\} \text{ be the indeterminates.}$$

Then we find readily that

$$PK = \frac{ab + bx}{a}, \text{ and } mL = \frac{bcf + bxy}{ac}.$$

These two lines are the heights of the triangles MPm , Pmp which form the base of the element of the solid.

Hence the triangle $MPm = \frac{abg + bgx}{ac} \times dx = z^2$, and the triangle $Pmp = \frac{bcf + bgx}{2ac} \times dx = \delta^2$

For finding, then, one of the altitudes of the elementary solid, we seek in the first place AQ , by the analogy,

$$OD - AB : AD :: HR : AQ, \text{ or in letters,}$$

$$q - m : c :: m : \frac{cm}{q - m}, \text{ from which we get}$$

$$PT = \frac{cm + qx - mx}{c} = \gamma. \text{ In like manner}$$

$$MX = \frac{cn - nx}{c} = \phi.$$

It is now easy to find the elementary solid by means of the lines already known and the formula in the last article.

Employing the denominations for the quantities already laid down, the parts of this formula become

$$\begin{aligned} & \frac{abcdn + bcgdx - abgdx - bgdx^2}{3ac^2} \times dx = \frac{2z^2\phi}{3} \\ & + \frac{bc^2fm + bcfdx + bcfmx + bgmx - bgmx^2}{3ac^2} \times dx = \frac{2\delta^2\gamma}{3} \\ & + \frac{bc^2fn + bcgdx - bcfnx - bgdx^2}{6ac^2} \times dx = \frac{\delta^2\phi}{3} \\ & + \frac{abgcm + abgqx - abgdx + bcgmx + bgqx^2 - bgmx^2}{6ac^2} \times dx = \frac{\epsilon^2\phi}{3} \end{aligned}$$

The integral of these quantities collected and condensed as much as possible is

$$\begin{aligned} & \frac{2abgn + abgm + bcfm + bcfn}{6ac} \times x + \\ & \frac{2bcgn - abgn - abgm - bcfn + bcfm + bcfq + abgq + bcgm}{12ac^2} \times x^2 \\ & + \frac{bgq - bgm - bgn}{6ac^2} \times x^3. \text{ If we suppose } x = c, \text{ and make} \end{aligned}$$

the necessary reductions, we shall obtain for the volume of the whole solid contained within these boundaries,

$$\begin{aligned} \text{ABCDORSC} &= \frac{bg}{2} \times \frac{2n + q + m}{6} + \\ &+ \frac{bcf}{2a} \times \frac{2m + 2q + n}{6} \\ &+ \frac{bcg}{2a} \times \frac{2q + m + n}{6}. \end{aligned}$$

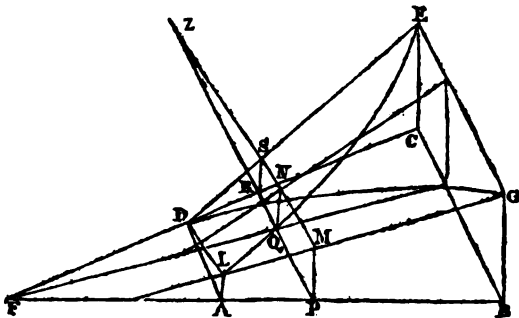
W. W. R.

He then goes on to construct a solid bounded wholly by plane faces, which shall be equal to the *solid gauche*: but as this is an operation of the most elementary kind, and is not, moreover, made conducive to any further important investigation we think it unnecessary to dwell upon it here.

This is, I believe, the first time that any part of a surface of the second order has been cubed.

To find the equation of the surface.

Let $AB = a$; $BC = b$; $DC = c$; $AD = d$, and $AF = f$; the



heights $AL = m$, $BG = n$, $CE = q$; $AP = x$; $PQ = z$; $QN = y$; then we shall have

$$AK = \frac{am}{n-m}, PM = \frac{am + nx - mx}{a}, RS = \frac{qx}{a}; RP = \frac{fd + dx}{f}.$$

Suppose the line MS prolonged towards Z , till it meets PR , also prolonged as far as necessary; then we shall find $RZ =$
 $(fd + dx)qx$

$\frac{f \times (am + nx - mx - qx)}{f \times (am + nx - mx - qx)}$, and find QN or y by the proportion
 $RZ : RS :: ZQ : QN$; or putting for these their analytical value, and reducing, we obtain
 $adxy + adfy = qd\lambda^2 + qdfx + (fd + dx - fz)(am + nx - mx - qx)$;
 an equation which appertains to an hyperbolic curve surface.
 C. Q. F. D.

Cor. 1. If we suppose z has a constant ratio to RP , so that

$$r : d :: z : \frac{fd + dx}{f}, \text{ and deduce the value of } z \text{ from}$$

this, and insert it in the general equation, and we shall get

$$\frac{n-m}{a} \cdot x + \frac{r}{ad} \cdot (mx - nx + qx - am) = y; \text{ the}$$

equation of a straight line, from which it follows that the *corps gauche* is such, that it may be equally generated by the motion of the ruler upon the sides ED , GL or upon LD , GE ; for since we have supposed $z : RP :: r : d$, it will follow that the lines RP , AD , and therefore the lines AD , BC are passed over in the same time.

Cors. 2, 3, 4. Are concerning the sections of the surface.

Cor. 5. It will follow, again, from this generation and Cor. 1. upon the double mode of generation, that of all curve surfaces which can terminate a body, this is the least possible, that is, it is most approximative to a plane. For we can apply a ruler and make it move in two different directions, through the whole extent of the surface; and it would become absolutely plain, if we could move the line upon the surface into a third position [always resting upon these two.]

"We may remark that this surface has much *analogy* to the annulus", engendered upon the lathe by the revolution of a cylinder, to the axis of motion of which we present the edge of the tool obliquely. We find in this body as well as in that which we have examined, the different sections of a cone; and we can also apply to the curve surface a straight line which shall touch it in all its extent, in two different directions."

Three important steps are here made. 1st. The *equation* of the surface in terms of three variables. 2nd. The discovery of the *double genesis* of the surface. 3d. The *analogy* between the *surface gauche* and the *hyperbolic cylindroid* of Sir Christopher Wren.

M. Mauduit promised to resume the enquiry in a second memoir, but I do not find that he ever accomplished his purpose. This is the more to be regretted, as he would doubtless have been able then to give a general and a more simple investigation of the equation; and have also shewn not only the analogy between the *surface gauche* and *hyperbolic cylindroid*, but also that *the latter is only a particular case of the former of these surfaces*.

Were we to follow the order of time we should now be brought to the consideration of the paper of Euler in the Petersburg Academy Transactions for 1771, upon Developable

Surfaces, and to the various papers of Monge upon the same class of surfaces, upon the evolute surfaces of curves of Double Curvature, the Construction of Arbitrary Functions which enter into the integrals of partial differential equations, and upon "Ombres et Penombres,"—all in the *Mémoires des Savans Etrangers*. We should more especially be led to consider the memoir inserted in the ninth volume of that work, p. 382, "*Sur les propriétés de plusieurs des Surfaces Courbes*," one part of which is expressly devoted to *surfaces gauches*. It appears, however, from the very different purposes for which both Monge and Euler wrote from those we have been considering, as well as the essentially different method of conducting the enquiry, that it will be more advisable to pass at once to the paper of M. Tinseau, also inserted in the ninth volume of the *Mémoires Etrangères*. Our business has hitherto been exclusively with the *surfaces gauches*,—we shall, after finishing this, trace the history of the *surfaces developables* in the same manner: and then finally consider the two classes in conjunction with respect to contrast and similarity.

M. TINSEAU, 1774.

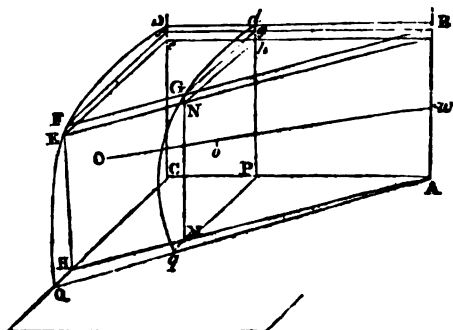
M. Tinseau considers the movable line as *enfilading* the three directrices, which may be any curves whatever: and the surfaces he calls "*surfaces gauches*." This notion of the method of generating the rule-surfaces was first made use of by Monge, in his Memoire of 1771: and was a very obvious consequence of the discovery of Mauduit, of the possibility of generating *surfaces gauches* in two different manners already mentioned (at Cor. 1. p. 26.)

For let A, B, C, D be the four sides of a twisted quadrilateral (*quad. gauche*, Fr.) taken in order, and let G, G' denote the two generatrices that pass through any point of the *surface gauche*. Then if G' become fixed in any one of its positions, the other generatrix G must always glide along it as well as along B, D; since all these three lines are upon the surface described by G. Monge, moreover, had employed any curves of double curvature as directrices; and Wallis and Brakenridge had employed a curve instead of one of the rectilinear directrices, long before this. Three conditions are necessary to confine the generatrix so as to describe a surface; and it was obvious that these lines may be of any kind whatever. When, therefore only two lines were given, it became necessary, as M. Tinseau states, to adopt some other condition. He employs, exactly as Wallis and Brakenridge had done, the condition

that the moving line shall be always parallel to a given plane. He considers this plane to be situated horizontally; takes a straight line for one of the directrices, which he denominates the "*axe*"; and for the other "*directrix*" (he so designates it) he takes any curve whatever. He states that the surface generated will belong to a class of surfaces known in architecture by the name of "*Conoids*." He distinguishes these conoids into "*right*" and "*oblique*," according as the *axe* is perpendicular to the horizon or not: and remarks that every such surface is infinitely extended, and divided into two perfectly equal parts, by the *axe*, as a cone is by its summit.

When the directrix is a semicircle in a certain position, the figure is clearly the cono-cuneus of Wallis: and when a straight line in any position, it is the anonymous solid of Brakenridge, subsequently called the *hyperbolic paraboloid*, or the *plane gauche* by the indefatigable Hachette.

In the annexed figure he supposes ACQ to be the horizontal plane, and AB the *axe*. Then cutting the conoid by a plane parallel to the *axe*, we shall have a section such as DEQ . Then considering the solid comprised between the horizontal plane ACQ , the plane of the base $CDEQ$, the conoidal surface, and any plane $ABDC$ drawn through the *axe* and a point D of the base: we "shall find the figures possessed of properties which merit attention from their analogy with those of the prism and the pyramid."



Theor. 1. "The areas of the sections $CDEQ$, $p d n q$ parallel to one another and to the *axe*, are proportional to their distances from the *axe*." For imagine an infinity of planes infinitely close to one another, which cut the solid in a series of triangles, and the two sections $CDEQ$, $p d n q$ in parallel lines. These planes will divide the two sections into the same number of minute parallelograms of the same altitude, two and two, as $E e f F$, $N n g G$, &c. Then

$$E e f F : N n g G :: E e : N n :: E D : B d,$$

that is in the ratio of their distances from the *axe*: and therefore the sum of the areas $E e f F$ and the sum of the areas $N n g G$, that is

the sections $CDNQ$ and $p\delta Nq$ are proportional to their distances from the axe.

Theor. 2. "The solid $ABDCQCA$ is equal to the product of its base $CDNQ$ by half its perpendicular height."

For we may regard it as composed of an infinity of layers comprised between sections parallel to the base. Let $p\delta Nq$ be one of these sections, x its perpendicular distance from the axe, s the base, h the height of the conoïde, dx the thickness of one of the layers which has $p\delta Nq$ for base; then this layer

will be (Theor. 1.) $\frac{Bx}{h} \times dx$. The solid, then, is $\int \frac{Bx dx}{h} =$

$\frac{Bx^2}{2h}$ (the constant being zero since we integrate from the axe):

and when $x = h$, $\frac{Bh}{2}$ will be the value of the entire solid.

We may easily deduce, he remarks, from this the solidity of a trunk of a conoïde terminated by bases parallel to one another and to the axe, and that the layers supposing them of the same thickness, will increase as the ordinates of a triangle.

Theor. 3. The centre of gravity of the conoïde is at two-thirds,* reckoning from the axe, of the horizontal line ow drawn from the centre of gravity of the base to the axe.

"For demonstrating this, I shall in the first place shew that the line ow passes by the centres of gravity of all the sections parallel to the base: in fact the distance of the centre of gravity of the section $p\delta Nq$ from the horizontal plane (parallel to the axe AB) is

$$\frac{\int NGgn \times MN}{\int NGgn} = \frac{\int \frac{x}{h} \times EFfe \times EH}{\int \frac{x}{h} \times EFfe} = \frac{\int EFfe \times EH}{\int EFfe};$$

that is to say the same as that of the centre of the base; and hence the horizontal plane drawn by the centre of gravity, o , of the base contains the centres of all the sections parallel to this base.

"The distance of the centre of gravity of the section $p\delta Nq$

* There is, in the original, a typographical error in the statement of this—"aux tiers."

from the plane $ABDC$ (taken parallel to the straight line ca) is =

$$\frac{\int NGgn \times \frac{Nn}{2}}{\int NGgn} = \frac{\int \frac{x}{h} \times EFfe \times \frac{Ec}{2} \times \frac{x}{h}}{\int \frac{x}{h} \times EFfe} = \frac{x}{h} \times \frac{\int EFfe \times \frac{Ec}{2}}{\int EFfe};$$

that is to say, that it is to the distance of the centre of gravity, o , of the base, as $x : h$. This centre is, then, in the plane drawn by the axe and the centre of gravity of the base. Hence, the intersection of these two planes, that is, the horizontal line ow drawn from the centre of gravity of the base to the axe, contains the centres of all the sections parallel to the base; and consequently of all the layers that are terminated by the sections, and finally of the *conoïde* itself.

"It only remains to find the position of this centre upon the line ow . Now any layer whatever, parallel to the base is (Theor.

I.) = $\frac{Bx dx}{h}$, and its moment about the point w is = $\frac{Bx dx}{h} \times$

$o'w$: but $o'w : x :: ow : h$; and hence, this moment is $\frac{Bx^2 dx}{h^2} \times$

ow . Therefore,

$$\frac{\int \frac{Bx^2 dx}{h^2} \times ow}{\int \frac{Bx dx}{h}} = \text{distance of centre of gravity of the}$$

solid from the point w ; or bearing in mind that since we integrate from the axe the constant is nul, we have,

$$\frac{\left(\frac{Bx^3 \times wo}{3h^2} \right)}{\left(\frac{Bx^2}{2h} \right)} = \frac{2}{3} ow \times \frac{x}{h} = \frac{2}{3} ow. \text{ Which was to}$$

be proved.

We may also deduce from this that the layers are as the ordinates of the triangle. We may, moreover, find readily the centre of gravity of a layer of a *conoïde* terminated by bases parallel to one another and to the axe, upon the straight line which joins the centres of gravity of its two bases."

I have given these three propositions at length, because they are exceedingly beautiful, and also because they have not been noticed by any other writer that I know of. The volume of the hyperboloid of one sheet found by M. Mauduit would be a

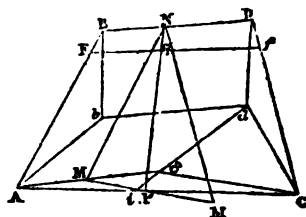
very suitable completion of this branch of the enquiry: but unfortunately it would occupy more room than can well be spared for it in the present sketch. It is more especially the case, as the subsequent part of M. Tinseau's *mémoire* will scarcely admit of compression without despoiling it of its value. *Almost the whole of it, too, has been in one form or other brought forward by subsequent writers as new.*

M. Tinseau goes on to remark that a singular property of these *conoïdes* is the "simplicity of their equations, when we take for co-ordinates the straight lines AB, AC, CQ. In fact, if N be any point in the surface, and NM = z , MP = y , AP = x its co-ordinates; we have $z = \text{EH}$, but $\text{EH} = \phi(\text{CH})$, ϕ indicating any function whatever, given by the nature of the base; then

$z = \phi(\text{CH})$. But $\text{CH} = \frac{y \times \text{AC}}{x}$, and hence

$$z = \phi\left(\frac{y \times \text{AC}}{x}\right).$$

He then proceeds to deduce from the preceding principles, the principal properties of the *surface gauche*. This he calls the *quadrilatère gauche*; in which he is peculiar, for all other writers call the *assemblage of any four straight lines traced upon this surface*, not the included portion of the surface itself, the *quadrilatère gauche*. "Let," then says he, "AB, CD be two indefinite straight lines situated in different planes and projected on the horizontal plane in ab and cd . If we know the angles CAB , BAB , ACD , and DCD , the length of AC, these two lines are given in position in space. There are also several other ways of fixing these positions.



Parallel to the horizontal plane CAB , let an infinity of straight lines be drawn, such as BD, Ef , &c. which pass by the lines AB, CD; we shall then have a surface of the conoïdal kind, the axe of which is AB, and directrix CD. This surface extends itself to infinity in every direction, since by all the points of AB and CD prolonged to infinity we may find horizontal straight lines, as BD, and prolong these also to infinity. Then by all the elements AC, Ef , BD of this surface we can draw parallel planes, and as these parallel planes cut in proportional parts the lines situated or not in the same plane, the lines AB, CD prolonged indefinitely are cut by the elements of the sur-

face into parts proportional to one another and to the lines AB , CD . Hence it is clear that the surface whose formation we have given is the same with that which is known by the name of *surface gauche*, *plan gauche*, or *quadrilatère gauche*, taken according to its common definition.*

It is quite clear from this that M. Tinscau had formed to himself a contracted view of the general character of *surfaces gauches*. In many structures, indeed, two of the sides of the quadrilateral might be parallel to the same plane, thereby forming the *hyperbolic paraboloid* of which he speaks. In many others, however, no such condition is implied, or even always possible (as in sloping terraces for instance) and in this case we have a more general, and in some respects a different class of surfaces produced,—the *hyperboloid of one sheet*. M. Tinscau treats only of the former: M. Mauduit only of the latter, class.

"The equation of this surface is extremely simple," says he. "Let A be the origin of co-ordinates, AC the axe of x ; draw DD parallel and equal to AB , and hence terminated at the plane CAB ; draw also CD ; let N be any point of the surface and BD the element upon which it is situated; draw MN parallel to AB and MP to CD ; and finally put $AP = x$, $PM = y$, $MN = z$, $AC = a$, $DDC = \kappa$, $DCD = L$. Then $CD = \frac{ay}{x}$; but $CD : DD$

$$:: \sin \kappa : \sin L; \text{ hence } \frac{ay}{x} = \frac{z \sin \kappa}{\sin L}; \text{ or } y = \frac{xz \sin \kappa}{a \sin L}.$$

"It follows from this equation that all the plane sections which we can make in a *quadrilatère gauche*, are either straight lines or hyperbolas.*

"Let 1st. $z = h$, it becomes $y = \frac{hx \sin \kappa}{a \sin L}$ which only shews what we already know from the genesis of the surface: viz. that all the horizontal sections are straight lines passing by the corresponding points of the lines AB , CD . I call *direct elements* of the surface, the straight lines of which it is composed parallel to the horizon.

"Let 2nd. $x = k$; then it becomes $y = \frac{hz \sin \kappa}{a \sin L} f$; hence

* It will hereafter appear that a class of *parabolic sections* can be made in this surface. The inference of M. Tinscau is too hastily drawn. An error, indeed, has crept into M. Tinscau's investigation, which will be pointed out presently. I have given it as it stood in his *Mémoire*, in preference to a more correct form, to shew the real amount of his error.—T. S. D.

the sections parallel to CDD , or to the two directrices, are also straight lines: then the surface of a *quadrilatère gauche* is composed, not only of horizontal straight lines but again of other straight lines parallel to the plane which is parallel to the two directrices. I call *transverse elements* of the surface, the lines of which it is composed parallel to the plane CDD .

"All the transverse elements being situated in parallel planes they cut the direct elements into parts proportional to the sides AC , BD : and for the same reason the direct elements AC , BF , BD , &c. cut the transverse elements AB , NP , CD , into parts proportional to the two sides AC , BD . Then in the *quadrilatère gauche*, $ABDC$, we obtain the same surface if we divide the opposite sides AB , CD into parts proportional to these sides, as if we divide the two other sides AC , BD into parts proportional to themselves, and draw straight lines through the points of division. We may then regard indifferently as directrices either the sides AB , CD or the sides AC , BD . We may also deduce from it this

Theorem 4. Any direct element of the quadrilatère gauche and a transverse element cut one another into parts proportional to the corresponding parts of the directrices respectively.

Thus $NP : \varphi P :: BF : AF$; and

$$FQ : \varphi f :: AP : PC."$$

He then deduces from it the mechanical principle of the uniform motion of the centre of gravity of two bodies in uniform motion—the principle employed in the solution of the problem which gave rise to these notes, the 504th of the Repository. Its details are unnecessary here, and we pass on to a more interesting speculation.

"We have already seen that all the direct elements of a *quadrilatère gauche* are parallel to one plane, and that the transverse ones are parallel to a second plane. These planes have a certain inclination to one another, which I proceed to determine. Put $AC = a$, $AB = DD = b$, $BAb = D\delta d = p$, $Dcd = q$, $Cab = m$, $ACd = n$; then we shall have $Bb = Dd = b \sin p$,

$$Ab = D\delta = b \cos p, CD = \frac{b \sin p}{\sin q}, cd = \frac{b \sin p}{\tan q}.$$

The point δ is the summit of a triangular pyramid whose faces are $D\delta d = p$, $d\delta c$ which it is easy to calculate, and $D\delta c$ which makes with $d\delta c$ the angle sought. For calculating $d\delta c$, I prolong $d\delta$ to i , and in the triangle $c\delta i$, I have the angle $i = m$, $c = n$, and the third angle, therefore, $= 180 - m - n$: then $\sin c\delta d = \sin(m + n)$,

$$\text{and } \cos c\delta d = -\cos(m + n). \text{ Moreover } cd = \frac{b \sin p}{\tan q} \text{ and}$$

$\delta d = b \cos p$. But in the triangle $d\delta c$, we have

$$\tan d\delta c = \frac{cd \cdot \sin c\delta d}{\delta d - cd \cdot \cos c\delta d}; \text{ hence}$$

$$\begin{aligned} \tan d\delta c &= \frac{\frac{b \sin p}{\tan q} \cdot \sin(m+n)}{b \cos p + \frac{b \sin p}{\tan q} \cos(m+n)} = \\ &= \frac{b \sin p \sin(m+n)}{b \cos p \tan q + b \sin p \cos(m+n)} = \frac{\sin(m+n)}{\cot p \tan q + \cos(m+n)}. \end{aligned}$$

We know, then, in the pyramid two faces and the right angle which they contain, or in the corresponding spherical right-angled triangle we know the two sides, $AB = p$ and $AC = d\delta c$ and require one of the angles c .

We have, then,

$$\tan c = \frac{1}{\sin AC \cot AB} = \frac{\tan AB}{\sin AC}; \text{ then, } \tan c = \frac{\tan p}{\sin d\delta c};$$

$$\text{but } \sin = \frac{\tan}{\sqrt{1 + \tan^2}}; \text{ and hence}$$

$$\begin{aligned} \sin d\delta c &= \frac{\sin(m+n)}{\cot p \tan q + \cos(m+n)} \div \sqrt{1 + \frac{\sin^2(m+n)}{\{\cot p \tan q + \cos(m+n)\}^2}} \\ &= \frac{\sin(m+n)}{\sqrt{\{(\cot p \tan q + \cos(m+n))^2 + \sin^2(m+n)\}}}. \text{ Hence} \\ \tan c &= \frac{\tan p \sqrt{\{(\cot p \tan q + \cos(m+n))^2 + \sin^2(m+n)\}}}{\sin(m+n)} \end{aligned}$$

or since $\tan \times \cot = 1$, we have, finally,

$$\tan c = \frac{\sqrt{\tan^2 p + 2 \tan p \tan q \cos(m+n) + \tan^2 q}}{\sin(m+n)}.$$

“ To find when the two *directing planes** of a *quadrilatère gauche* are perpendicular to one another we have only to suppose $\tan c = \infty$, which gives $\sin(m+n) = 0$, and consequently $m+n = 0^\circ$ or 180° , that is when the projections of the two opposite sides upon the plane parallel to the two other sides are parallel. The two planes, again, are perpendicular when one of the angles p and q or both of them, is a right angle, as is

* This appropriate term *plans directrices*, has neither been defined, nor used by any previous writer. It is constantly employed by the modern French geometers.

$y = M\pi - P\pi = y' - \frac{x'}{\cos m}$, and consequently

$$\frac{y' \cos m - x'}{\cos m} = \frac{x'z}{g \sin m \tan p},$$

which is the rectangular equation of the surface.

"*dgQ* being always the horizontal plane, if through the axe *RGF* prolonged and *GQ* a plane be made to pass it will cut the surfaces along an edge *Fa'π'*, perpendicular to the axe and parallel to *GQ*. Through this edge draw a horizontal plane, which meets in the points *b', m'*, &c. the ordinates *Bb, NM*, &c. and through these points *b', m'*, &c. draw the straight lines *b'a', m'π'*, &c. parallel to *Mπ*; denote the ordinates *Fπ', π'm'*, *m'N* by *x, y, z* respectively. Let the edge *AB* meet *Fπ'* in *a'*, and put *fa' = ca = e*; *a* being the point where *AB* meets *Fπ'*.

Then we shall have $y = \frac{xz}{e \tan p}$. This equation of the surface

is, except that of the plane, the most simple possible.

"We may easily find the distance *FG* ($=f$) of this new horizontal plane. For, let $x = fa' = Ga = e$, $y = a'A' = aA = g \cos m$, and z will be $=f$. Then

$$g \cos m = \frac{ef}{e \tan p}; \text{ or } f = g \cos m \tan p = \frac{a \cos m \tan p \tan q}{\tan p - \tan q},$$

by giving g its value found in above.

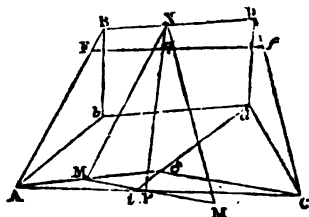
"We can readily discover the value of the angle *BAC*, and hence that which the plane *BAC* makes with the horizontal *CAB*: and thus we may find the equation of the *quadrilatère gauche* referred to the plane drawn by two of its contiguous sides. But it is easy to find it directly: though on account of its greater complexity compared with that already given we shall omit it here, and more especially as it does not in any degree furnish us with new views or properties of the surface.

"We may desire to know the *minimum* of all the elements *AB, NP, CD*, &c. of the twisted quadrilateral. In the triangle *NMP* the angle *NMP* is constant; let it be called *M*. Let also $NM = AB = b$,

$$MP \text{ or } y = \frac{bx \sin K}{a \sin L};$$

$$\text{but } NP = \sqrt{NM^2 + MP^2 - 2NM \cdot MP \cdot \cos M}$$

$$= \sqrt{b^2 + \frac{b^2 x^2 \sin^2 K}{a^2 \sin^2 L} + \frac{2b^2 x \sin K \cos M}{a \sin L}}$$



which must be a *minimum*. Hence $2ax \sin K - 2a \sin L \times \cos M dx = 0$, and therefore $x = \frac{a \sin L \cos M}{\sin K}$. But $\cos M =$

$-\cos(K + L) = \sin L \sin K - \cos L \cos K$; and hence

$$x = a \sin L \{ \sin L - \cos L \cot K \}.$$

"This determines the *position* of the least element. To find its *magnitude* we may proceed thus. By substituting for x its value just found, we have,

$$\begin{aligned} NP &= b \sqrt{1 + \frac{\sin^2 K \times a^2 \sin^2 L \cos^2 M}{a^2 \sin^2 L \sin^2 K} - \frac{2 \sin K \cos M \times a \sin L \cos M}{a \sin L \sin K}} \\ &= b \sqrt{1 + \cos^2 M - 2 \cos^2 M} = b \sin M. \end{aligned}$$

The equation $NP = b \sin M$ gives $NP : NM :: \sin M : 1$; that is to say, the shortest rectilinear element of the *quadrilatère gauche* is that which makes with its oblique projection a right angle."*

M. l'insertu then goes on to give another solution of the problem of finding the shortest distance between two lines, adapted to the practice of cutting stones, &c. but being foreign to our present purpose it must, elegant as it is, be omitted here.

"*Theor. 5. We can exactly determine the volume contained by portion of the surface of a quadrilatère gauche ABCD, by planes drawn in any manner through its four sides, and by any fifth plane whatever.*

"For forming the conoïde $ABDdCA$, it is $= ddc \times \frac{h}{2}$ (by *Thor. 2.*); but whatever be the inclination of the planes drawn by the four sides, they only increase or diminish the conoïde by solids whose faces are plane, and therefore whose volumes are rigorously cubable. Hence, &c.

"*Theor. 6. We can exactly determine the position of the centre of gravity of the same figure as above, viz. *Theor. 5.* and its demonstration is of the same kind.*"

It is worthy of remark that though one of the lines of the second order is quadrable,†—that though arcs of two of them can be

* It is easily deducible from this, that if another line NM' be drawn from N to meet the plane in M' , and $M'P$ be joined; then M, P, M' are in one straight line. *Euc. 14. I.*

† Archimedes first squared the parabola. *Arch. Op. Oxon.* pp. 16–34.

assigned whose difference reckoned from given points is rectifiable*—that though one surface of the second order is capable of *cylindrical* or *conical* perforation, so as to leave quadrable and cubable remainders:†—yet no one seems to have noticed that *cubable portions can be cut by planes from a surface of the second order*, except MM. Tinseau and Mauduit in the cases before us. Nor is it less remarkable that no subsequent writer has taken the slightest notice either of this property or of the *memoires* which contain it.

M. Tinseau proceeds:—"I have expanded my enquiries upon this last species of *surfaces gauches*, both because it is the most simple, and because it enters as an element into the composition of all *surfaces gauches*, which are nothing else than system of faces of *trapèzes gauches*, of which one dimension is finite, and the other infinitely small."

This is better expressed by later writers, whom we shall presently notice—"That any surface *gauche*, whatever be its directrices, may be touched through the whole extent of any one of its edges, by a *surface gauche* of the second degree."

M. Tinseau then in the close of his paper examines the figure formed by taking, instead of the two linear directrices we have been considering, two curves of any description and in any position. These he calls *directrices*; and the surface generated by the line which constantly rests upon them and is constantly parallel to the same plane, he calls a *paralleleloid*. In fig. 12, $\triangle CQR$ is the horizontal plane DEQ , BER the directrices. If these be parallel, or conceived to be the result of cutting the *paralleleloid*, already formed by means of parallel planes, then he names them the *bases* of the solid *gauche*, and the *solid gauche* itself is the name appropriated to the solid included by the two bases, the surface, the horizontal plane, and a plane $ABCD$ passing through any one of the edges or rectilineal elements of the surface.

Fig. 12.

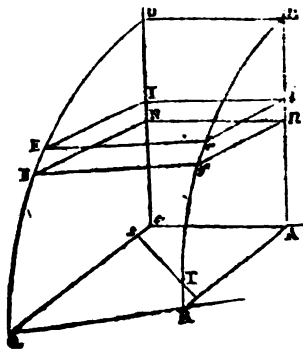
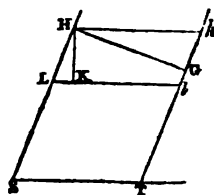


Fig. 13.



* Mathematical Repos. Vol. I. Part II. p. 9.

† MM Viviani, Bossut, &c. and Messrs. Woodhouse and Ivory, have perhaps completed this enquiry.

"*Theor. 7.* The solid *gauche*, $\Lambda B E R Q C D E Q$ is equal to the product of its perpendicular height by half the sum of the two bases.

"For imagine an infinity of horizontal planes infinitely close to one another, which cut the solid along a series of trapezia; let $E I i e$, $F N n f$ be two of these consecutive trapezia; the stratum of the solid which they include is equal to one of these trapezia multiplied by the thickness dx ; but ST being the perpendicular distance of the straight lines ΛR , CQ , the trapezium

$F N n f = ST \times \frac{FN + fn}{2}$, and the elementary stratum is, there-

fore, $= ST \times dx \times \frac{FN + fn}{2}$. Through ST (figs. 12 & 13.)

imagine a plane perpendicular to the two bases, which cuts them in the parallels SH , Th , and the two trapezia in the parallels Hh , Ll ; and finally, draw HK perpendicular to Hh , and HG to SH . Then HK will be $= dx$, and $HG = h$, the perpendicular height of the solid. The stratum will be, then, $ST \times$

$KH \times \frac{FN + fn}{2}$. But $HK : HL :: HG : Hh$. Then $HK \times ST$

$= HL \times GH$; and the stratum is $=$

$\frac{FN + fn}{2} \times HL \times HG = \frac{F E I N + f e i n}{2} \times h$. Hence the

solid $= \frac{h}{2} \int (F E I N + f e i n) = \frac{h}{2} (C D E Q + \Lambda B E R)$; or calling

s the solid, and B, b , the bases, we have finally

$$s = \frac{h}{2} (B + b)."$$

"*Theor. 8.* In the paralleloïde, as in the conoïde, the area of any section parallel to the bases depend solely upon the area of these bases, and its distance from them.

For, let b' be such a section, and x its distance from B and calling s' , s'' the portions of the solid included between b'

and each of the sections B, b : then $s' = \frac{B + b}{2} \cdot x$, $s'' =$

$\frac{b + b'}{2} (h - x)$; hence $(B + b) \frac{h}{2} = \frac{B + b'}{2} \cdot x + \frac{b + b'}{2} (h - x)$,

and hence $b' = \frac{B(h - x) + bx}{2}$. Hence, &c.

" We shall now seek the centre of gravity of the solid, and first its distance from the base B

" But the stratum (by the last) which has h' for base is $\frac{B(h-x) + bx}{h} \times dx$, then $\frac{B(h-x) + bx}{h} \times x dx$ is its moment with respect to the plane B; and hence, the distance of the centre of gravity of the solid from the plane of the base B is

$$\frac{\frac{1}{h} \int (Bhx dx - Bx^2 dx + bx^2 dx)}{\frac{B + b'}{2}} = \frac{\frac{Bhx^2}{2} - \frac{Bx^3}{3} + \frac{bx^3}{3}}{h \times \frac{B + b}{2} \times x}$$

(the constant being null if we integrate from the base B). But this expression, making $x = h$, an $b' = b$ becomes,

$$\frac{\frac{Bh^3}{2} - \frac{Bh^3}{3} + \frac{bh^3}{3}}{h^2 \frac{B + b}{2}} = \frac{Bh + 2bh}{3(B + b)} = \frac{h}{3} \cdot \frac{B + 2b}{B + b}.$$

With respect to the distance of the centres of gravity from two other planes, they depend upon the particular equations of the two bases, and cannot be expressed by any general formula."

To find the equation of the paralleloïde.

" Let x, y, z , be the co-ordinate, x and y being horizontal. Whatever be the equation which expresses the mutual dependence of the co-ordinates, it is of the form $\phi(x, y, z) = 0$. Then, if we cut the surface of the paralleloïde by a horizontal plane, the section will be a straight line or a system of straight lines: but we know that, if in the equation of a surface, we make z a given constant, h , the equation between x and y which results is that of the horizontal section made in the surface at the distance h : hence since, in the present case, this last equation is that of a straight line, we must have

$$\phi(x, y, h) = Ax + By + C; \text{ and}$$

for another section at the distance h' we have

$$\phi(x, y, h') = A'x + B'y + C'.$$

From these we see that the coefficients A, B, C vary with the height h : and since h is in general represented by z , it is obvious that the coefficients are functions of z . Let ϕ', ϕ'', ϕ''' represent these functional characteristics: then we have for the equation of the paralleloïde referred to the horizontal plane

$$\phi(x, y, z) = x\phi'z + y\phi''z + \phi'''z = 0."$$

I should remark that in some cases, even where the sentences are marked as translated quotations, they are only paraphrases, and generally abbreviated paraphrases.

The next section will embrace the labours of Euler, Monge, and Mensnier: and the Third will contain those of Hachette and the *Eleves de l'Ecole Polytechnique*.

ARTICLE III.

ON THE STEREOGRAPHIC PROJECTION OF THE SPHERE.

(In a Letter from Mr. DAVIES to the Editor.)

MY DEAR SIR,

Bath, Aug. 10, 1830.

The importance so justly attached by Delambre to the history of the Stereographic Projection of the Sphere, induces me to add a few notes to my letter in your last number. They chiefly relate to omissions which arose from the haste with which the "*Historical Introduction*" of that letter was written.

I was desirous of doing justice to the memory of the almost forgotten Jordanus Nemorarius—perhaps the most original genius that appeared in the early part of the Thirteenth Century: but this is neither the time nor the place for such discussions. Perhaps it may on some future occasion form a separate paper for another "PART" of the Repository, if you think it likely to fall in with your established plan.

Believe me, my dear Sir,

Your faithful friend and obedient servant,

THO^s. ST^{ns}. DAVIES,

To PROFESSOR LEYBOURN, &c.

Royal Military College.

Notes on the Stereographic Projection.

I have expressed a belief that the method of proving the first property which was discovered by Jordanus Nemorarius must have led immediately to the discovery of the other two: * but

* Math. Repos. Vol. V. Part II. p. 147.

this does not seem to have been the course which the investigation actually took. The collection of Treatises on the Sphere,* published by Zieglerus, which contained the works on the planisphere by Ptolemy and Jordanus, made its appearance in 1544 (though the title page is dated MDXXXVI.); and these two tracts were again printed by Aldus, in 1558, the former having notes from the pen of Commandine, and the text of the latter considerably amended, though still left destitute of comment. Delambre, indeed, mentions an edition printed in 1507, but he had not seen it, nor have I after considerable search been able to find the slightest further evidence of the existence of such an edition. Did it, indeed, exist, we might fairly calculate upon finding it noticed in Panzer's *Annales Typographiques*: but neither there, nor in Murhardt's *Bibliotheca Mathematica*, nor in any of the Bibliothecal lists of our public or private libraries, nor yet in any well-arranged sale catalogues, that it has been my luck to consult, have I been able to find a trace of it. However, as it was then very much the custom to print several works in one volume, this edition of Jordanus might, perchance, be shrouded under the wing of some more fortunate author, whose work standing first in the volume gives a catalogue-title to the whole of its contents. Yet the publications of that year put down in Panzer do not, from their titles, seem likely to find an appropriate supplement in the *Planisphærium Jordani*. Another circumstance, too, seems to support the same view, viz that Stoflerus whose commentary on the Sphere of Proclus Diodachus was printed in 1534, ten years before the publication of Ziegler's Jordanus,—an author minutely acquainted with the state of science in his time, of great mental activity and unwearied even to the last in his pursuits—though he mentions in the course of his works no less than a hundred and seventy-six authors upon the sphere and its collateral studies,—yet he does not in any way allude to the existence of a work by Jordanus on the subject. Upon the whole, then, the impression on my mind is, that Delambre must have been misinformed respecting this edition—that no such edition does, or ever did, exist. I should, indeed, be glad to find myself mistaken, as perhaps such a one, if it exist, might furnish some little information respecting both the author and his works—information which in clearing up a difficulty in the early history of figurate arithmetic I have some time sought in vain.

* Printed at *Basle*, not at *Thoulousc*, as inadvertently stated at p. 144, *ub. sup.*

The ingenious method of projection invented by Jordanus* was not unnoticed by the writers of the seventeenth century: as, for instance, of Maurolycus,† the inventor of the secants, and the indefatigable Jesuit, Clavius Bambergensis.‡ They

* I say, "invented;" though Delambre speaking on this subject chuses to say, "On ne sait d'après quel auteur Jordanus l'a composé. Savait-il l'arabe? avait-il lu la traduction de Maslem? avait-il trouvé dans quelques auteurs inconnus une idée vague de cette projection? a-t-il trouvé de lui même une partie des constructions qu'il donne? C'est ce qu'il est difficile de conjecturer. Cela ne serait pas indifférent pour le gloire de l'auteur, qui pourtant pouvait connaître l'ouvrage de Proclus ou celui d'Ammonius. Au reste, il n'est pas le seul qui, vers cette époque, se soit occupé du planisphere que l'Arabes avaient répandu en Espagne." Hist. de l'Astr. du Moyen Age, p. 442. I hope shortly to exhibit ample proof of his adequacy to the construction of the planisphere.

† *Opuscula Mathematica*, p. 62, in the article "*De Instrumentis*," 4to. Venice, 1576.

‡ *Clavii Astrolabium*, pp. 284-6, Romæ, 1593. In this work, however, though the method of demonstration given by Jordanus is exhibited, the projection is on the plane of the equator as in that of Ptolemy. Delambre must have spoken from memory (which in this instance led him wrong) when he said that the only peculiarity of the method Jordanus was the plane of projection, which he employed. His peculiarity is that the demonstration altogether dispenses with the sub-contrary sections of the cone; but it may be applied to cases where the plane of projection is parallel to the equator as easily as where the equator itself is employed. Jordanus chose the polar tangent plane, and this, after all, is the simplest and most obvious method that has been contrived.

It has been usual to represent the work of Clavius as a cumbrous, inelegant, treatise, abounding with unnecessary refinements and distinctions which defy the resolution of the most ardent Geometer to read it. Perhaps to that class of readers who made the astrolabe and other branches of Descriptive Astronomy (Dialling especially) their chief study, the work might wear such an aspect: but to every man of good taste in the *Antient Geometry*, few works can offer more interest than this is calculated to afford. Many elegant propositions which I had supposed were the growth of later times, sprung up, I find, amongst the "heavy writers of the 16th century!"

In a work which teems with almost every kind of curious learning—scientific excepted—Dr. Clarke's "*Commentary on the Holy Bible*"—we find Clavius and his writings noticed in a rather unusual manner. "Since that time (that of the Roman dials) the science of dialling has been cultivated in most civilized nations; but we have no professed treatise on the subject before the time of the Jesuit Clavius, who in the latter part of the 16th century demonstrated both the theory and practice of dialling: but he did this after the most rigid mathematical principles, so as to render that which was before simple in itself, exceedingly obscure." It must not be deemed hypercriticism to remark that—whatever sense the word "*demonstrate*" may have in Anatomy (to shew or exhibit) and though this may be defensible etymologically; yet in every thing relating to mathematics it signifies to prove by

both give the demonstration with much commendation, and yet neither of them, nor yet Commandine, deduces from it the consequences which I had thought so obvious on contemplating the figure. There is, indeed, a great difference between deriving

means of the syllogism, and the word "prove" is held synonymous with "demonstrate." Substitute the word "prove," then, and try how the sentence reads. It was not however, for this that I quoted the passage, but to call attention to graver errors. He tells us, that "rigid mathematical principles" have "rendered that obscure which before was very simple." To the uninitiated, every thing is obscure—but is the ignorance of the reader to be a cause of censure to the mathematician who writes a profound geometrical treatise that transcends the capacity of the said ignorant reader? I trow otherwise. On this principle Dr Clarke has vitiated the sacred writings by his learned investigation of their signification—seeing that not one in a hundred of those who open his Commentary are sufficiently skilled in the various tongues from which he has drawn his illustrations, to understand him. I trust, however, that Geometers will still feel content to bear the reproach of "obscurity," rather than attempt the "simplicity" which has of late become so much the fashion in elementary philosophical works, and I regret to say in mathematical, too—a simplicity which by leaving out the difficult and mutilating the easy, invariably fails to cultivate any faculty save the memory, and scarcely that. The tendency of elementary books is now such as should put every one who feels an interest in the permanency of science upon his guard against their influence. One of the most fatal sources of "the decline of science in England," will prove, I am convinced, to be the relaxing of this austerity of true scientific rigour, and the attempts to dress up knowledge in the meretricious ornaments of the novelist or the juggler. In short—to convert science into a toy for grown children, and thereby relieve them from every tax upon the exercise of their mental faculties! Science may then become, as it is now becoming, fashion; but the next step is decay,—and the third, annihilation.

"Though we have," the Doctor continues, "useful and correct works of this kind from Rivard, de Parcieux, Dom. Bédos de Celles, Joseph Blaise Garnier, Gravesande, Emerson, Martin and Leadbetter, yet something more specific, more simple, and more general is a desideratum in the science of scialerics or dialling." Comment. II. Kings, Cap. 21. Without any great difficulty Dr. Clarke might have squared the number of names, by consulting a few Encyclopædias, instead of transcribing the confined list given in Hutton's Montucla, Vol. III. p. 342, 3.): for to any one who had studied a single work of these above mentioned, especially those of Gravesande or Emerson, such an assertion as that which he has used in the conclusion would have been too obviously false to have crept, even in the most incautious moments, into the passage. If those three qualities are not to be found in works on Dialling, I believe we shall find no subject that is not destitute of them altogether. All this depreciation of existing treatises, however, is to pave the way for the "*more specific, more general, and more simple*" method of drawing the hour lines upon the *Jerick Dials*, which the Rev. Philip Garrett proposes to give a little further on. This Rev. Gentleman, indeed, may plead guiltless of the sin

from a suitable figure the demonstration of a property already known, and discovering therein the property originally.

I believe these properties are not to be found in any continental work earlier than 1751, in Savérien's *Dictionnaire des Mathé-*

of employing "rigid mathematical principles" in his little treatise; but I fear it is scarcely, amidst all its simplicity, and its other virtues, less "obscure" than the methods of Clavius and his predecessor in the enquiry, Commandine. Whatever "*indubitable proofs*" the learned Doctor had had of his friend's "*rare knowledge in the science of gnomonics, and ingenuity in constructing every possible variety of dials*,"—certain it is, from the very phraseology employed in his treatise, that he is *totally ignorant* even of the simplest geometrical definitions, to say nothing of the theorems upon which an investigation of general methods must necessarily rest; and that the dial which he has foisted upon the Doctor is one of the most unblushing impostures that ever was practiced upon the liberal confidence of a scholar! I know not the author,—who, or what he is; but I speak plainly, for I feel ashamed that a work which does no common honour to the literary character of England should be degraded by such absurdities as the Rev. Philip Garrett's construction of the Jewish Dial!

The Doctor supposed there was a want of *generality* in the method of constructing dials, because the Jewish Dial did not submit to the same rules as the common dials for equal hours. A little real study of the question might have convinced him that they *were essentially different problems—as different as describing the sinoid and the circle*. The true nature, however, of the hour lines on these dials had been altogether misunderstood till about 1817, when Mr. Cadell read before the Royal Society of Edinburgh a short paper containing a graphic construction on a polar tangent-plane of the "lines which divide the semidiurnal arcs into six equal parts." See Ed. Trans. vol. VIII. p. 61—81. As however between the tropics the antique hour-lines differ but little from straight lines, these have generally been substituted for them; and the problem under this aspect has been long, and often solved. The most antient of these dials that is certainly, from authentic records, known to have existed, is, that upon the Tower of Andronicus Cyrrhestes at Athens, erected, there is reason to think, between four and five hundred years anterior to our era. Another brought home from Greece by Lord Elgin is believed to be that from the Pnyx, by which Demosthenes and other illustrious orators harangued the Athenian people. A third from the island of Delos, is minutely described by Delambre, and supposed by him to be of more modern fabrication; and two others exist in private collections in England. The first attempt to describe geometrically their construction that I have met with, is that of Commandine in his Commentary on the *Analemma* of Ptolemy, printed at Rome in 1602, by Paulus Manutius. The method he gives is sufficiently simple—sufficiently so, I should suppose, it would have been for Dr. Clarke, but that he had a better to bring forward—and it has been adopted into almost every general treatise on dialling during the seventeenth century. In Leybourn's *Dialling*, Editions of 1682 and 1700, p. 124-7; Good's *Art. of Shadows*, 1700, —, and 1714, &c. &c. Dr. Long,

matiques, tom. II. p. 330. This, however, I speak *only in part* from my own enquiries : and I rely more upon the fact—that Delambre whose mind ever was alive to this specific enquiry, could scarcely have missed it in his regular analysis of all the works which he conceived had contributed, even remotely, to the progress of astronomy, if it were to be found in any of them. From the greater facilities which he had in investigating the continental works than I can possibly hope to obtain, I should at once consider his opinion decisive without waiting a more minute investigation than he has given : whilst the relative difficulty he must have had in his search into English books must account for his not having discovered the property where it, unquestionably, first appeared.

Savérien to his enumeration of properties, one of which was quoted from the *Encyclopædié Méthodique* in my former letter (*Rep. ub. sup.* p. 147.) annexed the following remark :—“ Clavius est peut-être le premier qui ait enseigné la projection de la sphere. C'est ce qu'on peut conclure de la maniere abstraite dont il donne cette projection dans son *Traité De Astrolabio*. Tacquet en a parlé avec plus de clarté dans son *Optique*, p. 178, et suiv. (Tacq. Op tom. I.) & *Vitty dans un Ouvrage particulier écrit en Anglois*. On peut consulter aussi les *Elementa Matheseos Universa*, de M. Wolf, tom. IV.”

Is it not singular that Delambre, who, in 1787*, had read this passage and quoted the preceding one of the same article from Savérien—who must have had opportunities of consulting both Tacquet and Wolfius, though in vain—should have been so indif-

it is clear, mistook the character of the problem, *Astr. Vol. II* p. 509.

The opinion that such dials were constructed so as to shew the hour by a flight of steps does not appear to be quite so original as Dr. Clarke considers it, nor, though sanctioned by all the authority which his rendering of the Chaldaic *Maaloth* can give it, do I think it well established. Granting,—and who can doubt it?—that “ the shadow went back ten steps,” is the literal meaning; does not *the sun's daily path in the ecliptic in all the ancient languages bear literally, the same name?* The late Bishop Stock, in his translation of Isaiah, thinks this dial was similar to the *Gentar Murtur*, or Observatory of Delhi, in consequence of which he also proposed the notion of steps, prior to Dr. Clarke's having done so. However, waiving this, I shall in the second part of a paper, (the first of which is now in the possession of the Royal Society of Edinburgh) on the antique hour-lines, examine the conditions necessary for their construction : and, therefore, shall here close a note already, perhaps, longer than the subject deserves.

* See Mem. de l'Inst. tome Vème p. 394.

ferent about the English author whom Savérien calls "*Vitty*"? Though some apology might occasionally be offered for a Frenchman who substitutes V for W, the latter letter not being a component of his language, yet is there any apology for Savérien who by mis-writing the name of *Witty* mis-directs the future enquirer, when in the same sentence he puts *WOLFUS* with its proper initial?—And, when, indeed, as many parts of his Dictionary, (much more than he has the candor to acknowledge) shew that he was tolerably well acquainted with our writers? It seems, indeed, to be a system generally and tacitly practised by the French to so distort the names of foreigners (English especially) that it is almost impossible to recognize our own countrymen in their Gallic costume. I know there are many who attribute the practice to improper motives, but I should think it is, in part at least, attributable to that negligence of good etymological habits which ought to be cultivated not only by mere men of letters but by men of science also.

Though Delambre did not discover whom *Vitty* meant, and who *Witty* really was, he at last discovered the properties in a very common and very ill-digested book, viz. in Leadbetter's *Astronomy*.^{*} "Le seule chose," says he, "qui m'ait paru digne d'être extraité de ces deux volumes est le démonstration du second théorème général de la projection stéréographique, c'est-à-dire de l'égalité des angles sur la sphère et sur la projection. Cette démonstration est le plus ancienne que je connaisse : mais elle ne parait ni le plus claire ni le plus complète. Essayons de l'éclairer et de la compléter."

Before particularly attending to this demonstration it may be as well to go back with our history to its real origin. The property was first published by Dr. Halley in Phil. Trans. No. 202, in "An Easie Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line or Sum of the Secants." In 1706 Jones's *Synopsis Palmariorum* was published, where, at page 278, we find the passage, "the angles in this projection (stereographic) are equal to the corresponding ones on the sphere." Again, in 1708, the second volume of Derham's *Miscellanea Curiosa* was published, under the immediate influence of Halley, and the same paper was reprinted in it, p. 20-36. At the end of an Appendix to the same volume is a paper of ten pages by Hodgson, on the "Laws of Stereographic Projection," in which the property is again brought for-

^{*} Leadbetter's *Astr.* 1728, Vol. I. p. 61. See also *Hist. de l'Astr.* du 18^{ème} siècle, p. 87.

ward. I cannot at this moment refer to his (Hodgson's) "Complete Course of Mathematics," 2 vols. 8vo. 1710; but I think I recollect the property is also put down there. Again in the Rev. John Witty's "Treatise on the Sphere," a posthumous work, the printing of which was conducted by Hodgson, in 1714, we find a demonstration of the same property, pp. 56, 7. Other works, it is very possible, which I do not possess, published between the times when Halley and Leadbetter wrote, may also contain these properties; but enough has been done to shew that Leadbetter is not the author of them. A second edition of Witty's work was published by his friend Hodgson in 1734, but though several changes were made in the work by the editor, this proposition was left unaltered: and the edition of Leadbetter which Delambre uses bears the date of 1735. We are thus able to answer the enquiry of Delambre—"Leadbetter en est-il l'auteur?" and at the same time to shew the inaccuracy of his ingenious surmise, at least so far as regards *that* writer; though it might perhaps be more accurate when applied to *Dr. Hooke the real author of the proposition*—"il paraît plus probable qu'on aura remarqué ce théorème en construisant une mappemonde, soit sur le plan d'un méridien soit celui sur d'un horizon quelconque." (Ast. 18 siècle, p. 90.) He had also consulted Robertson's Demonstration in the first edition of the Navigation—in describing which he has, however committed an oversight. He states that Robertson employs the *subcontrary* section of the cone for effecting the demonstration. For proving the first property it is true he does employ the cone, but not for the second and third. That process, so far as I know, is altogether original.

He lastly refers to the demonstration in Emerson's well-known work on the Projections of the Sphere, (Prop. IV. sec. II.) which he admits to be "réellement complète et rigoureuse," though he objects to it on account of the "figure compliquée que jette quelque obscurité sur la démonstration." He states, very truly, that the demonstration is but a "completed reproduction of the demonstration of Leadbetter," and that it is complex when compared with that given by himself in the third volume of his *Traité de l'Astronomie théorique et pratique*, p. 681. Perhaps, too, I should remark that the substance of Delambre's own demonstration in the *Memoires of the Institute* (which is one of those in his *Astronomie*) may be seen very succinctly developed in Dr. Gregory's *Trigonometry*, p. 125—134; the properties on p. 127, were little known to the

English reader till they were introduced into that elegant little treatise.

Dr. Halley in his paper on the "Analogy" above referred to thus states the property:—"In the stereographic projections the angles under which the circles intersect one another are in all cases equal to the angles they represent: which is perhaps as valuable a property of this projection as that of all the circles on the sphere thereon appearing circles: *But this not being vulgarly known** must not be assumed without a demonstration." Misc. Curios. vol. II. p. 22-23.

After this demonstration he adds the following scholium: "This Lemma I had from Mr. A. de Moivre, though I understand from Dr. Hooke that he long ago had produced the same thing before the [Royal] Society. However, the demonstration and the rest of the discourse is my own."†

Halley was too well read in the writers of his own country (an easier matter then, than now) to have been ignorant of this proposition had it existed in any of them; and this is another reason for believing that it first appeared in the place above mentioned—a reason which, had my own examination of our early authors been even more limited than it has, would have been satisfactory to my own mind on the question. Again, the claims of Hooke, independently of the integrity of his character giving irresistible weight to his testimony, are established by another consideration, viz. that thirteen years elapsed between the time when the statement was made in the Transactions and the time of its repetition in Derham's *Miscellanea Curiosa*, and during the whole period the warmest friendship subsisted between Halley and Demoivre. Had the latter felt himself aggrieved by the statement of Hooke's claim in that scholium, he would have remonstrated, and have obtained justice from the candid pen of Halley. No traces of such a circumstance existing we are entitled to believe that Demoivre acceded to the truth of the statement.

The claims of Hooke to profound mathematical knowledge cannot be asserted; yet he was far from inexpert in those cases which came before him requiring the exertion of geometrical skill. The process by which Delambre conjectured that Leadbetter arrived at the proposition, I have already stated is very

* In the Trans. it was "*not being commonly known, &c.*"

† Phil. Trans. for 1695-6; Lowthorp's Abt. Vol. I. p. 665; Hutton's Abt. Vol. IV. p. 70; and *Misc. Curios.* II. p. 23.

One of the immediate consequences deduced by Halley from this proposition was, that—"the Stereographic Projection of the Loxodrome or Spiral of Pappus upon the equator is the equiangular spiral." This is the first notice I have found of that elegant relation.

likely to be that by which Hooke discovered it. His was a mind of great activity; but from his general studies he was more likely to seize upon those properties which present themselves in the course of a series of graphic operations than upon those which result from a long train of logical deductions. To such a mind such a property was more likely to present itself than to that of Demoivre, unskilled as he was in the antient geometry, and unused to the contemplation of magnitude under any other aspect than as it was presented by an algebraic expression. We have, indeed, yet to learn by what kind of processes an analyst of that day, even the most expert, could arrive at such a conclusion. I know of none, nor can I imagine any. *This beautiful property then, it appears to me, belongs incontestably to Dr. Hooke.*

Delambre has objected* to all the solutions he has seen except Emerson's and his own, that they only prove one case of the proposition, viz. when one of the circles in the question is a meridian. The completion is simple enough it is true, and it might have been given without much increase of length in the demonstration. There is too much predilection for short processes amongst geometers, and this brevity is too often attained at the expence either of completeness in the argument or of generality in the method employed. As this trifling incompleteness is to be found in the demonstrations of Halley, Hodgson, Wuy, and Leadbetter it may not be amiss to supply it. Let α , β be

the angles made by the two circles of the sphere with a meridian through their point of intersection; then P , P are the angles made by their projections and the projection of the meridian. Hence on the sphere and on the plane, we have

$$\alpha = P \text{ and}$$

$$\beta = P,$$

therefore adding or subtracting as the case may require, we have

$$\alpha \pm \beta = P \pm P$$

which completes the proof.

There are, however, in several of these demonstrations other little defects which it is desirable to remove.

1. Dr. Halley states, but does not prove it, that "the angle APD † shall be equal to the spherical angle contained between

* Astr. 186mo. siecle, p. 89.

† As the books containing these demonstrations are readily accessible to every one it is unnecessary to insert the demonstrations or the figures; and, therefore, I shall speak as though the figures were before us.

the planes APC, DPC." To say nothing of the inaccuracy of calling the dihedral angle $APC \wedge DPC^*$ a spherical angle,—*how are we to know that DP is at right angles to PC?* One of the readiest methods of shewing this, is, perhaps, to prove that the plane PAD touches the sphere at P; and hence, that DP is a tangent to the sphere, and perpendicular to the radius through P. Other methods, something different, might be readily invented.

2. In Jones's *Syn. Palm.* p. 273, no demonstration is offered, nor the method intimated.

3. In Hodgson's the π and the line cn is omitted in figure 6. In the demonstration of the equality of the angles, the triangles dmp, fmd , is written instead of dmp, dpf . He also assumes the rectangularity of the angles pdf, mdf :—an objection of the same nature as that which lies against Halley's demonstration.

4. In Leadbetter's demonstration, *Astr.* l. 61. the line ki should not only have been drawn at right angles to DK , but in the plane of the tangents bK, bi . It is also necessary to prove that ikl is a right angle. The demonstration will then be complete for the case when the meridian is one of the circles projected. This amendment has been suggested both by Delambre and Matthieu, *Astr.* 18ème, siècle, p. 88 and 92.

It may be worth while to observe in reference to an objection taken by Delambre to the lines bc, ca , which he deems "*inutile*," that if we admit *his* method of proving the equality of the angles FBK, BFK (or fbk, bfk of Leadbetter's figure) the line is unnecessarily introduced: but if we follow the *Euclidean* method of reasoning which has always prevailed in this country in demonstrations concerning angles in a circle, those lines are essential.

Leadbetter has also given another demonstration which I have in vain made reiterated attempts to understand. Of his first, too, we may remark either that it was taken from Witty's and clumsily altered to conceal its origin, or that it was taken from Hodgson's and its defects as clumsily removed.

5. To the demonstration of Witty nothing can be objected, except that the meridian is taken as one of the circles. It is amongst the most elegant of the geometrical demonstrations that has ever been given, and is almost identical with Delambre and Matthieu's correction of Leadbetter.

6. The processes of Robertson are in general very brief and

* Meaning the angle contained by the planes APC, DPC. Most foreign writers use a smaller angular sign placed above the line; thus $APC^{\wedge}DPC$. Several reasons induce me to prefer placing it *between* the letters denoting the planes, rather than *over* them.

perspicuous: but viewed as an attempt to demonstrate an isolated property of a spherical projection, this must certainly put in no such claim. This single property, however, was not the sole object he had in view, and perhaps as a *tout ensemble*, the series of propositions he has given on projections are as rigorous and as elegant as could possibly be given in the same compass.* He also takes the *meridian* for one of the circles in question.

7. Of Emerson's I have already spoken after Delambre: and Bishop Horsley's method led him to omit all mention of these properties.†

8. In Bonnycastle's Trigonometry is given a demonstration, which cuts short the only difficulties of this easy proposition at once. See page 401, 3rd edition. Unfortunately, however, this demonstration is no proof. For though he proves correctly that $FC = FC$, yet he is wrong in stating that "*in like manner* it may also be shewn that $GC = GC$," for the circles CBE , CKD are not "related in like manner" to the pole and plane of projection. The inference is hence unwarranted.

In these notes I have not alluded to the analytical demonstrations that have appeared in a few of the continental works, for they are far less simple in the principles employed, and they do not afford an expression elegantly interpretable into the second and third properties. Of the trigonometrical method of Delambre I have already expressed unqualified approbation. I shall therefore close the present enquiry by expressing a wish • that Gentlemen who have been in the habit of making notes of the works they read would *occasionally* take the trouble to enter a little into the historic details of different branches of mathe-

* I ought to except from this statement the very elegant (like every thing which comes from *his* pen) little treatise on projection, given by Mr. Lowry in the Appendix to the Fifth Edition of Dalby's Course, Vol. I. The process by which he has demonstrated the properties in question, is exceedingly similar to that of Jordanus, given in the last number of the Repository. This paper was "set up" before I saw that demonstration, or I should have noticed it more particularly.

† In a demonstration of a question which I re-proposed after Mr. Whitley, in the Gentleman's Diary, for 1828-9, (viz. Quest 1232) I attributed to Carnot the beautiful property that—"the solids under the sines of the alternate segments of a spherical triangle made by great circles passing through the angles and any point, are equivalent." I have since noticed the same property in Horsley's Mathematics, p. 224, who quotes Menelaus as the earliest author in whom he has found it. In Halley's Edition of Menelaus, Oxon, it is at p. 80. There is not, however, the slightest reason to think that Carnot was aware that he had been anticipated, much less that he had obtained it from any author whatever. Properly speaking, the two theorems put down by the Bishop, from Menelaus and Ptolemy, are but varieties of the same general property.

mathematical science, for the benefit of those who may take an interest in them. I am convinced, that much chagrin is suffered by men of enquiring minds from finding themselves anticipated in those discoveries upon which perhaps they have laboured long and hard, which might have been avoided had some such system of mutual communication been adopted as that I suggest, and of which I have here, as well as in my "History of Rule Surfaces," set something like an example. In the course of my reading I have often had occasion to remark the discovery and repeated re-discovery of the same propositions, and often in forms very similar, yet where there was not the least probability of plagiarism: and I have always regretted the disappointment which must be felt by the later enquirers, as well as the loss of time which might, with equal ease, have been devoted to the extension of the real boundaries of mathematical science. How many talented men in sickness of heart have sat down in desponding hopelessness with "there's nothing new under the sun—there's nothing left for me to do!" I will only add, that as a relaxation from severer labours, and as an inducement to some kind of regularity even under weariness of spirit, I have found it to be as gratifying at the time as it has been useful in the end. I consider it time saved from the wreck of our brief lives—and why, then, should not others view it so?

T. S. DAVIES.

Bath, 1830.

ARTICLE V.

HORÆ ARITHMETICÆ.—No. 8.

Memoranda.

The preceding part of this series of papers, (No. 4—7.) in which the principle of Transformation by Division was for the first time developed, were all dispatched between the beginning of June 1821, and the end of January 1822. Whatever were the circumstances which retarded their publication, the delay was most inauspicious to me, as having been among the causes of the failure of a paper of mine, which was read before the Royal Society, in June 1823, containing an improved and extended view of the same principle, but overloaded—as I was very conscious myself—with details which I had hoped to supersede by a mere reference to this Repository. Other reasons might have their influence also; but the consequence is, that a paper far superior, in every respect but neatness, to that which appeared in the Transactions of 1819, has lain upwards

3rdly. Making $R + r + r_1 = R_1 + r_1$ it is evident that in the same manner we have

$$\begin{array}{ccccccccc} A_2 + B_3 + C_2 + D_2 + E_2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & b_3 R_1 & c_3 R_1 & d_3 R_1 & \dots & \dots & \dots & \dots \\ 0 & A_3 r_1 & B_3 r_1 & C_3 r_1 & D_3 r_1 & \dots & \dots & \dots & \dots \\ \hline A_3 + B_3 + C_3 + D_3 + E_3 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

so that the method is *continuous*.

Investigated thus, the theory of this method possesses a beautiful simplicity. The balance of its *practical* merits must be thus stated :

The advantage is, that the depressed equation (see No. 6,) contains the remaining roots *unaltered*.

The disadvantages are (1.) that there are *no search-divisors* (No 4, p-46.) throughout the whole work, nor even proof-divisors ; (2) that the factors $R, R_1, R_2, \dots$ soon become of an *unwieldy* magnitude.

Both of these drawbacks are cognate with the previous benefit. And the proof is, that the roots being undiminished by the process, the operation begins as it were *ab integra*, at every step.

Mt. Bonnycastle so far relaxes the principle, in the sequel, as to introduce a facility into the use of the terms denoted by R ; a facility identical with my quadratic formula of commencement. Apparently, he did not perceive that in pursuing the same track which he had entered, the next step of abbreviation would have brought him to my cubic formula, and so on, until the whole of my process was developed.

The management of trinomial divisors in the tract which contains this method, is marked by the same characteristics. It is necessarily tentative throughout.

EVOLUTION.—Index an odd Number.

The improvements in approximation have, I fear, effectually excluded the following investigation of a new method of approximating to odd roots of numbers from any claim to be regarded as practically useful. A few years ago the case would have been different. At present I offer it as a matter of curiosity only.

The formula $(r + z)^n$ being *reciprocal*, may be reduced to lower dimensions when n is odd ; viz. to the $\frac{n-1}{2}$ th, and a quadratic.

As usual, put $s = r + z$, $P = rz$, $s_n = r^n + z^n$; and a well known theorem gives

$$s_n = s^n - n s^{n-2} P + \frac{n \cdot n-3}{1 \cdot 2} s^{n-4} P^2 - \frac{n \cdot n-4 \cdot n-5}{1 \cdot 2 \cdot 3} s^{n-6} P^3 + \dots$$

Hence we have

$$\frac{s^n - s_n}{n} = sP \left\{ s^{n-3} - \frac{n-3}{2} s^{n-5} P + \frac{n-4 \cdot n-5}{2 \cdot 3} s^{n-7} P^2 - \dots \right\}$$

Put $zr = sP$, and $n = 2m+1$; then, since $s^2 - P = z + r^2$, this formula reduces to

$$\begin{aligned} \frac{s^n - s_n}{nr} \text{ or } \frac{N - r^n - z^n}{nr} = \\ z \left\{ (z + r^2)^{m-1} + \frac{m-2 \cdot m-3}{2 \cdot 3} (z + r^2)^{m-4} r^2 z^2 + \right. \\ \left. \frac{m-3 \dots m-6}{2 \dots 5} (z + r^2)^{m-7} r^4 z^4 \dots \right\} = \\ z^m + \frac{m \cdot m+1}{2 \cdot 3} r^2 z^{m-1} + \frac{m-1 \cdot m+2}{4 \cdot 5} \Lambda r^4 z^{m-2} + \\ \frac{m-2 \cdot m+3}{6 \cdot 7} B r^6 z^{m-3} \dots (m-1) r^{2m-4} z^2 + r^{2m-2} z \end{aligned}$$

where Λ , B , &c. represent the next preceding coefficient respectively.

We have thus an equation of the m^{th} or $\frac{n-1}{2}$ order, from which the value of z may be found, correct to as many places as $\frac{z^n}{nr^{n-1}}$ contains zeroes in its commencement. Then the quadratic $z^2 + r^2 z = z$ gives the value of z , true to an equal extent.

Thus, we shall have, when

$$n = 3, z = \frac{N - r^3 - z^3}{3r} - r^3 = \frac{N - 4r^3 - z^3}{3r}$$

and the subsequent quadratic would lead to Halley's irrational formula for the cube root.

$$\text{If } n = 5, z^2 + r^2 z = \frac{N - r^5 - z^5}{5r}$$

which produces the formula for $N^{\frac{1}{5}}$,

$$\frac{1}{5}r + \sqrt{\left\{ \sqrt{\frac{4N - r^5 - z^5}{2 \text{ or } 4}} - \frac{r^3}{4} \right\}}$$

which quintuples the original accuracy of the root r .

If $n = 7, z^3 + 2r^2 z^2 + r^4 z = \frac{N - r^7 - z^7}{7r}$ and so of the rest.

W. G. HORNER.

Bath, Feb. 4, 1831.

ARTICLE VI.

Analytical Solution of the Prize Question in the present Repository, for the case of the Triangle.

By Mr. T. S. DAVIES.

Though in my solution of this question I have intimated my intention of giving two different methods of treating the general problem—one by the radial projection, and the other by Analytical Geometry—I have thought the following, which is something different from the analytical solution alluded to, might not be without interest. Its chief purpose is for comparison with the two solutions of that case given in the Gentleman's Mathematical Companion for 1813. The object of this comparison is to enforce by an example the greater simplicity and greater facility of operating by general methods than by isolated processes, and to shew that a superior solution can be obtained by far less exercise of skill than is there employed. Another advantage of general methods, is, that they produce symmetrical results, and thereby afford a ready check upon the accuracy of the work itself.

1. *Ellipse and Hyperbola.*

Let the ellipse or hyperbola referred to its centre and principal axes be denoted by

$$a^2y^2 + b^2x^2 = a^2b^2 \dots \dots (1).$$

For the sake of ease we shall take the ellipse only, as the solution for the hyperbola is obtained by a simple change of sign.

Let the angular points of any one of the inscribed triangles be x, y, x'', y'' ; then since these are in the conic section, we have

$$a^2y^2 + b^2x^2 = a^2b^2 \dots \dots (2)$$

$$a^2y''^2 + b^2x''^2 = a^2b^2 \dots \dots (3)$$

$$a^2y'''^2 + b^2x'''^2 = a^2b^2 \dots \dots (4).$$

Let α and β be the inclinations of the given straight lines to the axis of x : then the equations of the two sides of the triangle parallel to them, give

$$\frac{y - y''}{x - x''} = \tan \alpha, \text{ and } \frac{y - y'''}{x - x'''} = \tan \beta, \text{ or}$$

$$(x - x'') \sin \alpha = (y - y'') \cos \alpha \dots \dots (5)$$

$$(x - x''') \sin \beta = (y - y''') \cos \beta \dots \dots (6).$$

Subtracting (3) and (4) from (2) we obtain

$$b^2(x^2 - x_{II}^2) = -a^2(y^2 - y_{II}^2) \dots\dots(7)$$

$$b^2(x^2 - x_{III}^2) = -a^2(y^2 - y_{III}^2) \dots\dots(8).$$

Dividing (7) and (8) by (5) and (6) respectively, we get

$$b^2(x + x_{II}) \cos \alpha + a^2(y + y_{II}) \sin \alpha = 0 \dots\dots(9)$$

$$b^2(x + x_{III}) \cos \beta + a^2(y + y_{III}) \sin \beta = 0 \dots\dots(10).$$

Throwing open the vincula of (5), (6), (9), (10), we have

$$x, \sin \alpha - x_{II}, \sin \alpha - y, \cos \alpha + y_{II}, \cos \alpha = 0 \dots\dots(11)$$

$$x, \sin \beta - x_{III}, \sin \beta - y, \cos \beta + y_{III}, \cos \beta = 0 \dots\dots(12)$$

$$b^2x, \cos \alpha + b^2x_{II}, \cos \alpha + a^2y, \sin \alpha + a^2y_{II}, \sin \alpha = 0 \dots\dots(13)$$

$$b^2x, \cos \beta + b^2x_{III}, \cos \beta + a^2y, \sin \beta + a^2y_{III}, \sin \beta = 0 \dots\dots(14)$$

These give

$$y_{II} = - \frac{(a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)y, + 2b^2 \sin \alpha \cos \alpha \cdot x,}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \dots\dots(15)$$

$$x_{II} = \frac{(a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)x, - 2a^2 \sin \alpha \cos \alpha \cdot y,}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \dots\dots(16)$$

$$y_{III} = - \frac{(a^2 \sin^2 \beta - b^2 \cos^2 \beta)y, + 2b^2 \sin \beta \cos \beta \cdot x,}{a^2 \sin^2 \beta + b^2 \cos^2 \beta} \dots\dots(17)$$

$$x_{III} = \frac{(a^2 \sin^2 \beta - b^2 \cos^2 \beta)x, - 2a^2 \sin \beta \cos \beta \cdot y,}{a^2 \sin^2 \beta + b^2 \cos^2 \beta} \dots\dots(18)$$

Now the equation of the third side, or tangent of the locus sought, in terms of its extremities is

$$y(x_{II} - x_{III}) - x(y_{II} - y_{III}) = y_{III}x_{II} - y_{II}x_{III} \dots\dots(19)$$

Inserting in (19) the values of x_{II} , y_{II} , x_{III} , y_{III} from (15) ..(18) we have, after arranging the quantities according to x , and y ,

$$\begin{aligned} & \{a^2y(a^2 \sin \beta \sin \alpha - b^2 \cos \beta \cos \alpha) - a^2b^2x \sin(\alpha + \beta)\}y, \{ \\ & + \{b^2x(a^2 \sin \beta \sin \alpha - b^2 \cos \beta \cos \alpha) + a^2b^2y \sin(\alpha + \beta)\}x, \} = \\ & a^2b^2(a^2 \sin \beta \sin \alpha - b^2 \cos \beta \cos \alpha) \dots\dots\dots(23) \end{aligned}$$

or

$$y, = \frac{a^2b^2(\sin \beta \sin \alpha - b^2 \cos \beta \cos \alpha) - \{b^2x(a^2 \sin \beta \sin \alpha - b^2 \cos \beta \cos \alpha) + a^2b^2y \sin(\beta + \alpha)\}x,}{a^2y(a^2 \sin \alpha \sin \beta - b^2 \cos \alpha \cos \beta) - a^2b^2x \sin(\beta + \alpha)} \dots\dots(24).$$

Giving this the temporary form of $y, = \frac{q^2 - px,}{r}$ and eliminating y , between it and (2), we get

$$(a^2p^2 + b^2r^2)x,^2 - 2a^2pq^2x, = a^2(r^2b^2 - q^2) \dots\dots(25).$$

Differentiating relative to x , we get

$$x, = \frac{a^2pq^2}{a^2p^2 + b^2r^2} \dots\dots(26).$$

Eliminating x , between (25) and (26), we have

$$q^2 = a^2 p^2 + b^2 r^2 \dots\dots (27).$$

Restoring the values of p, q^2, r , the equation (27) becomes, after sorting the terms and performing the obvious and simple reductions,

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \left\{ \frac{a^4 (\cos \beta - a - \cos \beta + a)^2 - 2a^2 b^2 (\cos^2 \beta - a - \cos^2 \beta + a) + b^4 (\cos \beta + a + \cos \beta - a)^2}{a^4 (\cos \beta - a - \cos \beta + a)^2 + 2a^2 b^2 (\cos^2 \beta - a + \cos^2 \beta + a) + b^4 (\cos \beta + a + \cos \beta - a)^2} \right\}$$

Which denotes a concentric, similar, and similarly situated ellipse. W.W.R.

A simple change of sign in the result will convert this into the case adapted to the hyperbola. We may now proceed to

2. The Parabola.

Refer it to its principal diameter, and having the origin at the vertex. Its equation is

$$y^2 = ax \dots\dots (1') \quad y_{II}^2 = ax_{II} \dots\dots (3')$$

$$y_I^2 = ax, \dots\dots (2') \quad y_{III}^2 = ax_{III} \dots\dots (4').$$

Then as before, we have

$$y_I - y_{II} = (x_I - x_{II}) \tan \alpha \dots\dots (5')$$

$$y_I - y_{III} = (x_I - x_{III}) \tan \beta \dots\dots (6')$$

$$y_I^2 - y_{II}^2 = a(x_I - x_{II}) \dots\dots (7')$$

$$y_I^2 - y_{III}^2 = a(x_I - x_{III}) \dots\dots (8')$$

Divide (7'), (8') by (5'), (6') and we have

$$y_I + y_{II} = a \cot \alpha \dots\dots (9')$$

$$y_I + y_{III} = a \cot \beta \dots\dots (10').$$

$$\text{Hence } y_{II} = a \cot \alpha - y_I \dots\dots (11')$$

$$y_{III} = a \cot \beta - y_I \dots\dots (12')$$

$$\text{From (3') and (11') we have } x_{II} = \frac{1}{a} (a \cot \alpha - y_I)^2 \dots\dots (13')$$

$$\text{From (4') and (12') we have } x_{III} = \frac{1}{a} (a \cot \beta - y_I)^2 \dots\dots (14')$$

The tangent to the locus sought is, as before,

$$y(x_{II} - x_{III}) - a(y_{II} - y_{III}) = y_{III}x_{II} - y_{II}x_{III} \dots\dots (15')$$

Substituting the values of $x_{II}, x_{III}, y_{II}, y_{III}$ given by (11') $\dots\dots (14')$ in this, we have after obvious reductions

$$y^2 - y_I \{ a(\cot \alpha + \cot \beta) + 2y \} + a^2 \cot \alpha \cot \beta + ax + ay (\cot \alpha + \cot \beta) = 0 \dots\dots (16').$$

Differentiate relative to y , and eliminate between the result and (16'). The next result will be

$$y^2 = ax - \frac{a^2}{2} (\cot^2 \alpha + \cot^2 \beta).$$

This is the equation of a parabola, whose axes coincide with (1), is equal to it in all respects, and whose vertex is distant from the vertex of (1) by the quantity $\frac{1}{2}(\cot^2 \alpha + \cot^2 \beta)$,
W. W. R.

ARTICLE VII.

RESEARCHES IN THE GEOMETRY OF THREE DIMENSIONS. By Mr. DAVIES, Bath.

On the Ebnoids; or Surfaces of Revolution of the Second Order.

The properties of these surfaces, though exceedingly elegant, simple, and easy of investigation, have been rarely touched upon by writers of our own country, and in no case examined with the attention which they deserve. These properties are in many cases so similar to those of their generating curves, and most others have so obvious an alliance with them, that it is singular, so fertile, and so superficial a system of enquiry should have been overlooked. I attribute this to the very abbreviated form in which solutions were necessarily drawn up for the different periodicals which kept alive the scientific spirit of this country during three-fourths of the last century. A neat and concentrated result, rather than an amplified dissertation, was more in accordance with the plan of these works; and hence, the most interesting corollaries, if they were not characterized by some extraordinary simplicity, were necessarily excluded. The consequence was, that though these temporary productions are often replete with curious and interesting results, yet even the authors themselves learned to consider them unworthy of development, and to estimate their solutions as most valuable in proportion to the compression employed in them. Another necessary result; was a cultivation of specific artifices to adapt the solution to the particular case under consideration, rather than the invention of general methods by which the solution of the whole class of correlative propositions could be effected. We see in all the periodicals of that epoch; illustration of this remark—it is stamped on every page. One grand result, however, arose out of this—the more ample cultivation of the Greek Geometry. It led to a more minute examination of the several cases of a proposition, than could have resulted from

the more general and comprehensive methods which our Continental neighbours were simultaneously cultivating. This may be thought contradictory, but the paradox is easily explained. Whilst amplification of a single proposition was forbidden, the natural corollaries became in their turn new propositions for future investigation. The demi-enigmatical form in which they were sometimes proposed—or the modification of one or other of the hypotheses—or the application of the principle to some physical or descriptive problem:—these served to conceal the real connexion between the old and the new propositions. The new one in its turn gave rise to others: and thus arose new methods of solution adapted to the particular cases. This was still more facilitated by the necessary defect of any general principle to connect the various truths—or any uniform system of methods alike adapted to all problems—or even any classification being possible, which should so arrange the different subjects of investigation as to enable the mathematician to see before hand by what kind of processes any particular problem might admit of solution. At the same time, as an exercise in the best and most useful employment of the syllogism—at least of enthymeme—it kept, as it always had done, in advance of every other. As a mental discipline, the mathematics, (so long as mathematics was confined wholly or principally to the Greek Geometry) formed the most efficacious of all: and though I value as much as any one the increase of power in physical enquiries, which is afforded by the modern analysis, I cannot but regret the vagueness of thinking which prevails amongst its devotees, in consequence of the relaxation of the wholesome exercise afforded by the writings of Euclid, Archimedes, and Apollonius.

For the examination of a geometrical problem, the Greek methods were better adapted than the modern. The English learned to reason concerning figure by means of its approximate representation—the Continental Geometers by means of its arbitrary symbol. It is true that Landen and Waring in this country attempted to cultivate the method of solving Geometrical problems by means of Algebra, and though in many cases they arrived at very curious results, there is none of that generality of method—nothing of that reach of mind and systematic skill which laid down principles of solution rather than peculiar artifices adapted to special cases—to be found in any of their writings. They have in short, combined the inconveniences of both methods, without the slightest trace of the peculiar advantages which either might afford. The whole advantage of analytical Geometry has originated in one sole circumstance—the employ-

ment of co-ordinates. All other algebraical attempts at geometrical solution have led to results as complex as they were deficient in symmetry and difficult to obtain. A few isolated cases, it is true, might be adduced as exceptions to this remark : but it is so generally true that it would be difficult to find in the labours of twenty years in any one of the works of that period, as many indisputable exceptions. All the results to which the two authors above named have arrived are proofs rather of their dexterity, in the invention of algebraic expedients than of any comprehensiveness of mind which could seize the general principle that pervades in common a number of different processes.

The difficulties of interpretation, too, of analytical results, so long as those results proceeded from isolated processes, returned with every new example, and indeed depended for their solution upon the peculiarities of that example : and hence, till general methods of notation and investigation were devised, it was impossible to contrive any general methods of interpretation. Even in the present stage of analytical Geometry, this is by far the most difficult part of the problem : and though much has been done towards reducing into a regular system the more prominent geometrical interpretations of a result, yet the more delicate and often far more interesting indications are either passed over in silence, or rendered the subject of the most ludicrous mistakes. Mr. Babbage in his elaborate essay on the "Influence of Signs," has noticed a remarkable specimen of the former—and (amongst many others which might be readily instanced) may be cited the absurd interpretations which Messrs. Warren, Mourey, and Peacock,* have *revived* of the geometrical signification of $\sqrt{-1}$.

* I say "*revived*"—for it is well-known that the same notion—that $\sqrt{-1}$ signifies an angle—is not new, however much Mr. Warren may value the presumed discovery. Dr. Wallis, (who might well be excused for any error of interpretation, writing as he did in the infancy of science) was the first, I believe, who imagined that $\sqrt{-1}$ indicated a point *without the line* upon which + and — were estimated. It is worthy of remark, too, that he was the author of the mistaken interpretation of the signification of + and — in the Geometry of Position, which makes such a figure amongst the difficulties that Carnot imagines he has fixed upon the modern Algebraic Geometry. The notion that $\sqrt{-1}$ signified *perpendicularity* was proposed by a foreign author whose name I am unacquainted with—but its fallacy was exposed by Fongenex in the Turin *Mélanges*, 1760-1, pp. 122-6. In 1805, M. Bué (an Emigrant French Clergyman, resident here) guided, I have no doubt, by the hint of Wallis, came to the same conclusion. His paper was published in the Phil. Trans. 1806, and a

Since the time of Wallis no writer except Emerson seems to have taken much notice of the conoids, and he but very little. In our own times half a dozen properties have been appended to a little Geometrical Treatise on Conics, published anonymously at Cambridge. Even this little, coming from a University which has banished solid Geometry (except under an algebraical form) from its lecture rooms, is gratifying: and the execution of the work altogether is as good as we could expect from such a source. There is also in the Transactions of the Cambridge Philosophical Society (Vol. III. p. 185) a paper by Mr. Pierce Morton, upon the "Focus of a Conic Section." What that gentleman means by stating that he is "not aware of any simple and direct mode by which the existence of so remarkable a point (the focus) is established," I do not profess myself able to understand. Passing by this as a verbal oversight, I wish to remark that this paper was read in March 1829—two years and a half after the principal proposition in it had been announced in the Ladies' Diary, and eighteen months after the solution of a far more general one had been given in the Gentleman's Diary, viz. where the conoids generally had been substituted for the cone in Mr. Pierce Morton's

masterly refutation of it given in the Edinburgh Review, 1808, by Professor Playfair. About the same time, M. Argand published, in Paris, a little work devoted to the same view of the signification of the sign in question. In 1813 M. Frangais, who had been led into the speculation by a remark in a letter of Legendre to his brother, published in the *Annales des Mathematiques*, tom. IV. p. 60—72, a paper containing but a slightly different course of enquiries—definitions—conclusions—from those since brought forward by Mr. Warren. Other papers were published by Frangais and Argand in the same work—and a leaning towards the same view was shown by M. Gergonne, the learned and acute Editor of the *Annales*. A letter of M. Servois however, seems to have suppressed the enquiry, as we find no further mention of it except a last reply of Argand, adding new absurdities to the accumulation already made. Mr. Gompertz also seduced by the vague speculations of Bué was also led to waste his valuable powers upon this fruitless subject. In one or other of these authors there will be found sufficient to remove all claim to originality in Messrs. Warren, Mourey, and Peacock. How far they are indebted directly to the authors already cited is another question, and one which it would be scarcely proper to discuss here. The whole system of enquiry will be discussed, however, in a forthcoming volume of Miscellaneous Dissertations. One of these dissertations will be "An Enquiry into the principles which should regulate our judgment in deciding upon Mathematical plagiarism." It will be easy to furnish instances of every principle there contended for from a certain class of mathematical writers in this country.

disertation. I confess I was surprised to find so simple a subject as that which formed the fourth in order of difficulty in the *Ladies' Diary*, made the subject of a formal paper in the transactions of so learned a body as the University Philosophical Society, and I could not but remark how much the importance of any subject depends upon the manner in which it is ushered into the world! That Mr. Morton was aware of the *Diary* solution I do not insinuate: that paper bears on its face one of the tests of independence of the *Gentleman's Diary* solution—*less generality*. But upon turning to Quetelet's *Brussels Correspondence*, tom. III. p. 270, the intimate connexion—the almost identity between Mr. Morton's paper and that of M. Bobillier—looks less favourable. This is a question of little moment however, as Mr. Morton's paper though original would be deemed but a slight accession to our knowledge on the subject of conoids.

The most curious and interesting assemblage of these properties that has ever been given to the world, is that by M. Charles, in the late volumes of the *Transactions of the Royal Academy of Brussels*, and in some other scientific journals, and likewise in a reprint of his principal *Mémoire* at Paris. His method of investigation differs in most cases from mine. Waving the claim to originality, I shall be content to have an opportunity of introducing to the notice of Englishmen this curious collection, and more especially should I be so happy as to succeed in inducing them to pursue the same enquiry still farther. My own attention was led specifically to this subject when studying the beautiful property which has been alluded to in the *Gentleman's Diary*, and I then discovered many of the dependant and related properties. I had added to them at different times before M. Charles's papers fell in my way: and from the first intended to make them form a section of this series of papers. There will be, then, no necessity to mark which are here published for the first time—which have been anticipated by M. Charles,—or which are taken at once from his papers. Those who are curious in Philosophical or Mathematical History will of course turn to both: and by those who are not so, the explanations which it would require may be deemed unnecessary and even irrelevant. I have only to state that my reason for generally preferring in this enquiry the method which has borne the title of geometrical is—that after making the experiment upon a great number of the propositions, I have found it shorter, simpler, and more intelligible than the analytical method.

ARTICLE VII.*

ON SPHERICAL GEOMETRY.

By Mr. THOMAS STEPHENS DAVIES, Bath.

The usual method of establishing any property of figures formed by great circles on the sphere has been to prove that the analogous property holds amongst corresponding lines on a plane, and then project them gnomonically on the sphere itself. In all cases where the property is a *projective† one*, this method (with some limitations‡ it is true) can be legitimately applied. When also, a single less circle of the sphere enters into the enunciation of the proposition, the same process, also under the same slight limitation, is equally applicable. It is thus that a considerable number of spherical loci, and of spherical porisms may be established with great facility: and were it not that there is an immense number of enquiries equally curious and far more difficult, that cannot be made to yield to this treatment, that method may perhaps be deemed sufficient.§ It might, however, with much propriety be urged that

* Our readers will perceive that we have numbered this Article as a repetition of the last; the reason is this—

The length to which the communications of Mr. Davies on Spherical Geometry and Spherical Loci extended, rendered it impossible to insert even the portions of them which we have done in this number, without reserving to a future occasion the section of that gentleman's paper on the *Geometry of Three Dimensions*, which we had begun in our last half sheet: and on account of the greater importance attached by the author to his latter communications, we have arranged that the former should "stand over" to make room for them in the present number.—ED.

† "Une figure dont les parties n'auront entre elles que des dépendances indestructibles par l'effet de la projection, sera appelée *figure projective*."

Ces dépendances elles-mêmes, et en général, toutes les relations ou propriétés qui subsistent à la fois dans la figure donnée et dans ses projections seront appelées également *relations* ou *propriétés projectives*. Poncelet, *Traité des Prop.* p. 5.

‡ That is when the point projected is not more than $\frac{\pi}{2}$ from the point

of contact of the sphere and plane of projection. It might, indeed, even then by some subordinate considerations *be made to apply*: but the method is certainly far from elegant, and only to be tolerated when it is the *only* one that can be devised.

§ The case where two less circles enter into the proposition belongs to this class: as they cannot both be gnomonically projected on the same plane into circles.

projection though a *ready* is not an *elegant* method of investigating any class of properties : and this doctrine is so generally admitted in England that in the demonstration of properties of the Conic Sections, projection is rarely admitted where other methods can be devised—or at all events, it is universally considered an *inferior* process. Nor am I aware that any attempt has yet been made (with one slight exception to be noticed presently) to establish a method of treating even determinate problems on the sphere, with any thing like appropriate and promising methods. Our spherical trigonometry, in all its improvements, has had an exclusive reference to facility of computation—and has very rarely turned aside to consider any of those elegant and striking properties of spherical figures that force themselves upon our notice at every turn. This is the more extraordinary, as the processes by which they are established are so simple, and the calculations so habitual to those who have attended ever so little to physical researches, that it can only have arisen from the attention of geometers never having been specially directed to that object.

A paper of mine which is now printing in the Transactions* of the Royal Society of Edinburgh, is devoted to the principles of such an enquiry, as far as the doctrine of spherical loci can be treated without reference to the properties of the differential coefficients : and another which will be shortly furnished to the same Society, takes up the subject, under this aspect also. The co-ordinates I have employed are the longitude, and either latitude or polar distance of the points in the curve, as the case may seem best to admit of. Still, the great variety of curious and often unexpected results that I have obtained, far exceed the possible limits of such dissertations as those, even though they themselves greatly exceed the usual length of papers in the Transactions of Learned Societies. I have, on that account, availed myself of the kind permission of the Editor of the Mathematical Repository, to employ his pages for introducing to public notice, such of my results as could not be conveniently inserted in the memoirs referred to : and I trust I am not too sanguine in hoping that the readers of that work will enter upon the investigation of spherical problems with that ardor which the opening of a boundless subject of enquiry never fails to produce in liberal and enlightened minds.

SECTION FIRST.

OF GREAT-CIRCLE TRANSVERSALS.

I.

Def. As in plano a line AB is said to be harmonically divided in C, D when the proportion

$$AC : CB :: AD : DB$$

subsists amongst its segments; and as the lines drawn from ABCD to any point P on the plane were called by Maclaurin, *Harmonicals*: so, when an arc AB of a great-circle is divided in C, D, so that

$$\sin AC : \sin CB :: \sin AD : \sin DB$$

the arc is said to be divided harmonically and arcs drawn to any point P on the sphere are called *spherical harmonicals*.

Our purpose is to shew the analogy that subsists between the *plane and spherical harmonicals*. A few of these have already been pointed out by Carnot in his *Essai sur la Théorie des Transversales*, p. 86: and I shall avail myself of his labours on this part of my subject, at the same time remarking that his theorems are only simple corollaries from a theorem laid down by Menelaus, and which formed the basis of Spherical Trigonometry in the hands of Ptolemy.

"The theory of rectilinear transversals which we have laid down may be extended without difficulty to spherical transversals; that is to say, that all the properties which have been established concerning plane polygons, may be extended to polygons formed of great-circle arcs, traced on the sphere, by substituting the sines of these arcs for the sides of the plane polygons.

"For example, if a spherical triangle ABC be cut by an arc of a great-circle transversal, abc, which meets them all; then each of the arcs AB, AC, CB will be cut into two segments, and the product of the sines of three of these segments will be equal to the product of the three others, in taking them so that these segments have not common extremities. That is to say, that we shall have

$$\sin Ac \cdot \sin Ba \cdot \sin Cb = \sin Ab \cdot \sin Bc \cdot \sin Ca \dots (A)$$

i 2

Fig. 1.

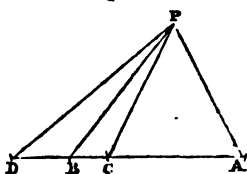


Fig. 2.

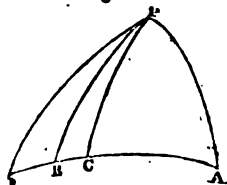
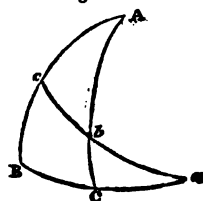


Fig. 3.



In fact, by the proportionality of the sines of the angles to the sines of their opposite sides, in spherical triangles, the triangles Acb , Bac , $Ca b$ give

$$\sin AC : \sin Ab :: \sin b : \sin c$$

$$\sin Ba : \sin Bc :: \sin c : \sin a$$

$$\sin cb : \sin ca :: \sin a : \sin b.$$

Multiplying these three equations, and effacing the common terms of the products, we shall have equation (A).*

"Similarly, if upon the surface† [*l'aire*] of a spherical triangle ABC , we take the point D , and through this point draw three great-circle arcs Aa , Bb , Cc , passing also through the angles, these arcs will cut each of the sides of the triangle into two segments, and the product of the sines of the three non-adjacent segments of these sides will be equal to the product of the sines of the other three non-adjacent segments.

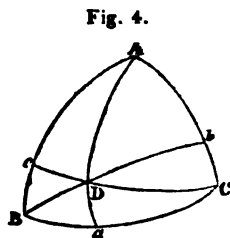


Fig. 4.

That is, we shall have,

$$\sin Ac . \sin Ba . \sin Cb = \sin Ab . \sin Ca . \sin Bc \dots (B).$$

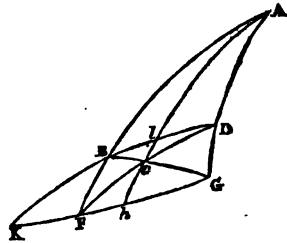
"In the same manner, if we have a spherical quadrilateral

* This is the theorem of Menelaus, vid. Sphaeric. lib. III. pr. 1, 2. and Ptolemæi Almagest, lib. I. pr. 16. The property has been referred to its proper author by Bishop Horsley, Elem. Treat. Math. p. 224, and by him only, so far as I know amongst the moderns. Brianchon pointed it out as being in Ptolemy, in the Correspondence of the Polytechnic School, II. p. 257. Carnot, I am quite convinced, was ignorant that it or even the *lemmata* of Menelaus had before been discovered: or he would have distinctly stated, that *these properties which form the basis of his processes*, were not from his own investigation.

† This seems as if Carnot did not consider the property to hold except when the point D was within the field of the triangle: it is universal, however, and the same demonstration applies to it in every case,

$ABCD$ and that we trace its three diagonals AB , CD , FG ,* we shall find that each of its diagonals is cut by the two others into two segments, whose sines are proportional: that is to say, for example, that in designating by l and h the points of intersection of the diagonal AC by the two others, BD , GF , we shall have

Fig. 5



$$\sin Al : \sin Ah :: \sin cl : \sin ch. \dots\dots(c)$$

"I do not think it necessary after all that has been said concerning rectilinear transversals, to enter into the detail of the demonstration of this proposition, nor of pursuing further the analogy which subsists between the rectilinear and spherical transversals: on which account I leave the investigation of these consequences to the sagacity of the reader." *Essai*, pp. 86-87.

It does not appear that the subject was ever resumed by Carnot, and I am inclined to doubt whether he ever went a step further than this point, as there are at no very remote stages of the enquiry interruptions to the analogy he seems to have in view. There are, it is true, analogies, but these analogies do not exist amongst the sines, but amongst other functions of the great-circle segments. All the projective properties which refer to points of intersection and contact, independent of metrical relations, and the ratios of metrical quantities, do indeed in all cases exist unbroken, and it will appear presently that relations of this kind (or having an analogy to this) which could never have been inferred from projective considerations, nor even suggested by them, are very numerous.

M. Poncelet too, has suggested that the various properties of his "centre of harmonic means" may be referred to the sphere by gnomonic or central projection: but this conclusion is doubtful in itself, and certainly if the property be true, it cannot be established by the method of projections, without some auxiliary spherical theorems. In his *Traité*, too, M. Poncelet had proposed to establish spherical properties by means of projection—which however, is no more than had been long before practised in England by several distinguished geometers.

* See my note to Solution of Question 524, of the present number of the Repository.

I do not recollect that any other author has taken up the subject in this way: though it should be remarked that Carnot had in his *Géométrie de Position*, in 1803, taken the same view of the spherical transversals that he did in the work already quoted from, (1806), but he did not deduce any other conclusions than those which he has so luminously explained in the extract already given. See *Géom. de Pos. Arts.* 246-9, 344-6.

II.

It has been shewn by Carnot, *Essai*, p 77, that if $AB : BC :: AD : DC$,

then $\sin APB : \sin BPC :: APD \sin DPC$. [*see Fig. 1. p. 75.*]

It is also true that in the spherical harmonicals, we shall have $\sin APB : \sin BPC :: \sin APD : \sin DPC$. [*see Fig. 2, p. 75.*]

To take the most general view of this we shall first prove the following.

Lemma.

If any four great-circles PA, PB, PC, PD be cut by a transversal great-circle, then

$$\frac{\sin AC \sin BD}{\sin BC \sin AD} = \frac{\sin BPD \sin APC}{\sin APD \sin BPC}.$$

For by spherical properties

$$\sin ACP = \frac{\sin AP \sin APC}{\sin AC} \dots\dots(1) \quad [\textit{see Fig. 2.}]$$

$$\sin PCB = \frac{\sin BP \sin CPB}{\sin CB} \dots\dots\dots(2)$$

$$\sin ADF = \frac{\sin AP \sin APD}{\sin AD} \dots\dots\dots(3)$$

$$\sin BDP = \frac{\sin BP \sin BPD}{\sin BD} \dots\dots\dots(4).$$

Equating (1, 2) on account of their being the sines of supplementary angles,

we find,
$$\frac{\sin AP \sin APC}{\sin AC} = \frac{\sin BP \sin CPB}{\sin CB} \dots\dots(5)$$

From (3, 4) we get
$$\frac{\sin AP \sin APD}{\sin AD} = \frac{\sin DFB \sin BP}{\sin BD} \dots\dots(6)$$

From (5) we derive
$$\frac{\sin AP}{\sin BP} = \frac{\sin CPB \sin AC}{\sin APC \sin CB} \dots\dots\dots(7)$$

From (6) also, $\frac{\sin AP}{\sin BP} = \frac{\sin BPD \sin AD}{\sin APD \sin BD} \dots\dots\dots(8).$

Or finally equating (7, 8) we have

$$\frac{\sin CPB \sin AC}{\sin APC \sin CB} = \frac{\sin AD \sin BPD}{\sin BD \sin APD} \dots\dots(9)$$

which is, in fact, the same property as above enunciated.

We may also remark that when the circles radiating from π are given

$$\text{then, } \frac{\sin AC}{\sin CB} : \frac{\sin AD}{\sin DB} = \text{constant.}$$

This is analogous to the property of rectilinear radiants, made by Brianchon, the foundation of his method of treating Lines of the second order. See his little Tract on that subject, 1817, p. 7.

We may now prove the theorem itself, or rather derive it as a corollary from the lemma :--

$$\text{For by lem. } \frac{\sin AC \sin CPB}{\sin CB \sin APB} = \frac{\sin AD \sin BPD}{\sin BD \sin APD};$$

$$\text{and by hypoth. } \frac{\sin AC}{\sin CB} = \frac{\sin AD}{\sin BD}$$

$$\therefore \frac{\sin CPB}{\sin APC} = \frac{\sin BPD}{\sin APD} \dots\dots\dots(10)$$

$$\text{or } \sin APC : \sin CPB :: \sin APD : \sin BPD \dots\dots\dots(11).$$

Again, we learn at once, that if any great-circle transversal AD intersect the spherical harmonicals AP, BP, CP, DP, it will itself be divided harmonically by them.

For, in this case we have by hypoth.

$$\sin CPB : \sin APC :: \sin BPD : \sin APD,$$

and by Lem.

$$\sin AC \sin CPB : \sin CB \cdot \sin APC :: \sin AD \sin BPD : \sin BD \cdot \sin APD.$$

$$\text{Hence } \sin AC : \sin CB :: \sin AD : \sin DB \dots\dots\dots(12).$$

The theorem of Carnot follows from this by putting the evanescent arcs for their sines.

III.

For the purpose of more briefly designating these several quantities we shall occasionally express the angles formed by the spherical harmonicals by Greek capitals, and the segments of the transversal which subtend them, by corresponding lower-

case Greek letters. Thus, if the angle $\angle APB = 2\kappa$, $\angle CPB = \Theta$, and $\angle BPD = \Theta_{//}$: then $AB = 2\kappa$, $BC = \theta$, and $BD = \theta_{//}$.

In this case, we shall have

$$\begin{aligned}\cot \theta - \cot \theta_{//} &= 2 \cot 2\kappa \\ \cot \Theta - \cot \Theta_{//} &= 2 \cot 2\kappa \\ \tan^2 \kappa &= \tan \frac{\kappa - \theta}{\kappa + \theta_{//}} \\ \tan^2 \kappa &= \tan \frac{\kappa - \Theta}{\kappa + \Theta_{//}}.\end{aligned}$$

For 1st. we have by the general definition of harmonicals

$$\sin(2\kappa - \theta) : \sin \theta :: \sin(2\kappa + \theta_{//}) : \sin \theta_{//} \dots (1)$$

or expanding and dividing out, it is

$$\begin{aligned}\sin 2\kappa \cot \theta - \cos 2\kappa &= \sin 2\kappa \cot \theta_{//} + \cos 2\kappa, \text{ or} \\ \cot \theta - \cot \theta_{//} &= 2 \cot 2\kappa \dots (2).\end{aligned}$$

This is analogous to the theorem first given by Maclaurin in his *Treatise on the General Properties of Curve Lines*, Algebra, Ed. 4th. pp. 389 and 458.

2nd. As by the property established in II. 11. we have

$$\sin 2\kappa - \Theta : \sin \Theta :: \sin(2\kappa + \Theta_{//}) : \sin \Theta_{//} \dots (3)$$

and here as above we obtain the stated result.

3rd. By composition and division of (III. 1) we have

$$\begin{aligned}\sin \frac{2\kappa - \theta}{\kappa + \theta} + \sin \theta : \sin \frac{2\kappa - \theta}{\kappa + \theta} - \sin \theta :: \sin \frac{2\kappa + \theta}{\kappa + \theta_{//}} + \sin \theta_{//} \\ : \sin \frac{2\kappa + \theta}{\kappa + \theta_{//}} - \sin \theta_{//} \dots (4)\end{aligned}$$

that is

$$\sin \kappa \cos \kappa - \theta : \sin \kappa - \theta, \cos \kappa :: \sin \kappa + \theta_{//}, \cos \kappa \cos \kappa + \theta_{//}, \sin \kappa \dots (5)$$

or by reduction it becomes

$$\tan^2 \kappa = \tan \frac{\kappa - \theta}{\kappa + \theta_{//}} \dots (6).$$

In like manner by composition and division of (3) and subsequent reduction

$$\tan^2 \kappa = \tan \frac{\kappa - \Theta}{\kappa + \Theta_{//}} \dots (7).$$

The equations (2) and (4) have well known analogies in plano, viz.

$$\frac{1}{EC} - \frac{1}{ED} = \frac{2}{AB};$$



$$\text{and } EC \cdot ED = EB^2.$$

The equations (3) and (7) are identical with properties of the corresponding angles in rectilinear harmonicals.

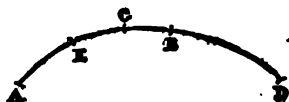
IV.

Lemma. (Corresponding to Euc. II. 5, 6.). Let any arc be bi-

sected in E , and divided unequally in C ; or produced to D ; then,

$$\sin^2 EB - \sin^2 EC = \sin AC \cdot \sin CB$$

$$\sin^2 ED - \sin^2 EB = \sin AD \cdot \sin DB.$$



$$\begin{aligned} \text{For } \sin^2 EB - \sin^2 EC &= (\sin EB + \sin EC)(\sin EB - \sin EC) \\ &= 2 \sin \frac{1}{2}(EB - EC) \cos \frac{1}{2}(EB + EC) \cdot 2 \sin \frac{1}{2}(EB + EC) \sin \frac{1}{2}(EB - EC) \\ &= \sin (EB + EC) \sin (EB - EC) \\ &= \sin AC \sin CB \dots\dots\dots (1). \end{aligned}$$

$$\begin{aligned} \text{Again, } \sin^2 ED - \sin^2 EB &= (\sin ED + \sin EB)(\sin ED - \sin EB) \\ &= \sin (ED + EB) \sin (ED - EB) \\ &= \sin AD \sin DB \dots\dots\dots (2). \end{aligned}$$

To apply this :—

By the general property of harmonicals we have

$$\sin AD \sin CB = \sin AC \sin BD \dots\dots\dots (3),$$

and from (1, 2) by addition

$$\sin^2 ED - \sin^2 EC = \sin AD \sin DB + \sin AC \sin CB \quad (4)$$

$$\therefore \sin^2 ED - \sin^2 EC = \frac{\sin AC}{\sin CB} (\sin^2 DB + \sin^2 CB) \dots\dots\dots (5),$$

$$\sin^2 ED - \sin^2 EC = \frac{\sin CB}{\sin CA} (\sin^2 DA + \sin^2 CA) \dots\dots\dots (6),$$

Also multiplying (5) and (6) we find

$$(\sin^2 ED - \sin^2 EC)^2 = (\sin^2 DB + \sin^2 CB) (\sin^2 DA + \sin^2 CA) \dots\dots (7).$$

Further, we have the equation

$$\begin{aligned} \tan^2 ED - \tan^2 EB &= \frac{\sin (EB + BD) \sin (ED - EB)}{\cos^2 EB \cos^2 ED} \\ &= \frac{\sin AD \sin DB}{\cos^2 EB \cos^2 ED} \dots\dots\dots (8). \end{aligned}$$

$$\text{In like manner } \tan^2 EB - \tan^2 EC = \frac{\sin AC \sin CB}{\cos^2 EB \cos^2 EC} \dots\dots (9).$$

Subtracting (9) from (8) we find

$$\tan^2 ED - 2 \tan^2 EB + \tan^2 EC = \frac{\cos^2 EC \sin AD \sin DB - \cos^2 ED \sin AB \sin CB}{\cos^2 EB \cos^2 EC \cos^2 ED} \dots\dots (10).$$

$$\text{But as shewn in (III. 6.) } 2 \tan^2 EB = 2 \tan EC \tan ED \dots\dots (11).$$

Inserting (11) in (10) we get

$$\frac{\cos^2 EC \sin AD \sin DB - \cos^2 ED \sin AC \sin CB}{\cos^2 EB \cos^2 EC \cos^2 ED} = (\tan ED - \tan EC)^2 = \frac{\sin^2 CD}{\cos^2 EC \cos^2 ED} \dots\dots (12),$$

or cancelling the denominators it becomes finally

$$\sin^2 CD \cos^2 EB = \cos^2 EC \sin AD \sin DB - \cos^2 ED \sin AC \sin CB \dots\dots (13).$$

It may here be remarked that when the radius of the sphere becomes infinite, the sphere itself becomes a plane. The arcs, then, which any finite straight lines represent upon this infinite sphere are become zero: all the trigonometrical functions which we employ, then become the same trigonometrical functions of the evanescent arc, or of 0. The cosines and secants become unity; the sines and tangents become equal to the arcs themselves: the cotangents and cosecants become infinite. Making these substitutions in any of our equations we should find the corresponding plane property. The logic of such a process is I confess too much like that commonly employed in treating of differential *quantities*, to be strictly admissible: if however we substitute the doctrine of *limiting ratios* by which alone the doctrine of differential *coefficients* is justly established, we shall stand on safer ground. Still less disputable, however, is the direct *geometrical* investigation of the plane property as a distinct enquiry: and I only allude to the method at all on account of our ascertaining what the plane property *is* which corresponds to any spherical one, whilst it ought to be afterwards made the subject of independent demonstration.

This plan applied here gives

$$CD^2 = AD \cdot DB - AC \cdot CB$$

which is Prop. 7. Book VI. Leslie's Geometry.

The *analogues* of (5, 6, 7) do not seem to have been noticed. Their demonstrations will be too obvious to need dwelling on here.

V.

In the Ladies' Diary for 1817, a question of this class was proposed by Mr. Whitley, (Quest. 14.) and solved in the next number of that work, by projecting the corresponding lines gnomonically on the surface of the sphere. In the Gentleman's Diary for 1828, it was repropoed, to be demonstrated by the pure principles of spherics: and a solution, the only one that fulfilled that condition, was sent by myself, for which, as I there stated, I was chiefly indebted to the writings of Carnot. The passage in question of this illustrious French geometer has been already given in the early part of this paper (Art. I.): and I shall here recapitulate the solution I then formed out of it, in as much as the inverse propositions are of quite as frequent and extensive use as the direct ones, in those classes of enquiries where spherical harmonicals are concerned.

"If three great circles be drawn through the angles of a spherical triangle, and the same point on the surface of the sphere, meeting the sides in three points: three other great circles each passing through two of these points will intersect the sides of the triangle (produced or not as the case may require) in the circumference of one and the same great circle,"
Gent. Diary, 1828, p. 35.

SOLUTION.

"It will be convenient to premise the following *Lemma*.

1. Let a spherical triangle ABC be cut by any great circle $a'bc$, then we shall have

$$\sin AC \sin Ba' \sin cb = \sin CB \sin a'c \sin bA \dots\dots\dots (1)$$

For

$$\sin AC : \sin Ab :: \sin \angle b : \sin \angle C$$

$$\sin Ba' : \sin BC :: \sin \angle c' : \sin \angle a$$

$$\sin cb : \sin ca' :: \sin \angle a' : \sin \angle b.$$

Hence by compounding, we get the property stated in the lemma.

"*Cor.* Conversely, if $\sin AC \sin Ba' \sin cb = \sin CB \sin a'c \sin bA$, then $a'bc$ are in one great circle.....(2).

"2. If through any point P on the surface of the sphere, great circles be described, which also pass through the angles of the triangle ABC , and cut the opposites on a, b, c respectively, then

$$\sin AC \sin Ba \sin bC = \sin aP \sin BC \sin bA \dots\dots\dots (3).$$

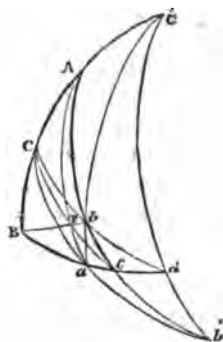
For the two spherical triangles BAA, CQA cut by the great circles C, b give respectively

$$\sin aP \sin AB \sin bC = \sin aP \sin BC \sin bA$$

$$\sin aP \sin CB \sin cA = \sin aP \sin aC \sin cB$$

whence as before we get the expression stated in the lemma.

"3. If through any point in a given great circle aa , which passes through one of the angles of a spherical triangle, great circles be drawn through the remaining angles, and cutting the opposite sides in b, c , respectively: then the great circle bc



shall always pass through the same point a' in the great circle BC , and divide it so that

$$\sin BA : \sin AC :: \sin Ba' : \sin a'C \dots\dots\dots(4).$$

For by Lem. 2. $\sin BA : \sin AC :: \sin BC \sin Ab : \sin CA \sin bc$ and by Lem. 1. $\sin Ba' : \sin a'C :: \sin BC \sin Ab : \sin CA \sin bc$ whence by equality of ratios, we have the property stated.

“ *Demonstration of the Proposition.* By Lemma 3, we have

$$\sin BA : \sin AC :: \sin Ba' : \sin a'C$$

$$\sin CB : \sin bA :: \sin cb' : \sin b'A$$

$$\sin CA : \sin CB :: \sin c'A : \sin c'B.$$

Hence $\sin Ba' \cdot \sin cb' \cdot \sin c'A = \sin a'C \cdot \sin b'A \cdot \sin c'B$, by Lemma (2).

Therefore by Cor. Lem. 1, the points a' , b' , c' are in the same great circle. *Q. E. D.*

“ The method of investigation employed in the foregoing solution is similar to that of M. Carnot, in his *Essai sur la Théorie des Transversales*, p. 86; or *Geom. de Position*, Art. 246.” *Gent. Diary*, 1829, p. 85.

VI.

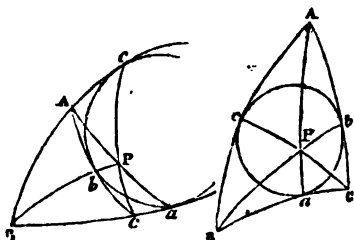
This method of proceeding will establish a considerable number of spherical properties, analogous ones to which in plano, are already well known. Some of them, too, have been established by other independent processes, as well as by gnomonic projection, on the sphere itself.

1^{mo}. Let great circles be drawn from the angles of a spherical triangle, to the points of contact of the opposite sides with the inscribed or escribed*) circle, these will pass through the same point P .

For as $Ac = Ab$, $Bc = Ba$, $ca = cb$, we have

$$\sin Ac \sin Ba \sin cb = \sin cb \sin ac \sin ba \dots\dots(1).$$

Hence (V. 2.) the three arcs pass through the same point P .



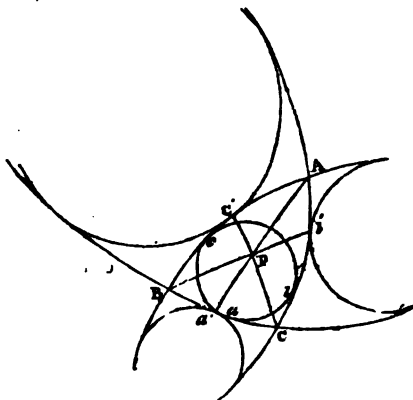
* “Ex-inscribed” is the term which some French Geometers employ. How they overlook the absurdity of this combination of the propositions, I cannot tell.

2^{do}. The great circles which pass through the angles A, B, C and the points of contact of the three inscribed circles a, b, c also pass through the same point P' ,

For describe the circle acb in the triangle.

Then by the properties of the figures

$$\begin{aligned} \Delta b &= Cb' \\ \Delta c &= Bc' \\ \therefore Cb' &= Bc' \\ BC &= \Delta c' \\ Ba &= Ca' \\ \therefore \Delta c' &= Ca' \\ Ca &= Ba' \\ Cb &= Ab' \\ \therefore Ba' &= Ab'. \end{aligned}$$



Hence we have at once

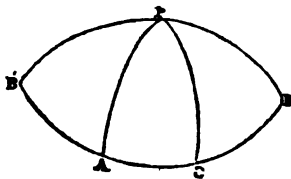
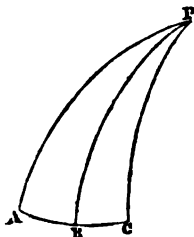
$$\sin Cb' \sin \Delta c' \sin Ba' = \sin Bc' \sin Ca' \sin Ab' \dots (2).$$

Hence (V. 2.) the arcs Aa', Bb', Cc' pass through one point P' .

Cor. In all these cases, if great circles be drawn through the points of contact two and two, so as to cut the remaining sides in three other points, these three points are in one great circle, by (V).

3^o. Great circles bisecting the *interior* angles of a spherical triangle, pass through the same point: and if these cut the opposite sides, and great circles be drawn as in (V), they will intersect the opposite sides in one great circle: and finally, if great circles be drawn bisecting the *exterior* angles of the triangle, these shall meet the opposite sides in the same great circle.

Lemma. If an arc of a great circle PB bisect the angle of a triangle APC it will divide the base into segments whose sines have the same ratio as the sines of the sides have.



Case 1. Let the *internal* angle be bisected : then

$$\sin BPC : \sin PBC :: \sin BC : \sin CP$$

$$\sin APB : \sin PBA :: \sin AB : \sin AP,$$

$$\therefore \sin AB : \sin BC :: \sin AP : \sin PC \dots\dots\dots (3).$$

Case 2. Let the *external* angle be bisected : then,

$$\sin BPC : \sin PBC :: \sin CB : \sin CP$$

$$\sin APB' : \sin P'BA :: \sin AB' : \sin AP$$

$$\text{or } \sin APB : \sin PBA :: \sin AB : \sin AP.$$

Also, $\sin B'PB$ bisects the interior angle P , we have $APB' = CPB$; hence, finally, by compounding these, we have

$$\sin CP : \sin PA :: \sin CB : \sin BA \dots\dots\dots (4).$$

Dem. of the Prop.

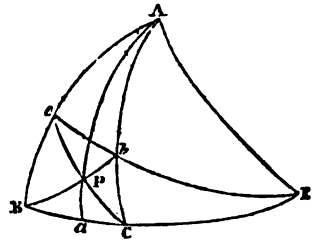
Case 1st. By the 1st. Lemma

$$\sin BA : \sin AC :: \sin ba : \sin ac$$

$$\sin AC : \sin CB :: \sin ac : \sin cb$$

$$\sin CB : \sin BA :: \sin cb : \sin ba$$

or compounding, it becomes



$$\sin ba \sin ac \sin cb = \sin ac \sin cb \sin ba \dots\dots (5).$$

Hence by conv. of IV. 2. the prop. is true.

Case 2nd. A case of the general proposition (IV) now that case (1) is established.

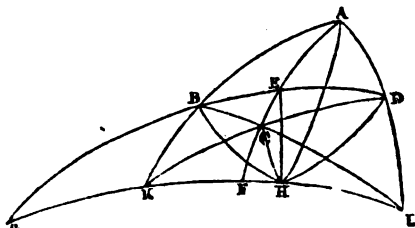
Case 3rd. Draw the great circle cb , cutting BC in E , and draw the great circle AE' to bisect the interior vertical angle at A , and meeting the base in E' (for the moment distinguished from E): and then we shall have

$$\sin ba : \sin ac :: \sin BE : \sin EC$$

$$\sin ba : \sin ac :: \sin BE' : \sin E'C.$$

Hence E, E' coalesce. The same is true of the other two bisecting circles, and hence these three points coalescing with the three former, are situated like them, in the circumference of the same great circle.

4th. From the intersection of Σ of any one pair of diagonals AC, BD of a spherical complete quadrilateral $ABCD$ * draw a perpendicular EH to the remaining diagonal KL . Then the angles BHD, CHA being formed, they will be bisected by EH .



Because AC is harmonically divided in Σ and F , (Art. IV.) and harmonicals are drawn to H , we have by (Art. II.),

$$\sin FHC : \sin CHE :: \sin FHA : \sin AHE \dots (6).$$

But, componendo et dividendo,

$$\sin FHC + \sin CHE : \sin FHC - \sin CHE :: \sin FHA + \sin AHE : \sin FHA - \sin AHE$$

whence

$$\frac{\sin FHC + \sin CHE}{\sin FHC - \sin CHE} = \frac{\sin FHA + \sin AHE}{\sin FHA - \sin AHE}, \text{ that is}$$

$$\tan \frac{1}{2}(FHC + CHE) \tan \frac{1}{2}(FHA - AHE) = \tan \frac{1}{2}(FHA + AHE) \tan \frac{1}{2}(FHC - CHE) \dots (7)$$

But $FHC + CHE = FHA - AHE = \frac{1}{2}\pi$; and hence

$$\tan \frac{1}{2}(FHA + AHE) = \cot \frac{1}{2}(FHC - CHE) \dots (8)$$

and hence

$$FHA + AHE + FHC - CHE = \pi$$

that is $FHA - AHE + FHC + CHE = \pi - 2AHE + 2CHE = \pi$

$$\text{and } \therefore CHE = AHE \dots (9)$$

which is the property enunciated above.

Cor. If upon a line EF as hypotenuse, a right angle EHF be constituted, and from the points of harmonical division, A, C , of that line, great circles be drawn to H , then CH and AH make equal angles with EH or with FH .

These conclusions might have been derived by shorter processes, but there are some difficulties in the expression of them, that render those we have employed rather preferable. Many other properties of this figure, and circles described in various ways, are exceedingly interesting: but they mostly require for their investigation some *local* considerations, which renders it necessary to defer them to a later part of this series of enquiries.

* See the note to my solution of Prop. 12. of the present number of the Repository. I give but one figure here, as they are all perfectly analogous to those mentioned in that solution and its note.

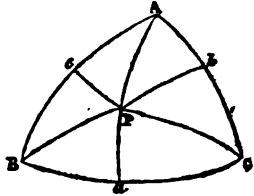
5¹⁰. Perpendiculars from any point P on the sphere upon the three sides AB, BC, CA, meeting them in c, a, b respectively give the equation

$$\cos BA \cos Cb \cos AC = \cos BC \cos Ab \cos Ca \dots (10).$$

For join AP, PC, CB by great-circle arcs. Then

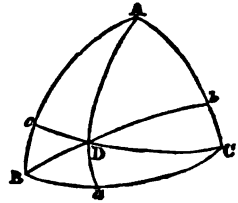
$$\begin{aligned} \cos BA \cdot \cos AP &= \cos BP = \cos BC \cdot \cos CP \\ \cos Cb \cdot \cos Pb &= \cos CP = \cos CP \cdot \cos AC \\ \cos AC \cdot \cos CP &= \cos AP = \cos Ab \cdot \cos PB \end{aligned}$$

omitting the middle vertical column, as being merely the mean term of the syllogism, and compounding the extreme columns, we have the property stated above.



Conversely:—when the equation (10) takes place amongst the segments of the sides made by three perpendiculars, those perpendiculars pass through one point on the sphere.

6¹⁰. If any three arcs passing through the angles A, B, C of a spherical triangle, and a point D on the sphere, divide the angles each into two segments, then the products of the sines of these segments taken alternately, will be equal.



$$\begin{aligned} \text{For } \sin BAD : \sin ABD &:: \sin BD : \sin AD \\ \sin DBC : \sin BCD &:: \sin DC : \sin BD \\ \sin DCA : \sin DAC &:: \sin DA : \sin DC. \end{aligned}$$

Hence compounding we find

$$\sin BAD \sin DBC \sin DCA = \sin ABD \sin BCD \sin DAC \dots (11).$$

Cor. We might at once have proved from this that the great circles which bisect the three angles pass through the same point: as we might by the last (Conv. of 5¹⁰.) have proved that the three perpendiculars from the middle of the sides pass through one point. We should thus be able, had it been our object to do so, admitting one or two simple trigonometrical properties, have been able to derive the whole system of spherical geometry by means of trigonometrical processes.

Continued on page 121.

ARTICLE VIII.

PROPOSITIONS IN LUNAR THEORY.

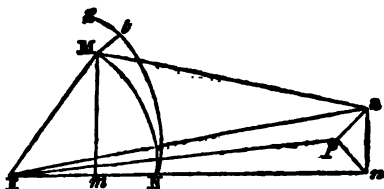
By the Rev. Mr. BRICE BRONWIN, of Danby, near Wakefield.

Very accurate results are not the object of the following memoir; which is intended to establish a Theory of the Moon upon indirect principles something like correct. All attempts of this kind, hitherto made, have in a greater or less degree failed. The Theory of Vince, almost from beginning to end, proceeds upon principles confused and erroneous. Indeed it is little better than a mass of nonsense. The celebrated Commentators on Newton have handled the subject with more ability; though they have fallen into great errors. No authors on this subject seem to have understood how to manage the tangential disturbing force, and therefore they have taken little or no notice of it. In general, however, this force produces the greater part of the inequalities of the moon's motion. The indirect methods of treating the Lunar Theory are easier and pleasanter than the direct ones; and, if cultivated, might perhaps be brought to the same degree of accuracy.

PROP. I.

To find the disturbing forces of the sun on the moon.

Let $\mathbf{E}$ the earth, $\mathbf{M}$ the moon, $\mathbf{S}$ the sun; and let $\mathbf{sn}$, $\mathbf{pn}$ be perpendicular to $\mathbf{En}$; $\mathbf{sp}$ perpendicular to $\mathbf{Ep}$, $\mathbf{np}$; also $\mathbf{mt}$ perpendicular to the ecliptic.



The force direct of $\mathbf{s}$ on

$$\mathbf{M} = \frac{s}{\mathbf{SM}^2}.$$

And in the directions $\mathbf{ME}$, $\mathbf{En}$, $\mathbf{np}$, $\mathbf{ps}$, it is

$$\frac{\mathbf{EM}}{\mathbf{SM}^3} \cdot s, \quad \frac{\mathbf{En}}{\mathbf{SM}^3} \cdot s, \quad \frac{\mathbf{np}}{\mathbf{SM}^3} \cdot s, \quad \text{and} \quad \frac{\mathbf{ps}}{\mathbf{SM}^4} \cdot s \text{ respectively.}$$

The force of $\mathbf{s}$ on $\mathbf{E}$ direct $= \frac{s}{\mathbf{SE}^2}$. And in the directions

$\mathbf{En}$, $\mathbf{np}$, $\mathbf{ps}$, it is

$$\frac{\mathbf{En}}{\mathbf{SE}^3} \cdot s, \quad \frac{\mathbf{np}}{\mathbf{SE}^3} \cdot s, \quad \text{and} \quad \frac{\mathbf{ps}}{\mathbf{SE}^3} \cdot s.$$

The differences of these and the former; or

$$\left(\frac{1}{SM^3} - \frac{1}{SE^3}\right) En \cdot s, \left(\frac{1}{SM^3} - \frac{1}{SE^3}\right) np \cdot s, \left(\frac{1}{SM^3} - \frac{1}{SE^3}\right) ps \cdot s$$

are disturbing forces in the three last directions.

Let D, D', D'' be the whole disturbing forces in the directions ME , perpendicular thereto in the plane of the moon's orbit, and perpendicular to that plane. Then, resolving the preceding forces into these, we shall find

$$D = s \cdot \left[\frac{EM}{SM^3} - \left(\frac{1}{SM^3} - \frac{1}{SE^3} \right) \cdot (En \cos MEN + np \sin MEN) \right];$$

$$D' = -s \cdot \left(\frac{1}{SM^3} - \frac{1}{SE^3} \right) \cdot (En \sin MEN - np \cos MEN);$$

$$D'' = s \cdot \left(\frac{1}{SM^3} - \frac{1}{SE^3} \right) ps; \quad D = \frac{Mt}{SM^3} \cdot s, \text{ if } D \text{ be the force perpendicular to the ecliptic.}$$

$$\text{But } \frac{1}{SM^3} = (SE^2 - 2SE \cdot ME \cos MES + ME^2)^{-\frac{3}{2}} = \frac{1}{SE^3} + 3 \frac{ME}{SE^4} \cos MES - \frac{3}{2} \frac{ME^2}{SE^5} + \frac{15}{2} \frac{ME^3}{SE^5} \cos^2 MES.$$

And by spherics $\cos MES = \cos MEN \cos SEN + \sin MEN \sin SEN$.

$$\begin{aligned} \text{Let } EM &= r, \quad ES = r', \quad MEN - SEN = v - v' = w, \\ \sin N &= s, \quad \sin MEt = \sigma. \quad \text{Then } \cos N = 1 - \frac{1}{2} s^2 \text{ nearly;} \\ En &= r' \cos v'; \quad np = r' \sin v' \cos N = \left(1 - \frac{1}{2} s^2\right) r' \sin v'; \\ ps &= r' \sin v' \sin N = sr' \sin v'; \quad \cos MES = \cos w - \frac{1}{2} s^2 \sin v \sin v'; \\ En \cos MEN + np \sin MEN &= r' \cos w - \frac{1}{2} s^2 r' \sin v \sin v'; \\ En \sin MEN - np \cos MEN &= r' \sin w + \frac{1}{2} s^2 r' \cos v \sin v'; \\ \text{and } Mt &= r\sigma. \end{aligned}$$

Substituting these values in the disturbing forces; we find, putting $s = \mu'$,

$$D = -\mu' \left\{ \frac{r}{2r'^3} (1 + 3 \cos 2w) + \frac{r^2}{8r'^4} (9 \cos w + 15 \cos 3w) \right. \\ \left. - \frac{3r\sigma^2}{4r'^2} (1 + \cos 2w - \cos 2v - \cos 2v') \right\}$$

$$\begin{aligned} \dot{v} &= r \mu' \left\{ \frac{3r}{2r^3} \sin 2v + \frac{1}{8r^4} (3 \sin w + 15 \sin 3w) - \frac{3rs^2}{4r^3} (\sin 2w - \sin 2v) \right\}; \\ \dot{v} &= \frac{3\mu'rs}{r^3} \cos w \sin v'; \quad \dot{v} = \frac{\mu'rs}{r^3}. \end{aligned}$$

Corol.

The velocity generated perp. to r by the force $\dot{v}$ in the time $dt = \dot{v} dt$. Therefore the velocity generated in the double area described in the time 1, or the differential of that double area $= \dot{v} r dt$. Consequently, the area itself $= h + \int \dot{v} r dt$.

But it is also $= r^2 \frac{dv}{dt}$. Therefore $r^2 \frac{dv}{dt} = h + \int \dot{v} r dt$.

And hence $r \frac{dv^2}{dt^2} = \frac{h^2}{r^3} + \frac{2h}{r^3} \int \dot{v} r dt + \frac{1}{r^3} \left(\int \dot{v} r dt \right)^2$;

which is the centrifugal force.

The centripetal force is $\frac{\mu}{r^2} + \dot{v}$.

In the following propositions, in order to abridge; we shall make $n - n' = m$; $\sin(nt - n't) = \sin mt$; $\sin(mt - cnt) = \sin mt$; $\sin(mt + cnt) = \sin mt$; $\sin(amt - cnt + c'n't) = \sin mt$; $\sin amt - agnt - cnt + c'n't) = \sin amt$; and so of others.

The cosines will be denoted in like manner. The angles cnt , $c'n't$, gnt will denote the moon's mean anomaly, the sun's mean anomaly, and the mean distance of the moon from her node: and these angles are sufficiently distinguished by the letters c , c' , g placed at the top, or at the bottom; according as they are to be added or subtracted. The quantity $(1 - c)nt$ represents the progression of the moon's apogee; and the quantity $(g - 1)nt$ the regression of her node.

PROP. II.

To find the progression of the Lunar Apogee.

We include here, and in the other elements, such equations only as vary little during half a revolution of the moon.

If two bodies be attracted by the forces P, f respectively towards their centres of force; and if they begin to fall from the

same distance, they will at the end of each of the small portions of time $d\tau$, dt be at the same distance from their respective centres if $Fd\tau^2 = fdt^2$. The times τ , t must flow uniformly, This, if not self-evident, is extremely easy to demonstrate.

If the bodies at the same time revolve about their centres; then for F and f we must take the difference of their centripetal and centrifugal forces.

Now if in $Fd\tau^2 = fdt^2$ we put for $d\tau$, dt their values in terms of R , $d\nu$; r , dv their radii-vectors and differentials of the angles described; we shall have the relation of $d\nu$ to dv , or the ratio of the angles included between equal radii. If therefore one of the orbits be an ellipse, described by the force $\frac{\mu}{r^2}$; the ratio $\frac{d\nu}{dv}$ will give us the progression of the apse, or the distance from apse to apse, of the other.

If one of the orbits be the moveable ellipse of Newton; then the centrifugal force and square of the time, or rdv^2 is obviously the same in both orbits. Therefore as the times $d\tau$, dt have a constant ratio, the centripetal forces must have a constant ratio likewise. This is the principle on which Newton has founded his reasoning in the 45th Prop. 9th Sect.

If we compare the moon's orbit with this moveable ellipse, and use Newton's Formula; such part of her centrifugal force as depends on the mean area, or is of the form $\frac{h}{r^3}$, h including angles that vary slowly, must alone be deducted. This is as far as the two orbits can be compared. And thus we may apply Newton's formula to the solution of our problem. Hence

$$F^2r + Rv^2 - Rv^2 \propto \mu r + r^3D - 2h \int D r dt.$$

The part required will go out of this last term of itself. In the above proportion the square of the disturbing force is left out. Putting $a - ae \cos cv$ for r , and neglecting the powers $ae \cos cv$ above the first; the mean terms must have the same ratio as the coefficients of $e \cos cv$. Under the term—mean-terms that vary slowly are included.

If $p = \frac{n'}{n} = \sqrt{\frac{\mu'a^3}{\mu a'^3}}$ nearly; and if for r , dt , and v we

put their well known elliptic values; we shall have

$$-\frac{3\mu'r^4}{2r'^4} \cos 2w = -\frac{3\mu'a^4}{2r'^4} \cos 2w + \frac{3\mu'a^4}{r'^4} \cos 2w;$$

$$\frac{3\mu'}{r^3} \int r^2 dt \sin 2w = \frac{3\mu'}{r^3} \int r^2 dv \sin 2w = - \frac{3\mu' a^3}{2r^3(1-p)} \cos 2w \\ + \frac{6\mu' a^2 e}{r^3(1-2p)} \cos 2w;$$

$$\cos 2w = \cos(2w - cv) \cos cv = - \cos(2mt - 2cnt) e \cos cv, \text{ or} \\ = - \cos 2mt \cdot e \cos cv;$$

leaving out $\sin(2w - cv) \sin cv$, which affects the eccentricity, and not the progression of the apogee,

$$\cos 2w = \cos 2mt \cos cv \approx \cos 2mt \cos cv \text{ nearly.}$$

By expansion therefore, and making $c = \cos 2mt$; our proportion becomes, neglecting small quantities depending on e^2 &c.;

$$G^2 - F^2 e \cos cv \propto 1 - \frac{\mu' a^3}{2\mu r^3} - \left(1 - \frac{2\mu' a^3}{\mu r^3} - \frac{12\mu' a^3 \cdot 1 + \frac{9}{8} p \cdot c}{\mu r^3}\right)$$

$\times e \cos cv$ very nearly, after dividing by quantities common to each term.

$$\text{Now } p^2 = \frac{r'^2}{a^2} = \frac{\mu' a^3}{\mu a^3}. \text{ If, therefore, we make } r' = a', \text{ and}$$

divide by $1 - \frac{\mu' a^3}{2\mu a^3}$, we shall have

$$G^2 : 1 :: F^2 : 1 - \frac{3}{2} p^2 - 12 \left(1 + \frac{9}{8} p\right) p^2 c; \text{ or}$$

$$\frac{F}{G} = 1 - \frac{3}{4} p^2 - \frac{21}{2} p^4 - 6 \left(1 + \frac{9}{8} p\right) p^2 \cos 2mt - 9 \left(1 + \frac{9}{4} p\right) p^4 \cos 4mt,$$

by extracting the root, and restoring the value of c . If we multiply this result by ndt and integrate we shall have the anomaly.

Then we may correct the above value of $\frac{F}{G}$ by putting instead

of cnt the value so found. We may correct again for the resulting equations. And thus we shall at length find, putting in

the complex parts of the coefficients $\frac{1}{13}$, or rather $\frac{2}{27}$ for p ,

and $1 - \frac{3}{2} p^2$ for c , to which it is nearly equal :

$$\frac{F}{G} = 1 - \frac{3}{2} p^2 - 6(1 - 2p) p^2 \cos 2mt + 22 p^3 \cos 4mt \text{ very nearly.}$$

But we have taken $r' = a'$, $v' = n't$; making now $r' = a'(1 - e' \cos c'n't)$, $v' = n't + 2e' \sin c'n't$, and correcting the principal of the new equations that arise: we have

$$c = \frac{F}{G} = 1 - \frac{3}{2} p^2 - \frac{9}{2} p^2 e' \cos c'n't - 6(1-2p)p^2 \cos \frac{2mt}{2c} - 19(1-2p)p^4 e' \cos \frac{2mt}{2c} + (1-2p)p^2 e' \cos \frac{2mt}{2c} + 22p^3 \cos \frac{4mt}{4c}.$$

In this last step it is necessary to consider the effect of integration on the two terms whose arguments are $\frac{2mt}{2c}$ and $\frac{2mt}{2c}$. The coefficients have been reduced and simplified. Thus $\frac{1}{1-p} =$

$1 + p$ nearly. The constant part $1 - \frac{3}{2} p^2$ is very nearly its true value as given by observation. And, if, in this constant part, we had retained e' , the solar eccentricity; we might have determined the secular variation of the progression.

Multiplying now by ndt , and integrating; after reducing a little the coefficients; we have finally the anomaly equal

$$cnt - \frac{9}{2} p e' \sin c'n't + 3(1 - \frac{1}{2} p) p \sin \frac{2mt}{2c} + \frac{19}{8} (1-p) p e' \sin \frac{2mt}{2c} - (1+p) p \sin \frac{2mt}{2c} - 6p^2 \sin \frac{4mt}{4c} =$$

$$cnt - 19' 30'' \sin c'n't + 12^\circ 22' 35'' \sin \frac{2mt}{2c} + 25' 18'' \sin \frac{2mt}{2c} - 4' 38'' \sin \frac{2mt}{2c} - 1^\circ 55' 24'' \sin \frac{4mt}{4c}.$$

These are near their true values. And it must be remembered that we have neglected small quantities in the coefficients. Thus the method of Newton in the 9th Sect. is fully adequate to the determination of the progression of the apogee, including the equations that vary slowly, and even the secular equation also. Mr. Vince has not attempted to find any of these equations by Theory. And one of the methods adopted by the Commentators of Newton for finding the greatest equation is downright nonsense. Not to mention that the tangential force, which produces the greater effect is wholly unnoticed.

PROP. III.

To find the eccentricity of the moon's orbit.

We shall attempt this by means of the greatest and least distances.

By mechanics the second differential of the space, divided by the square of the differential of the time; is equal to the accelerative force. And the accelerative force along the radius-vector is the difference of the centripetal and centrifugal forces. Hence this equation :

$$\frac{d^2r}{dt^2} = \frac{h^2}{r^3} - \frac{\mu}{r^2} - D + \frac{2h}{r^3} \int D r dt.$$

This, expanding the disturbing forces and reducing the coefficients, becomes

$$\begin{aligned} \frac{d^2r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} - 3\left(1 + \frac{1}{2}p + \frac{1}{2}p^2\right) \frac{\mu'a}{r^3} \cos 2mt \\ + \frac{21}{2} \left(1 + \frac{3}{2}p + 2p^2\right) \frac{\mu'a}{r^3} e \cos 2mt = 0, \text{ reserving only} \end{aligned}$$

such terms as will give equations varying slowly in the final result; and neglecting, or allowing for, some small quantities in the coefficients depending on e .

Putting $r' = a$, multiplying by adr , and integrating; we obtain

$$\frac{dr^2}{dt^2} + \frac{h^2}{r^3} - \frac{2\mu}{r} + \frac{\mu}{a} \left\{ 1 - \frac{15}{4}(1 + 4p)pe^2 \cos 2mt \right\} = 0;$$

where $\frac{\mu}{a}$ is the arbitrary of the integration.

Now at an apse $\frac{dr}{dt} = 0$. But if we eliminate dt , it will make no sensible difference in the result. Therefore at an apse $\frac{dr}{dt} = 0$. Consequently the remainder of the preceding equation is equal to nothing. The two roots or values of r which it gives are the greatest and least distances; and half their difference, divided by half their sum, gives the eccentricity. The eccentricity therefore

$$= \left(1 - \frac{h^2}{\mu a} \left\{ 1 - \frac{15}{4}(1 + 4p)pe^2 \cos 2mt \right\} \right)^{\frac{1}{2}}.$$

But $h^2 = \mu a (1 - e^2)$, denoting by e , the constant part of the eccentricity. Therefore the eccentricity

$$= e, \left\{ 1 + \frac{15}{4} (1 + 4p) p \frac{e^2}{e'} \cos 2mt \right\}^{\frac{1}{2}} = e + \frac{15}{8} (1 + 4p) p \frac{e^2}{e'} \cos 2mt - \frac{3}{2} p^2 \frac{e^4}{e'^2} \cos 4mt.$$

If we correct this for e and cnt , we shall find

$$e = e + \frac{15}{8} (1 + 7p) p e \cos 2mt - 2p^2 e \cos 4mt \text{ nearly.}$$

And if we correct again, by putting for r' and v' their elliptic values; and again correct the resulting equations, remembering that if the corrections were made before the integration, we should, for the angles $2mt$, $2mt$, $4mt$ divide by $\frac{2}{3}$, 1, and 4 respectively instead of 2 by that process, we shall ultimately have

$$\begin{aligned} e = e + \frac{15}{8} (1 + 7p) p e \cos 2mt + 6p e e' \cos 2mt - p e e' \cos 2mt \\ + 4p e e' \cos c'n't - 2p^2 e \cos 4mt = \cdot 0548553 + \cdot 0117223 \cos 2mt \\ + \cdot 0004134 \cos 2mt - \cdot 0000689 \cos 2mt + \cdot 0002756 \cos c'n't \\ - \cdot 0006138 \cos 4mt. \end{aligned}$$

The authors referred to at the conclusion of the last problem are again erroneous in this. Their principles and methods, especially those of Vince, are false and absurd; and it will be well for the student to be reminded that they are so.

PROP. IV.

To find the mean distance and mean motion.

Half the sum of the roots of the equation found in the last Prop. i. e. half the co-efficient of the second term, the sign changed, divided by the third gives the mean distance, which therefore

$$= a + \frac{15}{4} (1 + 4p) p a e^2 \cos 2mt. \text{ Or, corrected, for } e, \text{ it becomes}$$

$$a + \frac{15}{4} (1 + 6p) p a e^2 \cos 2mt \text{ very nearly.}$$

The equation here found corresponds to the same equation of the eccentricity, and in fact arises from it. For let a and c only vary in the well known equation $\frac{h^2}{\mu} = a(1 - e^2)$; and we find $\frac{\delta a}{a} = 2e\delta e$, which is the very relation which subsists between the coefficients above found.

The preceding equation of the mean distance resulted from forces whose arguments were very nearly nt or $2nt$. But there is another part of the mean distance depending upon mean forces; and which corresponds to the variation of a in the equation $\frac{h^2}{\mu} = a(1 - e^2)$, when μ , h , and a vary, but not e .

Assuming that the body does not fall to the centre, nor fly off from it; we must have the mean part of the centripetal force equal to the mean part of the centrifugal; or

$$\frac{M}{a^3} = \frac{N}{a^3}; \quad Ma = N; \quad \text{and } M\delta a + a\delta M = \delta N; \quad \text{or}$$

$$\frac{\delta a}{a} = \frac{\delta N}{N} - \frac{\delta M}{M}.$$

The mean part of $\frac{h^2}{r^3} = \frac{3\mu'h}{r^3} \int \frac{r^2 dt}{r^3} \sin 2w = \frac{h^2}{a^3}$
 $- \left\{ \frac{15}{4} p(1 + \frac{3}{2}p + \frac{9}{4}p^2) + \frac{27}{2} p^3(1 + \frac{7}{3}p) - \frac{27}{8} p^3(1 + p) \right\}$
 $\frac{h^2}{a^3} e^2 \cos 2mt$ very nearly. Therefore the mean distance arising from the mean centrifugal force is

$$a - \frac{15}{4} \left(1 + \frac{21}{4} p \right) p a e^2 \cos 2mt.$$

The mean part of $\frac{\mu}{r^2} = \frac{\mu'r}{2r^3} - \frac{3\mu'r}{2r^3} \cos 2w - \frac{3\mu'rs^2}{4r^3} \cos 2v$,
 or of the centripetal force, is

$$\frac{\mu}{a^2} - \frac{\mu'a}{2r^3} - \frac{9\mu'a}{4r^3} e^2 \cos 2mt - \frac{9\mu'as^2}{4r^3} \cos 2mt = \frac{\mu}{a^2}$$

$$\left\{ 1 - \frac{3}{2} p^2 e' \cos c'n't - \frac{9}{4} p^2 e'^2 \cos 2c'n't - \frac{9}{4} p^2 e^2 \cos 2mt - \frac{3}{4} p^2 s^2 \cos 2mt \right\}.$$

This gives the mean distance

$$= a + \frac{3}{2} p^2 a e' \cos c'n't + \frac{9}{4} p^2 a e'^2 \cos 2c'n't + \frac{9}{4} p^2 a e^2 \cos 2mt + \frac{3}{4} p^2 a s^2 \cos 2mt.$$

But the disturbing force D has the term $\frac{3\mu'rs^2}{4r'^3}$; and s is not constant. Its greatest equation is $+\frac{3}{8}ps\cos 2mt$ nearly, as will hereafter appear.

This will give in the centripetal force the mean term $+\frac{9\mu'as^2}{16a'^3}p\cos 2mt$, and in the mean distance the term $-\frac{9}{16}p^2as^2\cos 2mt$.

Collecting the several terms together; we at length find the mean distance

$$= a + \frac{3}{2}p^2a'e'\cos 2c'n't + \frac{9}{4}p^2ae'^2\cos 2c'n't + \frac{81}{16}p^2ae^2\cos 2mt + \frac{3}{4}\left(1 - \frac{3}{2}p\right)p^2as^2\cos 2mt.$$

To find the mean motion we have $n = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}$, and

$\frac{\partial n}{\partial \mu} = \frac{1}{2} \frac{\partial \mu}{\mu} - \frac{3}{2} \frac{\partial a}{a}$. Hence, substituting for the variations of μ and a , n will become

$$n \cdot \left\{ 1 - 3p^2e'\cos c'n't - \frac{9}{2}p^2e'^2\cos 2c'n't - \frac{279}{32}p^2e^2\cos 2mt - \frac{3}{2}\left(1 - \frac{3}{2}p\right)p^2s^2\cos 2mt \right\}.$$

And the mean motion will become

$$\int n dt = nt - 3pe'\sin c'n't - \frac{9}{4}pe'^2\sin 2c'n't + \frac{35}{8}\left(1 + \frac{3}{2}p\right)pe^2\sin 2mt + \frac{3}{4}\left(1 - \frac{3}{2}p\right)p^2s^2\sin 2mt = nt - 12'57''\sin c'n't - 10''\sin 2c'n't + 3'46''\sin 2mt + 1'22''\sin 2mt.$$

Some writers have contrived to make the equation depending on $\sin 2mt$ come out right, without taking account of the tangential disturbing force. Vince has made it a little more than a twentieth part of what it ought to be, and of a contrary sign. He finds an equation $A \sin 2w$: and then takes the mean longitude instead of the true; leaving out the equations $B \sin 2w$,

$c \sin 2w$: which, when the true longitude is converted into the mean, would each give such an equation as $D \sin 2mt$, amounting, if I recollect rightly, to ten times what he makes it. In fact he has left out the tangential force, which gives in the mean distance an equation of the third order, in the mean motion an equation of the second. He has also left unnoticed, or did not perceive the change in the mean distance, arising from the change in the eccentricity. This gives likewise in the mean motion an equation of the second order, nearly equal to the former, but of a contrary sign. The difference of the two is an equation of the third order, which is just what is found to take place in the direct solution of the problem.

It ought to be remembered that the epocha may vary also by a change in the disturbing forces. But it does not appear that that variation would be sensible by the method we have pursued; which has thrown it rather upon the mean motion.

If we would include all terms of the third order, we must take account of a term depending on the angle $2gv$. Thus, the quan-

$$\begin{aligned} \text{tity } \mu r + r^2 D - 2h \int D' r dt \text{ will give } & - \frac{3\mu' r^2 s^2}{4r'^3} \cos 2gv + \\ \frac{3\mu' h s^2}{2r'^3} \int r^2 dt \sin 2gv = & - \frac{3\mu' a^2 s^2}{2a'^3} \cos 2gv + \frac{7\mu' a^4 s^2}{2a'^3} e \cos 2gv \\ = \frac{5\mu' a^4 s^2}{a'^3} \cos 2gnt \cdot e \cos cv \text{ nearly. And this gives in } \frac{F}{G} \text{ the} \\ \text{term } & - \frac{5}{2} p^2 s^2 \cos 2gnt. \text{ In the anomaly } cnt \text{ it gives } - \frac{5p^2 s^2}{4(g-c)} \\ \sin 2gnt = & - \frac{1}{2} s^2 \sin 2gnt \text{ very nearly} = - 13' 56'' \sin 2gnt. \end{aligned}$$

$$\begin{aligned} \text{In } D - \frac{2h}{r^3} \int D' r dt \text{ we shall also find the terms } & - \frac{3\mu' r s^2}{4r'^3} \\ \cos 2gv + \frac{3\mu' h s^2}{2r'^3 r^3} \int r^2 dt \sin 2gv = & - \frac{3\mu' a s^2}{2a'^3} \cos 2gnt + \frac{21\mu' a s^2}{4a'^4} \\ e \cos 2gnt. \text{ This gives in } \frac{dr^2}{dt^2} \text{ the term } & \frac{\mu}{a} \cdot \frac{15p^2 s^2}{8(g-c)} e^2 \cos 2gnt \\ = \frac{\mu}{a} \cdot \frac{5}{6} s^2 e^2 \cos 2gnt. \end{aligned}$$

$$\begin{aligned} \text{In } e \text{ therefore it gives the term } & - \frac{5}{12} s^2 e \cos 2gnt = \\ - 0001854 \cos 2gnt. \end{aligned}$$

Consequently the anomaly
 $= cnt - 19' 30'' \sin c'n't + 12^\circ 22' 35'' \sin 2mt + 25' 18''$
 $\sin 2mt - 4' 38'' \sin 2mt - 1^\circ 55' 24'' \sin 4mt - 13' 56''$
 in $2gnt$.

And the eccentricity
 $= .0548553 + .0117223 \cos 2mt + .0004134 \cos 2mt -$
 $.0000689 \cos 2mt + .0002756 \cos c'n't - .0006138 \cos 4mt -$
 $.0001854 \cos 2gnt$.

PROP. V.

To find the equations of the centre.

These are $2e(1 - \frac{1}{8} e^2) \sin cnt + \frac{5}{4} e^2 \sin 2cnt + \frac{13}{12} e^3 \sin 3cnt$.

But if Δ denote the sum of the equations of the anomaly and mean motion, and ϵ the sum of the equations of e ; then we must put $cnt + \Delta$ and $e + \epsilon$ instead of cnt and e . And by Taylor's theorem, we shall have to add to the original equations those arising from the

expansion of $2\epsilon \sin cnt + 2\Delta e \cos cnt + \frac{5}{2} \epsilon e \sin 2cnt + \frac{5}{2} \Delta e^2 \cos 2cnt + \frac{13}{4} \epsilon e^2 \sin 3cnt + \frac{13}{4} \Delta e^3 \cos 3cnt + 2\Delta \epsilon e \cos cnt - \Delta^2 e \sin cnt + \&c$.

And thus we shall find this whole function to be
 $+ 6^\circ 17' 9'' \sin cnt + 12' 56'' \sin 2cnt + 37'' \sin 3cnt + 1^\circ 21' 37''$
 $\sin 2mt + 3' 18'' \sin 2mt - 39'' \sin 2mt + 2' 32'' \sin cnt - 1' 37''$
 $\sin cnt + 5' 33'' \sin 2mt + 42'' \sin 4mt + 20'' \sin 2mt - 1' 23''$
 $\sin 2gnt$.

PROP. VI.

To find the variation, or deviation from an elliptic movement.

We shall first suppose the undisturbed orbit a circle, and

the sun always at his mean distance. And we resume the equation

$$\frac{d^2 r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} + D - \frac{2h}{r^3} \int D' r dt = 0.$$

$$\text{or } \frac{d^2 r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} - \frac{\mu' r}{2a'^3} - 3(1 + \frac{1}{2} p) \frac{\mu' a}{a'^3} \cos 2mt = 0.$$

Let $r = a + \rho$. Putting this value for r , and making that part of the equation which does not contain ρ separately equal to nothing; we have

$$-\frac{h^2}{a^3} + \frac{\mu}{a^2} - \frac{\mu' a}{2a'^3} = 0;$$

$$\frac{d^2 \rho}{dt^2} + \left(\frac{3h^2}{a^4} - \frac{2\mu}{a^3} - \frac{\mu'}{2a'^3} \right) \rho - 3(1 + \frac{1}{2} p) \frac{\mu' a}{a'^3} \cos 2mt = 0,$$

If μ_1 be the absolute force in the disturbed orbit: then $\mu_1 = \mu - \frac{\mu' a^3}{2a'^3}$, and $\mu = \mu_1 (1 + \frac{\mu' a^3}{2\mu_1 a'^3}) = (1 + \frac{1}{2} p^2) \mu_1$. Whence by substituting for h and μ , we find the coefficient of ρ to be $(1 - \frac{3}{2} p^2) \frac{\mu_1}{a^3} = (1 - \frac{3}{2} p^2) n^2$. Therefore

$$\frac{d^2 \rho}{n^2 dt^2} + (1 - \frac{3}{2} p^2) \rho - 3(1 + \frac{1}{2} p) p^2 a \cos 2mt = 0.$$

And integrating, this gives

$$\rho = - \frac{3(1 + \frac{1}{2} p) p^2}{4(1-p)^2 - 1 + \frac{3}{2} p^2} a \cos 2mt = - (1 + \frac{11}{3} p) p^2 a \cos 2mt \text{ nearly.}$$

Again, we have found

$$r^2 dv = h dt + dt \int D' r dt; \text{ or } dv = \frac{h dt}{r^2} + \frac{at}{r^2} \int D' r dt = \frac{h dt}{r^2} + \frac{3n^2 dt}{4n(1-p)} \cos 2mt. \text{ And putting } v = v_1 + u; \text{ we have, } v_1, \text{ being the value of } v \text{ in the undisturbed orbit, and consequently}$$

$$dv_1 = \frac{h dt}{a^3},$$

$$du = - \frac{2h p dt}{a^3} + \frac{3n^2 dt}{4n(1-p)} \cos 2mt = \frac{11}{4} (1 + \frac{32}{11} p) \frac{n^2}{n} dt \cos 2mt$$

very nearly. Therefore $u = \frac{11}{8} (1 + 4p) p^2 \sin 2mt.$

But if we would include the term depending on $\cos 4mt$ in the value of u , we must put in the disturbing forces the values of φ and u already found; and we shall find to be added to du the term

$$\frac{3hdt}{a^3} \varphi^2 + \frac{5}{4} p^4 n dt \cos 4mt = \frac{15}{4} p^4 n dt \cos 4mt \text{ nearly.}$$

This gives in the value of u the term $p^4 \sin 4mt$.

If the undisturbed orbit be an ellipse of small eccentricity, its form must be similarly changed by the sun's action. The equations in this case must become those above found when e is made equal to nothing. Those equations respect only the situation of the moon with respect to the sun, without any regard to the place of its apside, node, &c. From these, and some other considerations, I conclude that if in the values of φ and u above found, we put the true elliptic longitudes for the mean, or $v - v' = w$ for $mt = nt - n't$; we shall have, either accurately, or at least very nearly the variation. I regret, however, that I have not been able to demonstrate this, or to determine with absolute certainty what the true variation in this case is. We shall, however, upon this principle, which is that adopted by Vince, have the variation very near the truth; and more near, if we add the greatest equation of the centre to the value of v . This equation is $+ 6pe \sin 2mt$ nearly.

$$\varphi = -\left(1 + \frac{11}{3} p\right) p^3 a \cos 2w;$$

$$u = \frac{11}{8} (1 + 4p) p^3 \sin 2w + p^4 \sin 4w;$$

$$v = nt + 2e \sin cnt + 6pe \sin 2mt;$$

$$v' = n't + 2e' \sin c'n't; \quad a' \text{ or } r' = a' (1 - e' \cos c'n't).$$

Making these substitutions, we shall find

$$\begin{aligned} u = & \frac{11}{8} (1 + 4p) p^3 \sin 2mt - \frac{11}{4} (1 + 4p) p^2 e \sin 2mt + \\ & \frac{11}{4} (1 + 4p) p^2 e \sin 2mt + 5(1 + 4p) p^2 e' \sin 2mt - \frac{3}{4} (1 + 4p) p^3 e' \\ & \sin 2mt + 8p^3 e \sin 4mt + (p + 22e^2) p^3 \sin 4mt = 34' 22'' \sin 2mt - \\ & 3' 44'' \sin 2mt + 3' 44'' \sin 2mt + 2' 6'' \sin 2mt - 19'' \sin 2mt + \\ & 38'' \sin 4mt + 11'' \sin 4mt. \end{aligned}$$

This is the deviation from an elliptic movement, neglecting

the disturbing forces of the fourth order, depending on the angles w and $3w$.

PROP. VII..

To find the equations depending on the angle w .

For this purpose we have the equation

$$\frac{d^2 r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} - \frac{\mu' r}{2r^3} - \frac{3\mu' r}{2r^3} \cos 2w + \frac{2h}{r^3} \int \frac{3\mu' r^2 dt}{2r^3} \sin 2w \\ - \frac{9\mu' r^2}{8r^4} \cos w + \frac{2h}{r^3} \int \frac{3\mu' r^2 dt}{8r^4} \sin w = 0, \text{ or}$$

$$\frac{d^2 r}{dt^2} - \frac{h^2}{r^3} + \frac{\mu}{r^2} - \frac{\mu' r}{2r^3} - \frac{3\mu' r}{r^3} \cos 2w - \frac{15\mu' r^2}{8r^4} \cos w = 0,$$

very nearly. And hence, as in the last prop. denoting by δr the variation of r we are in quest of; we shall have

$$\frac{d^2 \delta r}{dt^2} + (1 - \frac{3}{2} p^2) \delta r - \frac{3\mu'}{n^2 r^3} \delta(r \cos 2w) - \frac{15\mu' a^2}{8n^2 r^4} \cos w = 0.$$

$$\text{We have also } \frac{dv}{dt} = \frac{h}{r^2} - \frac{1}{r^2} \int \frac{3\mu' r^2 dt}{8r^4} \sin w = \frac{h}{r^2} +$$

$$\frac{3\mu' r^3}{8hr^4} \cos w \text{ very nearly. Hence}$$

$$\frac{d^2 \delta v}{dt^2} = -2 \frac{\delta r}{a} + \frac{3\mu' a}{8n^2 r^4} \cos w.$$

We here select such terms as are increased by integration; or rather such as are increased much more than any others would be.

Leaving out the term $\delta(r \cos 2w)$; the first equation integrated gives

$$\delta r = \frac{15}{16} (1 + p) p \frac{a^2}{a^2} \cos mt - \frac{5}{4} \frac{a^2}{a^2} e' \cos^c mt; \text{ and the second}$$

$$\delta v = -\frac{15}{8} (1 + 5p) p \frac{a}{a^2} \sin mt + \frac{5}{2} \frac{a}{a^2} e' \sin^c mt.$$

Now by means of these values, the term $\delta(r \cos 2w) = \delta r \cos 2w - 2r \delta v \sin 2w$ will give

$$\delta r = 3p^2 \frac{a^2}{a^2} \cos mt, \quad \delta v = -11p^2 \frac{a}{a^2} \sin mt.$$

Therefore, ultimately, we have

$$2r = \frac{15}{16} (1 + 6p) p \frac{a^2}{a'} \cos mt - \frac{5}{4} \frac{a^2}{a'} e' \cos^c mt,$$

$$2v = -\frac{15}{8} (1 + 10p) p \frac{a}{a'} \sin mt + \frac{5}{2} \frac{a}{a'} e' \sin^c mt = -2' 6'' \sin mt + 21'' \sin^c mt.$$

PROP. VIII.

To find the reduction to the ecliptic.

If AC be the moon's orbit, AB the ecliptic, and CB perp. to AB; then $\cos A = \cot AC \tan AB$; or $\sin AC \cos AB \cos A = \cos AC \sin AB$.

Let $AC = AB + s$, $\sin A = s$.

Then $\cos A = 1 - \frac{1}{2}s^2$; and by substitution in the preceding equation, we have

$$\sin AB \cos AB \cos A + \cos^2 AB \cos A \cdot s = \cos AB \sin AB - \sin^2 AB \cdot s.$$

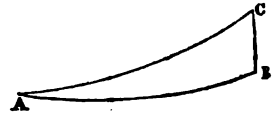
This gives $s = \frac{1}{4}s^2 \sin 2AB = \frac{1}{4}s^2 \sin 2gv$; which, with the sign changed, is the reduction.

The two principal equations of the node, which will hereafter be found; are $-\frac{3}{8}p \sin 2mt - \frac{3}{8}p^2 \sin 2mt$. Those of the inclination are $+\frac{3}{8}ps \cos 2mt - \frac{3}{8}p^2s \cos 2mt$. Theref. $2gv = 2gnt + 4e \sin cnt + \frac{9}{2}p^2 \sin 2mt + \frac{3}{4}p \sin 2mt + 12pe \sin^c mt$; adding three of the principal equations to the elliptic value of v , and deducting the equation of the node.

$$\text{Also } s \text{ becomes } s(1 + \frac{3}{8}p \cos 2mt - \frac{3}{8}p^2 \cos 2mt).$$

Substituting these values, we shall find the reduction

$$= -\frac{1}{4}s^2 \sin 2gnt + \frac{1}{2}s^2e \sin 2gnt - \frac{1}{2}s^2e \sin^c 2gnt - \frac{3}{16}ps^2$$



$$\begin{aligned} \sin 2mt - \frac{3}{2} p s^c \sin 2mt - \frac{3}{2} p s^c \sin 2mt - \frac{3}{2} p s^c \sin 2mt - \frac{3}{2} p s^c \sin 2mt = \\ - 6' 54'' \sin 2gnt + 45'' \sin 2gnt - 45'' \sin 2gnt - 24'' \sin 2mt \\ - 10'' \sin 2mt - 10'' \sin 2mt - 7'' \sin 2mt. \end{aligned}$$

Mr. Vince has made the coefficient of $\sin 2gnt$ as great affirmative as it ought to be negative, and has deduced it by a principle utterly inapplicable.

If we now collect all the equations together: we shall have the true longitude on the ecliptic, or

$$\begin{aligned} v = nt + 6^\circ 17' 9'' \sin cnt + 12' 56'' \sin 2cnt + 37'' \sin 3cnt \\ + 1^\circ 17' 53'' \sin 2mt + 39' 31'' \sin 2mt - 12' 57'' \sin c'n't - \\ 6' 54'' \sin 2gnt + 4' 4'' \sin 2mt + 2' 6'' \sin 2mt - 19'' \sin 2mt + \\ 3' 18'' \sin 2mt - 39'' \sin 2mt + 3' 46'' \sin 2mt + 1' 15'' \sin 2mt \\ + 2' 32'' \sin cnt - 1' 37'' \sin cnt - 2' 6'' \sin mt - 38'' \sin 2gnt - \\ 45'' \sin 2gnt + 21'' \sin mt - 10'' \sin 2c'n't + 42'' \sin 4mt + 38'' \\ \sin 4mt + 11'' \sin 4mt - 10'' \sin 2mt - 10'' \sin 2mt. \end{aligned}$$

Most of these are very near the truth; and none of them differ very widely from it. In finding them, it must be remembered, we have neglected small quantities in the coefficients. After all they are as accurate or more so, than the first theories by the direct method. But the object which I have here in view is not very accurate results, but to found a theory upon true principles.

PROP. IX.

To find the Moon's Latitude.

This will be most easily done by the direct method, and in terms of the true longitude. Then, after the longitude is found the latitude may be found also.

The force urging the moon perp. to the ecliptic is $\frac{\mu \sigma}{r^3} + D$.

Therefore by Mechanics,

$$\frac{d^2(r\sigma)}{dt^2} = -\frac{\mu\sigma}{r^2} - D; \text{ or } \frac{d^2(r\sigma)}{dt^2} + \frac{\mu\sigma}{r^2} + D = 0.$$

$$\text{Hence } \frac{d^2r}{dt^2} \sigma + 2 \frac{dr}{dt} \cdot \frac{d\sigma}{dt} + r \frac{d^2\sigma}{dt^2} + \frac{\mu\sigma}{r^2} + D = 0.$$

$$\text{But } \frac{d^2r}{dt^2} = r \frac{dv^2}{dt^2} - \frac{\mu}{r^2} - D; \text{ therefore}$$

$$r \frac{d^2\sigma}{dt^2} + 2 \frac{dr}{dt} \cdot \frac{d\sigma}{dt} + r\sigma \frac{dv^2}{dt^2} - D\sigma + D = 0.$$

If we change the independent variable from t to v ; we have

$$\frac{d^2\sigma}{dt^2} = \frac{d^2\sigma}{dv^2} - \frac{d\sigma}{dv} \frac{d^2v}{dt^2} = \frac{d^2\sigma}{dv^2} - \frac{1}{dt^2} \left(2 \frac{dr}{r} \frac{d\sigma}{dv} - \frac{Dr \frac{d\sigma}{dv} dt}{h + \int Dr dt} \right).$$

Therefore our equation becomes

$$r \frac{d^2\sigma}{dv^2} + r\sigma - D\sigma \frac{dt^2}{dv^2} + \frac{Dr^2 \frac{d\sigma}{dv} \frac{dt}{dv}}{h + \int Dr dt} + D \frac{dt^2}{dv^2} = 0.$$

Eliminating dt , and dividing by $r dv^2$, it is

$$\frac{d^2\sigma}{dv^2} + \sigma + \frac{\left(D - D\sigma + D \frac{d\sigma}{dv} \right) r^3}{\left(h + \int Dr dt \right)^2} = 0.$$

Or, if we neglect the square of the disturbing force; we have simply

$$\frac{d^2\sigma}{dv^2} + \sigma + \left(D - D\sigma + D \frac{d\sigma}{dv} \right) \frac{r^3}{h^2} = 0.$$

And restoring the values of D , D , D ; taking only the larger terms

$$\frac{d^2\sigma}{dv^2} + \sigma + \frac{3\mu' r^4}{2h^2 r^5} (\sigma + \sigma \cos 2w - \frac{d\sigma}{dv} \sin 2w) = 0.$$

$$\text{Putting } r = a(1 - \epsilon \cos cv + \frac{5}{4} p^2 \cos 2\pi v), \quad r' = a'(1 - \epsilon'$$

$\cos c'v') = a'(1 - \epsilon' \cos c'pv)$, $v' = pv + 2\epsilon' \sin c'pv$, which last is found by equating the values of nt and $n't$ expressed in terms of v and v' , we shall find

$$\frac{3\mu' r^4 \sigma}{2h^2 r^5} = \frac{3}{2} p^2 \sigma + \frac{9}{4} p^2 \epsilon' \sin gv + \frac{9}{4} p^2 \epsilon' \sin gv - \frac{15}{4} p^2 \epsilon^2$$

$$\sin \frac{2}{g} \frac{2cv}{g} + \frac{15}{4} p^2 s \sin \frac{2mv}{g} - 3p^2 se \sin \frac{gv}{c} - 3p^2 se \sin \frac{gv}{c}.$$

$$\frac{3\mu' r^4}{2h^2 r^3} (\sigma \cos 2w - \frac{d\sigma}{dv} \sin 2w) = -\frac{3}{2} p^2 s \sin \frac{2mv}{g} - \frac{21}{4} p^2 se'$$

$$\sin \frac{2mv}{g} + \frac{3}{4} p^2 se' \sin \frac{2mv}{g} + 3p^2 se \sin \frac{2mv}{g, c} + 3p^2 se \sin \frac{2mv}{g} - \frac{15}{4} p^2 se^2 \sin \frac{2mv}{2c, g}.$$

Collecting all the terms of these developments together ;

$$\frac{d^2 \sigma}{dv^2} + (1 + \frac{3}{2} p^2) \sigma - \frac{3}{2} p^2 (1 - \frac{5}{2} p) s \sin \frac{2mv}{g} + \frac{9}{4} p^2 se' \sin \frac{gv}{c} + \frac{9}{4} p^2 se' \sin \frac{gv}{c} - \frac{15}{4} p^2 se^2 \sin \frac{2cv}{g} - \frac{21}{4} p^2 se' \sin \frac{2mv}{g, c} + \frac{3}{4} p^2 se' \sin \frac{2mv}{g} - 3p^2 se \sin \frac{gv}{c} - 3p^2 se \sin \frac{gv}{c} + 3p^2 se \sin \frac{2mv}{g, c} + 3p^2 se \sin \frac{2mv}{g} - \frac{15}{4} p^2 se^2 \sin \frac{2mv}{2c, g} = 0.$$

Assuming $\sigma = s \sin \frac{gv}{c} + \Lambda s \sin \frac{2mv}{g}$; substituting this value in the preceding equation, and comparing the coefficients of the same angles : we find first $g^2 = 1 + \frac{3}{2} p^2$, or $g = 1 + \frac{3}{4} p^2$.

Putting this value for g , and $1 + \frac{3}{2} p^2$ for c in the coefficients, and reducing and simplifying them ; we shall have at length

$$\sigma = s \sin \frac{gv}{c} + \frac{3}{8} (1 - \frac{1}{6} p) ps \sin \frac{2mv}{g} - \frac{9}{8} p se' \sin \frac{gv}{c} + \frac{9}{8} p se' \sin \frac{gv}{c} + \frac{5}{12} se^2 \sin \frac{2cv}{g} + \frac{7}{8} p se' \sin \frac{2mv}{g, c} - \frac{3}{8} p se' \sin \frac{2mv}{g} + 3p^2 se \sin \frac{gv}{c} - p^2 se \sin \frac{gv}{c} - \frac{7}{2} p^2 se \sin \frac{2mv}{g, c} + \frac{6}{5} p^2 se \sin \frac{2mv}{g} - p se^2 \sin \frac{2mv}{2c, g}.$$

The latitude itself is $\sigma + \frac{1}{6} \sigma^2 = \sigma + \frac{1}{8} s^2 \sin \frac{gv}{c} - \frac{1}{24} s^2 \sin \frac{3gv}{c}$ nearly. And v is the longitude on the orbit in this proposition, and therefore $= v + \frac{1}{4} s^2 \sin \frac{2gv}{c}$, by applying the re-

duction, and reckoning v on the ecliptic. Consequently $\sin gv$ becomes $\sin gv + \frac{1}{8} s^3 \sin 3gv + \frac{1}{8} s^3 \sin gv$; and the latitude =

$$\begin{aligned} \sigma + \frac{1}{4} s^3 \sin gv + \frac{1}{12} s^3 \sin 3gv &= 5^\circ 8' 46'' \sin gv + 8' 32'' \\ \sin 2mv - 26'' \sin gv + 26'' \sin gv^c + 23'' \sin 2cv + 20'' \sin \\ 2mv - 9'' \sin 2mv^c + 12'' \sin 3gv + 17'' \sin gv - 6'' \sin gv - \\ 20'' \sin 2mv + 7'' \sin 2mv - 4'' \sin 2mv. \end{aligned}$$

These may be very easily expressed in terms of the mean longitude, by substituting for v its value given in Prop. 8. But we should by this step introduce some additional equations.

To find the inclination, we resume the equation $\frac{d^2\sigma}{dv^2} + \sigma + M = 0$, putting M for the function of the disturbing forces. Multiplying by $2d\sigma$, and integrating; we obtain

$$\frac{d\sigma^2}{dv^2} + \sigma^2 - \gamma^2 + \int 2Md\sigma = 0,$$

where γ is the arbitrary of the integration.

At 90° from the node $\frac{d\sigma^2}{dv^2} = 0$, and σ becomes the sine of the inclination.

Hence in this situation $\sigma = (\gamma^2 - \int 2Md\sigma)^{\frac{1}{2}}$.

$$\begin{aligned} \text{But } \int 2Md\sigma &= -\frac{3}{4} \left(1 - \frac{3}{4} p\right) p s^3 \cos 2mv + \frac{3}{4} p^2 s^3 \cos 2gv \\ &+ \frac{3}{4} p^2 s^3 \cos 2mv. \end{aligned}$$

Whence it appears that $\gamma = s$, and the sine of the inclination

$$\sigma = s + \frac{3}{8} \left(1 - \frac{3}{4} p\right) p s \cos 2mv - \frac{3}{8} p^2 s \cos 2mv + \frac{3}{8} p^2 s \cos 2gv.$$

And the inclination itself = $\sigma + \frac{1}{6} \sigma^3 = s + \frac{1}{6} s^3$ + the above equations; which, computing their numerical values, are

$$8' 12'' \cos 2mv - 35'' \cos 2mv + 35'' \cos 2gv.$$

Representing these coefficients by A, B, C ; and the coefficients of the equations of the node by A', B', C' : we must have the sine of the latitude $= \sigma = s(1 + A \cos 2mv + B \cos 2mv + C \cos 2gv)$

$$\begin{aligned} & \times \sin(gv - A' \sin 2mv - B' \sin 2mv - C' \sin 2gv) = s \sin gv \\ & - \frac{1}{2}(A + A')s \sin 2mv + \frac{1}{2}(A - A')s \sin 2mv - \frac{1}{2}(B + B')s \\ & \sin 2mv + \frac{1}{2}(B - B')s \sin 2mv - \frac{1}{2}(C + C')s \sin gv + \\ & \frac{1}{2}(C - C')s \sin 2gv. \end{aligned}$$

Hence $A + A' = 0, B - B' = 0, C - C' = 0; A' = -\frac{3}{8}p(1 - \frac{3}{4}p),$
 $B' = -\frac{3}{8}p^2, C = \frac{3}{8}p^2.$ Therefore the equations of the node
 are $-\frac{3}{8}(1 - \frac{3}{4}p)p \sin 2mv - \frac{3}{8}p^2 \sin 2mv + \frac{3}{8}p^2 \sin 2gv$
 $= -1^0 31' 1'' \sin 2mv - 7' 13'' \sin 2mv + 7' 13'' \sin 2gv.$

These are only a few of the greatest equations of the inclination and node; and have been investigated here, merely because we had occasion for them further back, where they were given without investigation.

PROP. X.

To find the Equatorial horizontal Parallax.

Making $r = r + p$, and giving to r its elliptic value, and to p its deviation from that value; we have

$$\begin{aligned} \frac{a}{r} &= \frac{a}{r} - \frac{ap}{r^2} = 1 + e \cos cnt + e^2 \cos 2cnt + (1 + 2e \cos cnt) \\ & \times \left(\frac{5}{4}p^2 \cos 2w + \frac{39}{16}(1 + \frac{31}{13}p)pe \cos 2mt \right) = 1 + e \cos cnt \\ & + e^2 \cos 2cnt + \frac{39}{16}(1 + \frac{31}{13}p)pe \cos 2mt + \frac{5}{4}p^2 \cos 2mt + \\ & \frac{39}{8}pe^2 \cos 2mt. \end{aligned}$$

This quantity is proportional to the parallax, the constant

part of which is $57' 12''$. Therefore multiplying the last series of equations by this quantity, we shall have the parallax equal to

$$57' 12'' + 3' 8'' \cos cnt + 10'' \cos 2cnt + 40'' \cos 2mt + 26'' \cos 2mt + 2'' \cos 2mt.$$

We remarked in Prop. 2. how the secular variation of the progression of the apogee, arising from the variation of the earth's eccentricity might be determined. The secular variation of the mean distance and mean motion might have been determined in like manner. And the secular variation of the regression of the node after the usual manner.

The methods adopted in this Theory are concise and easy; and, if cultivated as the direct method has been, might perhaps be brought to the same degree of accuracy.

The student who may wish to pursue this subject, and to consult the direct methods of managing it; must be referred to the Memoirs of Clairault, D'Alembert, Euler, T. Simpson, Mayer, La Place, and Damoiseau.

ARTICLE IX.

By the Rev. Mr. BRONWIN.

To find the attraction of an oblate spheroid of small ellipticity on a point without it.

We shall find the sum of each particle of matter attracting divided by the distance of that particle from the point attracted. Then it is obvious and well known that the partial differential co-efficients of this sum or of v , relative to the co-ordinates of the point attracted, will be the attractions in the directions of those co ordinates.

Let CE , CB be the semi-polar and semi-equatorial diameters, D the point attracted, mqn a plane perpendicular to CD , pq any line in this plane drawn from CD to the surface of the spheroid. And let $CB = t$, $CE = c$, $CP = a$, $\angle PCB = n$, $pq = r$, $\angle qp\pi = \phi$, $\pi = y'14159$, and $t = c(1 + \epsilon)$.

being put for the co-ordinates of the point D to the origin C , or
 $\cos \pi = \frac{a}{d}$, $\sin \pi = \frac{h}{d}$; we shall have for the whole spheroid

$$v = \frac{4\pi c^3}{3} \left\{ \frac{1 + 2\varepsilon}{d} + (a^2 - 2h^2) \frac{c^2}{5d^3} \right\}.$$

By taking account of the terms depending on δr^2 and ε^2 , we might easily find that part of v which depends upon the square of the ellipticity.

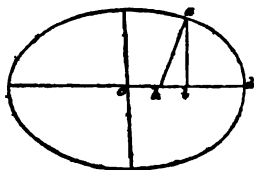
If the density of the spheroid be variable, and composed of a number of homogeneous shells indefinitely thin; we must take the differential of v , making c to vary, or c and ε , if the surfaces of the shells be dissimilar: this being multiplied by ρ , a variable density, and the integral taken; will give v for a spheroid of variable density.

But it is plain that $\frac{4}{3} \pi \int \rho d(c^3 \cdot 1 + 2\varepsilon)$ is equal to the mass of the spheroid, the same as when the density is uniform. The difficulty therefore is reduced to the finding of

$$\frac{4}{15} \pi \int \rho d(c^2 \varepsilon).$$

The spheroid must be supposed capable of changing its form by revolving around its axis. We will suppose it therefore fluid; in which case the resultant of all the forces, centripetal and centrifugal, will, at the surface, be perpendicular to the surface. This is known from the first principles of hydrostatics.

Let ak be perpendicular to the surface, al perpendicular to cd the semi-transverse axis, and the centrifugal force at the distance unity from the axis $= p$. Then at the distance cl it is $cl \cdot p$. And D being now supposed to coincide with a , let the whole forces in the directions al , ak be P and H . Also let m = mass of the spheroid, $x = \frac{4}{15} \pi \int \rho d(c^2 \varepsilon)$.



Then $al : lk :: P : H$, and $lk = \frac{H}{P} \cdot al$. But by conics $cl : kl :: 1 + 2\varepsilon : 1$, and $lk = \frac{cl}{1 + 2\varepsilon}$. Equating these values of lk , and putting for al and cl their values h and a ; we have

$$(1 + 2\varepsilon) \frac{H}{a} = \frac{P}{h}. \text{ Now}$$

$$H = - \frac{dv}{da} - ap = m \frac{a}{d^3} + x(3a^2 - 12h^2) \frac{a}{d^7} - ap;$$

$$P = - \frac{dv}{dh} = m \frac{h}{d^3} + x(9a^2 - 6h^2) \frac{h}{d^7}.$$

Substituting these values of H and P , and neglecting the powers of ε above the first; we have

$$6 \frac{x}{d^5} + p - 2\varepsilon \frac{m}{d^3} = 0.$$

Let n = the centrifugal force at the equator divided by gravity $= pc(1 + \varepsilon) \div \frac{m}{(1 + 2\varepsilon)c^2} = \frac{c^2 p}{m}$ nearly. Then $p = n \frac{m}{c^2}$. And $d = c$ nearly.

Hence our equation becomes

$$6 \frac{x}{c^2} + nm - 2m\varepsilon = 0, \text{ and}$$

$$x = (\varepsilon - \frac{1}{2}n) \frac{c^2 m}{3}.$$

Consequently in the case of a variable density, we have

$$V = \frac{m}{d} + (\varepsilon - \frac{1}{2}n) \frac{c^2 m}{3} \cdot \frac{a^2 - 2h^2}{d^5}.$$

The attractions, which in the directions of the co-ordinates a and h are $-\frac{dv}{da}$ and $-\frac{dv}{dh}$ respectively; are now easily found.

It may be observed here, that since $\frac{a}{d} = \cos n = \cos DCB$, $\frac{h}{d} = \sin n = \sin DCB$; $\frac{a^2 - 2h^2}{d^3} = \frac{1}{d^3} (1 - 3 \sin^2 DCB)$. When therefore $\sin^2 DCB = \frac{1}{3}$, the second term in V vanishes, and

the attraction becomes that of a sphere of the same mass.

When the point D is within the spheroid; in integrating between limits, we must make $\sqrt{d^2 - 2dc} + c^2 = c - d$, and not $d - c$, as when D was without it. And moreover for the inscribed sphere we must have $+d - c \cos \varphi$ instead of $-d +$

$c \cos \varphi$; and must integrate first from $\varphi = 0$ to $d - c \cos \varphi = 0$ afterwards as before. Thus for the protuberant matter we have

$$v = \frac{4\pi a}{3} \left\{ 2c^2 + (\cos^2 n - 2 \sin^2 n) \frac{d^2}{5} \right\},$$

and for the sphere $v = 2\pi (c^2 - \frac{1}{3}d^2)$. For the whole spheroid therefore

$$v = \frac{2\pi}{3} \left\{ 3c^2 (1 + \frac{4}{3}\epsilon) - a^2 (1 - \frac{2}{5}\epsilon) - h^2 (1 + \frac{4}{5}\epsilon) \right\}.$$

$$\text{Then } -\frac{dv}{da} = \frac{4\pi a}{3} (1 - \frac{2}{5}\epsilon), \quad -\frac{dv}{dh} = \frac{4\pi h}{3} (1 + \frac{4}{5}\epsilon).$$

These are precisely the attractions of a spheroid similar to the given one, whose surface passes through the point D. And thus we have obtained the attraction within very little compass whether the point attracted be situated without or within the spheroid.

ARTICLE X.

On the Expansion of Functions.

By the Rev. Mr. BRICE BRONWIN.

In the investigation of general formulæ for the expansion of functions, it has been usual to assume as a certain fact; that that expansion would proceed by integer powers of the leading quantity. This is the case in the Theorems of Maclaurin and Taylor. Whereas it is only a few quantities which can be thus developed. Many will proceed by fractional powers. And some by no power at all of the leading quantity, except as it is augmented or diminished by some given quantity. We shall attempt in what follows something more general.

Let us suppose $fx = ax^a + a_1x^{a_1} + a_2x^{a_2} + \dots$; where the exponents $a, a_1, \&c.$ are whole or fracted, rational or irrational; but ranged according to their order of magnitude. And first let the series be an ascending one. Then dividing by x^a , we have $\frac{fx}{x^a} = a + a_1x^{a_1-a} + a_2x^{a_2-a} + \dots$. The exponents $a_1-a, a_2-a, \&c.$

$a_2 - \alpha$ are now affirmative. Make $x = 0$, which give $a = \frac{f_0}{o}$.

Differentiating; $\frac{d}{dx} \left(\frac{fx}{x^a} \right) = (\alpha_1 - \alpha) a_1 x^{a-1} + (\alpha_2 - \alpha)$

$a_1 x^{a-1} + \dots$. Or dividing by x^{a-1}

$$\frac{1}{x^{a-1}} \cdot \frac{d}{dx} \left(\frac{fx}{x^a} \right) = (\alpha_1 - \alpha) a_1 + (\alpha_2 - \alpha) a_2 x^{a-1} + \dots$$

And again making $x = 0$, we find

$$a_1 = \frac{1}{\alpha_1 - \alpha} \cdot \frac{1}{o^{a-1}} \cdot \frac{d}{do} \left(\frac{f_0}{o} \right). \text{ In the same manner}$$

we may determine the other co-efficients. The indices $\alpha, \alpha_1, \&c.$ are so determined that x^a may be the factor of this description which divides fx , x^{a-1} the factor which divides $\frac{d}{dx} \left(\frac{fx}{x^a} \right)$;

and so on. Thus the indices are determined as we proceed together with the co-efficients. And we have

$$fx = \frac{f_0}{o} \cdot x^a + \frac{1}{\alpha_1 - \alpha} \cdot \frac{1}{o^{a-1}} \cdot \frac{d}{do} \left(\frac{f_0}{o} \right) \cdot x^{a-1} + \frac{1}{\alpha_2 - \alpha_1 \cdot \alpha_2 - \alpha} \cdot \frac{1}{o^{a-1}} \cdot \frac{d}{do} \left(\frac{1}{o^{a-1}} \cdot \frac{d}{do} \left(\frac{f_0}{o} \right) \right) \cdot x^{a-2} + \dots$$

If $fx = \sqrt{a^2 x - x^4} = \sqrt{x} \cdot \sqrt{a^2 - x^3}$; we make $x^a = x^{\frac{1}{2}}$, or $\alpha = \frac{1}{2}$. Then $\frac{fx}{x^a} = \sqrt{a^2 - x^3}$, and $\frac{d}{dx} \left(\frac{fx}{x^a} \right) = -$

$$\frac{3}{2} \frac{x^2}{\sqrt{a^2 - x^3}}. \text{ We make again } x^{a-1} = x^{\frac{1}{2}}, \text{ or } \alpha_1 = \frac{1}{2},$$

And so on in all the succeeding steps. This seems, in theory at least, to be the natural and legitimate mode of development.

If we would have a descending series, we might make $v = \frac{1}{x}$, and $fx = \varphi v$. Then expanding this last by the powers of v , we should have the development required. But we will suppose

$$fx = \frac{a}{x} + \frac{a_1}{x^1} + \frac{a_2}{x^2} + \dots, \text{ the exponents } a, a_1, \&c.$$

being ranged according to the order of their magnitude.

Now multiply by x^a , and we have

$$x^a fx = a + \frac{a_1}{x^{1-a}} + \frac{a_2}{x^{2-a}} + \dots. \text{ The exponents } a_1 -$$

$a, a_2 - a, \&c.$ are all affirmative; therefore let $x = \infty$, and $a = \infty^a f \infty$.

$$\frac{d}{dx} (x^a fx) = -(a_1 - a) \frac{a_1}{x^{a_1 - a + 1}} - (a_2 - a) \frac{a_2}{x^{a_2 - a + 1}} - \dots;$$

$$x^{a_1 - a + 1} \frac{d}{dx} (x^a fx) = -(a_1 - a) a_1 - (a_2 - a) \frac{a_2}{x^{a_2 - a_1}} - \dots.$$

Again make x infinite, and

$$a_1 = -\frac{1}{a_1 - a} \cdot \infty^{a_1 - a + 1} \cdot \frac{d}{dx} (\infty^a f \infty).$$

And proceeding in the same manner

$$a_2 = \frac{1}{(a_2 - a_1)(a_2 - a)} \cdot \infty^{a_2 - a_1 + 1} \cdot \frac{d}{dx} \left(\infty^{a_1 - a + 1} \frac{d}{dx} (\infty^a f \infty) \right).$$

The indices $a, a_1, \&c.$ must be determined so that ∞^a is the reciprocal of the factor; in $f \infty^{a_1 - a + 1}$ the reciprocal of the factor contained in $\frac{d}{dx} (\infty^a f \infty)$, &c.; so that the coefficients $a, a_1, \&c.$ are rendered of finite magnitude.

These are natural and general methods of development, if the function admits of being developed by the powers of any proposed quantity. If it does not admit of this, or if it be a logarithmic function; we shall proceed in exactly the same manner, only putting $x \pm a$ in the place of x , a being any assumed quantity. But now, in order to determine the coefficients; we make $x \pm a = 0$, or $x = \mp a$.

And thus we shall find, if $fx \simeq \log x$;

$$\log x = \log a + \frac{1}{a} (x - a) - \frac{1}{2a^2} (x - a)^2 + \frac{1}{3a^3} (x - a)^3 - \dots$$

$$\text{If } a = 1; \log x = \frac{1}{1} (x - 1) - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 - \dots$$

We may develop a function of x or ϕx in terms of another function of x or fx . This may be effected exactly as above. And if we suppose that the powers of fx are integer, and that n is the value of x which makes $fx = 0$; we shall find $a = \phi n$,

$$a_1 = \frac{d\phi n}{dfn} = \phi_1 n, \quad a_2 = \frac{1}{2} \cdot \frac{d\phi_1 n}{dfn} = \phi_2 n, \text{ \&c. } \text{ Therefore}$$

$$\phi x = \phi n + \phi_1 n \frac{fx}{1} + \phi_2 n \frac{fx^2}{1 \cdot 2} + \phi_3 n \frac{fx^3}{1 \cdot 2 \cdot 3} + \dots$$

But if the powers of fx cannot be integer, which will appear from the coefficients becoming infinite; we shall find

$$a = \frac{\phi n}{fn^a} = \psi n, \quad a_1 = \frac{1}{a_1 - a} \cdot \frac{1}{fn^{a_1 - a - 1}} \cdot \frac{d\psi n}{dfn} = \frac{1}{a_1 - a} \psi_1 n,$$

$$a_2 = \frac{1}{a_2 - a_1} \cdot \frac{1}{a_2 - a} \cdot \frac{1}{fn^{a_2 - a_1 - 1}} \cdot \frac{d\psi_1 n}{dfn} = \frac{1}{a_2 - a_1} \cdot \frac{1}{a_2 - a} \cdot \phi_2 n, \text{ \&c. }$$

And here the indices $a, a_1, \text{ \&c. }$ will be determined as before.

Thus, fn being nothing, fn^a will be the greatest power of fn which will divide ϕn ; or such a power as will render the quotient $\frac{\phi n}{fn^a}$ finite. In like manner $fn^{a_1 - a - 1}$, $fn^{a_2 - a_1 - 1}$ must

be factors of $\frac{d\psi n}{dfn}$, $\frac{d\psi_1 n}{dfn}$, &c.

In determining the coefficients we shall sometimes fall upon fractions whose numerators and denominators vanish at the same time. But the method of determining their values in this case is well-known.

As we may expand $\frac{d\phi x}{dx}$ in terms of fx , so we may in terms

of ϕx . For this is only putting $\frac{d\phi x}{dx}$ in the place of ϕx , and ϕx in the place of fx .

And hence also, putting $\int \phi x dx$ in the place of ϕx ; we may develop this integral by the powers of fx . Here the first term, being a definite integral, will become an arbitrary constant. And putting ϕx for fx , we shall have $\int \phi x dx$ in terms of ϕx .

By the preceding formulæ we may sometimes obtain series more converging than if they proceeded by the powers of x . And we may also transform a series, if we can render it of

swifter convergency, by making ϕx equal to the series to be transformed, fx equal to the quantity by the powers of which it is to proceed.

In Mr. Spence's Essay on Logarithmic Transcendents, we have

$$l^n(1+x) = \int \frac{dx}{x} \int \frac{dx}{x} \dots \dots \int \frac{dx}{1+x}.$$

If we would apply our formula to the development of this function, in terms of lx or $\log x$; we must make $fx = lx$,

$x=1$; and we find $\phi n = l^n(2)$, $\phi_1 n = l^{n-1}(2)$, $\phi_2 n = \frac{1}{2} l^{n-2}(2)$, &c.

$$\begin{aligned} \text{Therefore } l^n(1+x) &= l^n(2) + l^{n-1}(2) \cdot lx + l^{n-2}(2) \cdot \frac{lx^2}{2} + \\ &+ l^{n-3}(2) \cdot \frac{lx^3}{2 \cdot 3} + \dots l^2 \cdot \frac{lx^{n-1}}{2 \cdot 3 \cdot 4 \dots n-1} + \frac{1}{2} \cdot \frac{lx^n}{2 \cdot 3 \dots n} + \\ &+ \frac{1}{2^2} \cdot \frac{lx^{n+1}}{2 \cdot 3 \dots n+1} - \frac{1}{2^3} \cdot \frac{lx^{n+3}}{2 \cdot 3 \dots n+3} + \frac{1}{2^2} \cdot \frac{lx^{n+5}}{2 \cdot 3 \dots n+5} - \end{aligned}$$

If we would expand the same function in terms of $lx - la$

or $l\left(\frac{x}{a}\right)$; we shall find

$$\begin{aligned} l^n(1+x) &= l^n(1+a) + l^{n-1}(1+a) l\left(\frac{x}{a}\right) + \frac{1}{2} l^{n-2}(1+a) l\left(\frac{x}{a}\right)^2 \\ &+ \frac{1}{2 \cdot 3 \dots n} \cdot \frac{a}{a+1} l\left(\frac{x}{a}\right)^n + \frac{1}{2 \cdot 3 \dots n+1} \cdot \frac{a}{(a+1)^2} l\left(\frac{x}{a}\right)^{n+1} + \\ &+ \frac{1}{2 \cdot 3 \dots n+2} \cdot \frac{a(1-a)}{(1+a)^3} l\left(\frac{x}{a}\right)^{n+2} + \dots \end{aligned}$$

In like manner the circular function $c^n(x)$ of Spence may be expanded. This will give

$$\begin{aligned} c^n(x) &= c^n(a) + c^{n-1}(a) l\left(\frac{x}{a}\right) + \frac{1}{2} c^{n-2}(a) l\left(\frac{x}{a}\right)^2 + \dots \\ &+ \frac{1}{2 \cdot 3 \dots n} \cdot \frac{a}{1+a^2} \cdot l\left(\frac{x}{a}\right)^n + \frac{1}{2 \cdot 3 \dots n+1} \cdot \frac{1-a^2}{(1+a^2)^2} \cdot 2 l\left(\frac{x}{a}\right)^{n+1} + \dots \end{aligned}$$

These examples may be more curious than useful. We cannot however say to what curious speculation may lead. It is seldom that useful things are discovered when we are in direct search of them.

We may transform the series $x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ into one

proceeding by the powers of $\frac{x}{1+x}$, and we shall find it

$$\frac{x}{1+x} + \frac{1}{2} \left(\frac{x}{1+x} \right)^2 + \frac{1}{3} \left(\frac{x}{1+x} \right)^3 + \dots$$

As another example take

$$l(1+x) = l(1+x) + \frac{1}{2^2} l(1+x)^2 + \frac{1}{2^2 \cdot 3^2} l(1+x)^3 + \\ - \frac{4}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2} l(1+x)^5 + \frac{2 \cdot 3 \cdot 45}{2^2 \cdot 3^2 \cdot 4^2 \cdot 5^2 \cdot 6^2 \cdot 7^2} l(1+x)^7 + \dots$$

We may apply our method to the expansion of a function of two quantities x and y , and may make the expansion in terms of two functions of x and y .

Let $z = \psi(x, y)$, $u = \phi(x, y)$, $v = f(x, y)$.

We will suppose that the expansion can be made by integer powers and products. Imagine now x and y eliminated from the value of z , and we have $z = \phi(u, v)$. This expanded is

$$z = z_0 + \frac{1}{1} \left\{ \frac{dz_0}{du} u + \frac{dz_0}{dv} v \right\} + \\ \frac{1}{1 \cdot 2} \left\{ \frac{d^2 z_0}{du^2} u^2 + 2 \frac{d^2 z_0}{du dv} uv + \frac{d^2 z_0}{dv^2} v^2 \right\} + \dots$$

In z_0 and its partial differential coefficients u and v are to be made zero.

$$\frac{dz}{dx} = \frac{dz}{du} \frac{du}{dx} + \frac{dz}{dv} \frac{dv}{dx}, \quad \frac{dz}{dy} = \frac{dz}{du} \frac{du}{dy} + \frac{dz}{dv} \frac{dv}{dy}.$$

These will give

$$\frac{dz}{du} = \frac{\frac{dz}{dx} \frac{dv}{dy} - \frac{dz}{dy} \frac{dv}{dx}}{\frac{du}{dx} \frac{dv}{dy} - \frac{du}{dy} \frac{dv}{dx}}, \quad \frac{dz}{dv} = \frac{\frac{dz}{dx} \frac{du}{dy} - \frac{dz}{dy} \frac{du}{dx}}{\frac{dv}{dx} \frac{du}{dy} - \frac{dv}{dy} \frac{du}{dx}},$$

Thus we have $\frac{dz}{du}$, $\frac{dz}{dv}$ in functions of x and y . To abridge

let them be represented by u_1, v_1 ; or $\frac{dz}{du} = u_1$, $\frac{dz}{dv} = v_1$.

$$\frac{du_1}{dx} = \frac{d^2 z}{du^2} \frac{du}{dx} + \frac{d^2 z}{du dv} \frac{dv}{dx}, \quad \frac{du_1}{dy} = \frac{d^2 z}{du^2} \frac{du}{dy} + \frac{d^2 z}{du dv} \frac{dv}{dy},$$

$$\frac{dv_1}{dx} = \frac{d^2 z}{dv^2} \frac{dv}{dx} + \frac{d^2 z}{du dv} \frac{du}{dx}, \quad \frac{dv_1}{dy} = \frac{d^2 z}{dv^2} \frac{dv}{dy} + \frac{d^2 z}{du dv} \frac{du}{dy}.$$

These equations will give $\frac{d^2z}{du^2}$, $\frac{d^2z}{du dv}$, $\frac{d^2z}{dv^2}$ in functions of x and y ; and we might proceed in the same manner to find the differential coefficients of a higher order. In the final results the values m and n of x and y which render $u = 0$, $v = 0$ must be put for those quantities.

We might develope in the same way when the number of variables is greater than two.

If $z = \phi x$ is only given implicitly by an equation $\psi(z, x) = 0$, we may still expand z by the powers of ϕx . But general formulæ in this case are not good. Particular cases are best resolved by particular processes, or artifices.

La Grange in his Theory of Analytical Functions applies Mac-laurin's Theorem to the solution of differential equations. But it

will often fail as he uses it. If we have $f\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) =$

0; we must first put $z = yx^a$, and then proceed to apply the Theorem to the equation so transformed. The reason is obvious. The series for y will in general be of the form $ax^a + a_1x^{a+c} + a_2x^{a+2c} +$; and if multiplied or divided by some fractional power of x , will become of the form of Mac-laurin's series. And this fractional exponent will easily be determined by making it such as to prevent the coefficients from becoming infinite, or any way contradictory or inconsistent.

But in order to render the Theorem more certain of application; it may be necessary also to put $x = v^p$; in order to render all the indices of the development integer.

And taking a any quantity at pleasure, it will be more general to expand by the powers of $x - a$ instead of x . The quantities p and a must be determined as α .

I shall not now enter particularly into this subject, though it might be worth the while; as it might lead to a general method of solving differential equations by series.

ARTICLE XI.

FOUR INDETERMINATE PROBLEMS.

By Mr. JOHN DAVY, of St. Just, in Penwith, Cornwall.

PROBLEM I.

To find the sides, in whole numbers, of three right-angled triangles, such, that their perimeters shall be equal to each other, and their areas in arithmetical proportion.

SOLUTION.

Let $\frac{m^2 + n^2}{m}$, $\frac{m^2 - n^2}{m}$, $2n$; $\frac{p^2 + q^2}{p}$, $\frac{p^2 - q^2}{p}$, $2q$; and $\frac{r^2 + s^2}{r}$, $\frac{r^2 - s^2}{r}$, $2s$; denote the sides of the three triangles,

respectively; the second being those of the mean. Then by summing their sides severally, we shall find the perimeter of the first $= 2(m + n)$, of the second $2(p + q)$, and of the third $= 2(r + s)$; therefore by the conditions of the problem, we shall have $2(m + n) = 2(p + q) = 2(r + s)$; hence $n = p + q - m$, and $s = p + q - r$. Also when their areas are in arithmetical

proportion, we shall have $2(p^2 - q^2) \frac{q}{p} = (m^2 - n^2) \frac{n}{m} +$

$(r^2 - s^2) \frac{s}{r}$: from which by writing therein, for n and s , their

values as found above, and reducing, we shall have $q^2 + 2pq =$

$\frac{(p^2 + 2mpr) \times (m + r) - 6mp^2r}{(m + r)p - 2mr}$; adding p^2 to each side of this

equation, in order to complete the square on the first, and then ex-

tracting the root, we shall have $q = \sqrt{\left(\frac{2p - m - r}{\frac{1}{2}(m + r)p - mr}\right) \times mpr} - p$.

Now to render the fraction under the radical sign somewhat

more simple, assume $m = \frac{2t - u}{2t}v$, $p = \frac{2t}{2t + u}v$, and $r =$

$\frac{2t + u}{2(t + u)}v$; and substituting therein for each letter its assumed

value, we shall ultimately obtain $q = \sqrt{\left(\frac{2t-u}{t+u}\right)v - \frac{2}{2t+u}v}$;

hence $\frac{2t-u}{t+u}$ must be a square; let its root be $\frac{b}{a}$ then will

$$\frac{2t-u}{t+u} = \frac{b^2}{a^2}; \text{ which gives } t = \frac{a^2 + b^2}{2a^2 - b^2}u, \text{ or taking } t = a^2 + b^2,$$

we shall have $u = 2a^2 - b^2$; where a and b may be any convenient numbers at pleasure, and v the least number divisible by $2t$, $2t+u$, and $2(t+u)$, or some of their component parts, leaving no remainder.

Ex. 1. If we take $a = 1$, and $b = 1$, then $t = 2$, $u = 1$, $2t = 4$, $2t+u = 5$, and $2(t+u) = 6$; hence $v = 60$: and therefore $m = 45$, $n = 15$, $p = 48$, $q = 12$, $r = 50$, and $s = 10$; consequently

50, 40, 30, are the sides of the greater triangle,

51, 45, 24, are those of the mean, and

42, 48, 20, are those of the less, which numbers answer the conditions required.

Ex. 2. If $a = 3$, and $b = 4$, then $t = 25$, $u = 2$, $2t = 50$; $2t+u = 52$, and $2(t+u) = 54$; hence $v = 17550$: consequently $m = 104400$, $n = 38280$, $p = 126075$, $q = 16605$, $r = 137924$, and $s = 4756$; therefore after having divided each side by 2, we shall have

9698, 7150, 6552,

9699, 7176, 6525,

9700, 7200, 6500; for the sides of another set of triangles that will answer the conditions, and thus we may get as many sets as we please.

PROBLEM II.

To find the sides, in whole numbers, of three right-angled triangles, such that their perimeters shall be equal to each other, and their areas in geometrical proportion.

SOLUTION.

Let $\frac{p^2 + r^2}{p+r}v$, $\frac{p^2 - r^2}{p+r}v$, $\frac{2pr}{p+r}v$; $\frac{p^2 + s^2}{p+s}v$, $\frac{p^2 - s^2}{p+s}v$, $\frac{2ps}{p+s}v$; and $\frac{p^2 + t^2}{p+t}v$, $\frac{p^2 - t^2}{p+t}v$, $\frac{2pt}{p+t}v$; denote the sides

of the three triangles, respectively; the second being those of the mean. Then by adding their sides together severally, the perimeter of each will be found $= 2pv$: hence the first condition will evidently be satisfied. Also that their areas may

be had in geometrical proportion, we must have $\left(\frac{ps-s^2}{p+s}\right)^2 = \frac{pr-r^2}{p+r} + \frac{pt-t^2}{p+t}$: but as this equation in the present form,

appears rather intricate for solving, assume $r = \frac{psu}{pu-1}$; and

$t = \frac{(ps+s^2)u-s}{(p+s)u}$; and having written therein their assumed

values in their stead, it then becomes $\left(\frac{ps-s^2}{p+s}\right)^2 = \frac{(p-s)u-1}{(p+s)u+1} \times$

$\frac{psu}{pu-1} \times \frac{(p^2-s^2)u+s}{(p+s)^2u-s} \times \frac{(ps+s^2)u-s}{(p+s)u}$: from which, by

reduction, we get $u = \frac{2p^2-ps+s^2}{(p^2-ps-s^2) \times (p-s)}$: this value of

u being written for it, in each of the preceding assumed equa-

tions, we shall have $r = \frac{2p^2-ps+s^2}{p^2+2p^2s-s^2}ps$, and $t = \frac{p^2+4ps-s^2}{2p^3+p^2s-s^3}ps$;

where p and s may be assumed at pleasure:—however we shall find that the triangles resulting from these letters only, will mostly come out in fractional numbers; therefore when it so happens, take v = the least number divisible by each of their denominators, leaving no remainder, and then they will be had in whole numbers.

Ex. 1. Take $p = 3$, and $s = 1$, then $r = \frac{12}{11}$, and $t = \frac{15}{16}$;

hence the triangles in fractional numbers are $\frac{137}{55}v$, $\frac{105}{55}v$,

$\frac{88}{55}v$; $\frac{5}{2}v$, $\frac{4}{2}v$, $\frac{3}{2}v$; and $\frac{281}{112}v$, $\frac{231}{112}v$, $\frac{160}{112}v$; therefore

taking $v = 6160$, we shall have

15344, 11760, 9856, for the sides of the greater triangle,

15400, 12320, 9240, for those of the mean, and

15455, 12705, 8800, for those of the less; in whole numbers: which will be found to answer the conditions required.

Ex. 2. If $p = 3$, and $s = 2$, then $r = \frac{96}{55}$, and $t = \frac{87}{40}$;

hence the triangles in fractions, will be $\frac{4049}{1595}v$, $\frac{3520}{1595}v$, $\frac{2001}{1595}v$;

$\frac{13}{5}v$, $\frac{12}{5}v$, $\frac{5}{5}v$, and $\frac{2441}{920}v$, $\frac{2320}{920}v$, $\frac{759}{920}v$; therefore taking $v = 298480$, we shall have

745016, 647680, 368184,

763048, 704352, 298480,

778679, 740080, 242121, for the sides of another set of triangles as required.

PROBLEM III.

To find the sides, in whole numbers, of three right-angled triangles, such, that their perimeters shall be equal to each other, and their areas in harmonical proportion.

SOLUTION.

Denoting the sides of the three triangles by $\frac{p^2 + r^2}{p + r}v$, $\frac{p^2 - r^2}{p + r}v$, $\frac{2pr}{p + r}v$; $\frac{p^2 + s^2}{p + s}v$, $\frac{p^2 - s^2}{p + s}v$, $\frac{2ps}{p + s}v$; and $\frac{p^2 + t^2}{p + t}v$, $\frac{p^2 - t^2}{p + t}v$, $\frac{2pt}{p + t}v$; as in the last problem; the perimeter of each

will be $2pv$: and their areas will be represented by $\frac{pr - r^2}{p + r}pv$,

$\frac{ps - s^2}{p + s}pv$, and $\frac{pt + t^2}{p + t}pv$. Also considering that the re-

ciprocals of all numbers in harmonical proportion, are always in arithmetical proportion, and that the second of these areas is

the mean; we shall therefore have $2 \left(\frac{p + s}{ps - s^2} \right) = \frac{p + r}{pr - r^2} +$

$\frac{p + t}{pt - t^2}$. In this equation put $s = p - r$, then by substitution

will $2 \left(\frac{p + s}{ps - s^2} \right)$ be changed into $2 \left(\frac{2p - r}{pr - r^2} \right)$; and therefore

$2\left(\frac{2p-r}{pr-r^2}\right) = \frac{p+r}{pr-r^2} + \frac{p+t}{pt-t^2}$: which reduced gives $p = \frac{3t+r}{3t-r}t$; where r and t may be taken at pleasure, and v as in the last problem.

Ex. 1. If r be taken $= 8$, and $t = 4$, then will $p = 20$, and $s = 12$; hence the triangles derived from these will be $\frac{116}{7}r$, $\frac{84}{7}v$, $\frac{80}{7}v$; $17v$, $15v$, $8v$; and $\frac{52}{3}v$, $\frac{48}{3}v$, $\frac{20}{3}v$; therefore taking $v = 21$, we shall have

348, 252, 240, for the sides of the greater triangle,

357, 315, 168, for those of the mean, and

364, 336, 140, for those of the less; which will satisfy the conditions required.

Ex. 2. Taking $r = 5$, and $t = 2$, we shall have $p = 22$, and $s = 17$; hence the triangles arising from these numbers, will be $\frac{509}{27}v$, $\frac{459}{27}v$, $\frac{220}{27}v$; $\frac{773}{39}v$, $\frac{748}{39}v$, $\frac{195}{39}v$; and $\frac{61}{3}v$, $\frac{60}{3}v$, $\frac{11}{3}v$; therefore taking $v = 351$, there will be had

6617, 5967, 2860,

6957, 6732, 1755,

7139, 7020, 1287, for the sides of three other triangles which will answer the conditions.

PROBLEM IV.

To find the sides and areas, in whole numbers, of three isosceles triangles, such, that their perimeters shall be equal to each other, and their areas in arithmetical proportion.

SOLUTION.

Let p^2r^2 denote each of the equal sides, $2(p^2 - r^2)$ the base, and $2pr$ the perpendicular thereon, of the first triangle; $p^2 + s^2$, $2(p^2 - s^2)$, $2ps$, those of the second and mean; and $p^2 + t^2$, $2(p^2 - t^2)$, $2pt$, those of the third. Then will the perimeter of each be $= 4p^2$: therefore one of the conditions is satisfied.

Moreover when their areas are in arithmetical proportion, we shall have $2(p^2s - s^3) = p^2r - r^3 + p^2t - t^3$: here by writing $s - t$ for r , and then reducing the equation, we shall have $p = \sqrt{(s^2 + 3st - 3t^2)}$; hence $s^2 + 3st - 3t^2$ must be a square;

let its root be $s + \frac{tv}{u}$; then will $s^2 + 3st - 3t^2 = s^2 + \frac{2stv}{u} +$

$\frac{t^2v^2}{u^2}$; from which we finally obtain $s = \frac{3u^2 + v^2}{3u^2 - 2uv} t$; conse-

quently $p = \frac{3u^2 + 3uv - v^2}{3u^2 - 2uv} t$, and $r = \frac{2uv + v^2}{3u^2 - 2uv} t$: where

u , v , and t , may be any convenient numbers, at pleasure.

Ex. 1. If we take $u = 1$, $v = 1$, and $t = 1$, then we shall have $r = 3$, $p = 5$, and $s = 4$; hence

the sides of the first triangle are 34, 34, 32; its area = 480;
those of the second or mean are 41, 41, 18; its area = 360;
and those of the third are 26, 26, 48; its area = 240;
which answer the conditions.

Ex. 2. If $u = 2$, $v = 1$, and $t = 8$, then $r = 5$, $p = 17$, and $s = 13$; from which

the sides of the first triangle are 314, 314, 528, its area 44880;
these of the second or mean 458, 458, 240, its area 53040;
and these of the third 353, 353, 450, its area 61200;
which will also answer the conditions required.

The first of these problems has been already ably answered by Mr. James Cunliffe; first, by a particular solution given in No. XXII. Davis's Math. Companion; and afterwards by the same solution, with a general one annexed, in No. XVIII. of your Repository: The others to me are original.

ARTICLE XII.

Solution to a Physical Question.

By Mr. W. S. B. WOOLHOUSE.

QUESTION.—A slender hollow tube of a given length (l) is charged with a perfectly expansive and compressible fluid to

a given density (h) and is caused to revolve about one of its extremities with a given angular velocity (ω) so that, in consequence of centrifugal force, each particle tends to the other extremity: it is required to investigate an expression for the true density at each point during the motion.

This question having been proposed by me in the Gentleman's Diary for the year 1831, and as the solution of Mr. Richard Taylor of Carlisle, which is the only one given in the present Diary, is incorrect in its principles, probably through Mr. T. misconceiving the philosophical nature of the question, I here purpose giving that solution which accords with my views, and, which was originally furnished by me for the Diary. Mr. T. supposes the fluid, with the tube revolving, to have a surface at some point C, to hence occupy exclusively an extreme portion CB of the tube, and also in this state to possess an uniform density h' , which suppositions are incompatible with the properties of fluids in any degree expansive or elastic. In consequence of these errors Mr. T.'s result gives a greater quantity of fluid existing in the tube than before the supposed motion, which certainly ought not to be the case.

SOLUTION.

According to the nature of the fluid its density will be universally measured by the force with which it is compressed. Let k be the density at any distance y from the extremity round which the tube revolves; then is kdy the mass of the element dy , and $kdy \times y\omega^2 = k\omega^2 ydy$ its centrifugal force, which being the additional pressure on the next element of the fluid must obviously be equal to dk , the element of the density. Hence

$$dk = k\omega^2 ydy \text{ and } \frac{dk}{k} = \omega^2 ydy$$

which gives

$$k = ce^{\frac{\omega^2 y^2}{2}} = c \left(e^{\frac{\omega^2}{2}} \right)^{y^2}.$$

Now to find the value of the constant c , we have the whole mass of the fluid $= \Sigma k dy = \Sigma c dy e^{\frac{\omega^2 y^2}{2}} = c \Sigma dy e^{\frac{\omega^2 y^2}{2}} = c \int_0^l dy e^{\frac{\omega^2 y^2}{2}}$ which must $\therefore = lh$.

$$\therefore c = \frac{lh}{\int_0^l dy e^{\frac{\omega^2 y^2}{2}}}.$$

$$\text{and } k = \frac{lh^0}{\int_0^l dy e^{\frac{\omega^2 y^2}{2}}} \cdot \left(e^{\frac{\omega^2}{2}}\right)^y, \dots\dots\dots (u)$$

where the integral in the denominator must be taken between $y = 0$ and $y = l$.

The function $e^{\frac{\omega^2 y^2}{2}}$ expanded is

$$1 + \left(\frac{\omega^2}{2}\right)y^2 + \frac{1}{2}\left(\frac{\omega^2}{2}\right)^2 y^4 + \frac{1}{2 \cdot 3}\left(\frac{\omega^2}{2}\right)^3 y^6 + \frac{1}{2 \cdot 3 \cdot 4}\left(\frac{\omega^2}{2}\right)^4 y^8 + \&c.$$

$$\therefore \int_0^l dy e^{\frac{\omega^2 y^2}{2}} = l + \left(\frac{\omega^2}{2}\right) \frac{l^3}{3} + \frac{1}{2}\left(\frac{\omega^2}{2}\right)^2 \frac{l^5}{5} + \frac{1}{2 \cdot 3}\left(\frac{\omega^2}{2}\right)^3 \frac{l^7}{7} + \frac{1}{2 \cdot 3 \cdot 4}\left(\frac{\omega^2}{2}\right)^4 \frac{l^9}{9} + \&c. \dots\dots\dots (b)$$

which and the equation (a) determines k .

By substitution

$$\frac{l}{\int_0^l dy e^{\frac{\omega^2 y^2}{2}}} = 1 - \left(\frac{\omega^2}{2}\right) \frac{l^2}{3} + \left(\frac{\omega^2}{2}\right)^2 \frac{l^4}{9 \cdot 10} \&c.$$

Therefore the general expression for the density k at any distance y is

$$k = h \left(e^{\frac{\omega^2}{2}}\right)^y \left\{ 1 - \left(\frac{\omega^2}{2}\right) \frac{l^2}{3} + \left(\frac{\omega^2}{2}\right)^2 \frac{l^4}{9 \cdot 10} \&c. \right\} \dots\dots\dots (c)$$

By taking $y = 0$, we find the density at the extremity which is the centre of rotation

$$= h \left\{ 1 - \left(\frac{\omega^2}{2}\right) \frac{l^2}{3} + \left(\frac{\omega^2}{2}\right)^2 \frac{l^4}{90} \&c. \right\} \dots\dots\dots (c)$$

In these expressions y and l must be taken as purely numerical quantities and to be consistent with our original suppositions, we are restricted to value them by a determinate unit of length.

For if δ denote the density of the fluid when compressed by a given force or pressure p we shall have, in consequence of our assuming the density equal to the force of compression, $\frac{p}{\delta} =$ the unit of measure which we must adopt in representing arithmetically the lengths of lines.

Thus the numerical value of l is $l \div \frac{p}{\delta} = \frac{\delta}{p} l$, and that

of y is similarly $\frac{\partial}{\partial p} y$; so that the literal value of equation (b) is

$$k = h e^{\frac{\omega^2}{2} \cdot \frac{\partial^2 y^2}{p^2}} \left\{ 1 - \frac{1}{3} \left(\frac{\omega^2}{2} \right) \frac{\partial^2}{p^2} l^2 + \frac{1}{90} \left(\frac{\omega^2}{2} \right)^2 \frac{\partial^4}{p^4} l^4 \&c. \right\} \dots (d),$$

in which any standard of measure may be used.

Hence also the density of the centre

$$\begin{aligned} &= h \left\{ 1 - \frac{1}{3} \left(\frac{\omega^2}{2} \right) \frac{\partial^2}{p^2} l^2 + \frac{1}{90} \left(\frac{\omega^2}{2} \right)^2 \frac{\partial^4}{p^4} l^4 \&c. \right\} \\ &= h \left\{ 1 - \frac{\omega^2}{2} \cdot \frac{\partial^2}{p^2} l^2 + \frac{\omega^4}{360} \cdot \frac{\partial^4}{p^4} l^4 \&c. \right\} \dots \dots (d'). \end{aligned}$$

In these expressions $\frac{\partial}{\partial p} y$ and $\frac{\partial}{\partial p} l$ are abstract numerical values determined by the expansibility of the fluid.

It perhaps may be worth while to take the present opportunity of adverting to two erroneous solutions which have accidentally occurred to me in scanning over the last number of the Repository.

The first, which is Mr. Jones's solution to his question (508), appears to have arisen entirely from inadvertence. Mr. Jones says, "therefore the motive force of the segment AP is $A \times$

$$\left\{ \frac{a+z}{r} - \frac{a+z}{\sqrt{(a+z)^2 + r^2}} \right\}."$$

Now this motive force should be simply $A \times \left\{ \frac{1}{r} - \frac{1}{\sqrt{(a+z)^2 + r^2}} \right\}$, and I think Mr. J.

has multiplied it by $a+z$, the length of the segment, forgetting

that the integral of $\frac{Ax}{(r^2 + x^2)^{\frac{3}{2}}} dx$, between any limits, gives

the joint force of the *whole* segment, included between the same limits, without any further operation. The same applies to the other segment BP ; and these alterations will, of course, materially effect the succeeding integration for the velocity which will come out in a hyperbolic logarithm and will correspond with the solution of Mr. Davies.

The other is Mr. Davies's solution to question (503), which I am more inclined to notice, as it involves a mistaken principle, and might very much mislead a student in its application

to many other deviations besides the particular one connected with the question itself. This objectional principle is set forth in the following paragraph which is taken from page 29.

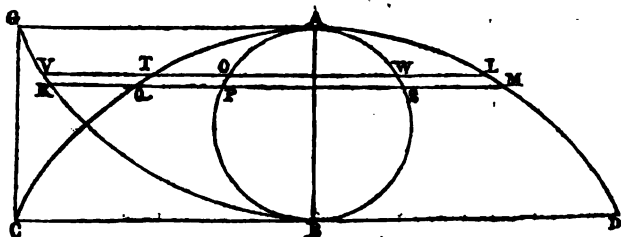
"In all cases of deviation we must of course suppose an independent variable in one curve equal to a corresponding variable in the other : and we can always take that variable as independent which may best suit the object we have in view. In cases, however, where both co-ordinates are expressed in terms of a third variable, it will almost always happen that the said third variable is the most convenient one to be equated in the two curves : and so it is with the case before us. We shall, therefore consider that $\theta = \theta'$, and seek the distance of the points Q, Q'."

Now these observations are without foundation ; for when two independent variables are thus equated a relation is at once fixed between any point in one curve and a corresponding point in the other. In the case of this question, under the assumption $\theta = \theta'$, and we may say universally, except in very particular cases indeed, a line joining a pair of these corresponding points will be oblique to both curves. That such a line cannot properly represent the deviation of one curve from the other at either of these points is, I think, very obvious. It appears to me very clearly that the only correct method of applying the analysis in such a case is to conceive that at any point in either curve the deviation of the other is precisely its nearest approach to that point, and therefore in cases of minute deviation, that the said deviation is measured by a normal line joining the two curves. And hence the question which requires the greatest deviation resolves itself into the finding of the greatest possible of these lines, or that which is perpendicular to both curves. The method of determining this is well known, and its application to this particular question may here be deemed unnecessary, as the solution in numbers of the resulting equations if not impracticable would be excessive tedious.

ARTICLE XIII.

By Lieutenant-Colonel GLOVER.

To prove that the Area of a Cycloid is three times the Area of its generating Circle.



Let CAD be a common cycloid, CD being its base, AB the diameter of its generating circle, and its axis, then will the three areas into which it is divided, viz. $CTAOB$, circle $AOBSA$ and $BWALD$ be equal to each other.

For, from A , parallel and equal to the semibase BC , draw AG , on which, as a second base, proceed to describe a reversed cycloid, which, if completed, would be manifestly equal to the cycloid CAD .

From the curve AD of the upright cycloid, draw ordinates LV , MR , &c. to meet the curve of the reversed cycloid, and conceive the whole space $GVBDLAC$ to be made up of these lines; abstract the circle from this mixed cycloidal space, and to find the area of $GVBOA$ and $DLAWB$, observe that in any of the parallel lines, VO , RP , &c.

$$VO = \text{arch } OB$$

and TO (or WL) = arch OA } by property of cycloid.

and adding these equals together, $VO + WL = \text{arch } (OB + OA)$ or half the circumference,

$$\text{likewise, } RP = \text{arch } PB$$

$$\text{and } QP \text{ or } SM = \text{arch } PA,$$

$$\text{and } RP + SM = \text{half the circumference as before.}$$

Now, half the circumference of the circle is equal to the semibase of the cycloid, and as the ordinates, or lines, are supposed to be drawn from the base GA to the base BD , it follows that the number of these lines must be represented by the distance between the bases GA and BD , that is, must be represented by the axis, and consequently the double space $GVBOA$,

DLAWB must be equal to a rectangle under this axis and semi-base, or the single space GVBOA, or DLAWB will be equal to a rectangle under the semiaxis and semibase; that is, will be equal to the product of half the circumference and half the diameter,

which is the area of the generating circle.

Therefore the cycloidal space is divided in three equal areas, which was to be proved.

ARTICLE XIV.

Certain Properties appertaining to a Plane Triangle briefly demonstrated.

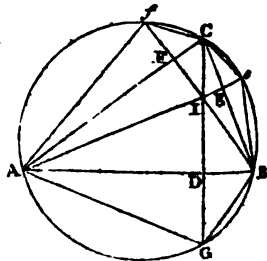
By Mr. JAMES CUNLIFFE, London.

PROPOSITION I.

If perpendiculars be drawn to the sides from the opposite angles of a plane triangle, these perpendiculars will intersect each other in the same point.

ACB is a plane triangle circumscribed by a circle; draw CD perpendicular to the base AB, produce CD to meet the circumference of the circle again in G, join AG, BG; upon DG, take DI = DG, through I, draw AIE, and BIF to meet BC and AC in E and F, respectively; then will AE and BF be at right angles to BC and AC respectively.

Fig. 1.



For CD is perpendicular to AB, and DI = DG and DB common, therefore the right angled triangles DBI and DBG are equal and similar, and therefore $\angle DBI$ ($\angle ABF$) = $\angle DBG$ ($\angle ABC$) = $\angle ACD$ Euc. 21. 3. that is $\angle ABF = \angle ACD$: but $\angle ADC$ is a right angle, and therefore $\angle AFB$ is a right angle, that is BF is perpendicular to AC.

Again $\angle BAG = \angle BAE = \angle BCD$ Euc. 21. 3. and $\angle ABE$ ($\angle DBC$) is common to both the triangles BAE and BCD, therefore these triangles are equiangular, and consequently AE perpendicular to BC.

Therefore the three perpendiculars from the angles upon the opposite sides of any plane triangle all intersect in the same point.

The demonstration of the foregoing property might have been effected in a different way, but the method pursued above appeared to be best adapted to what follows.

Cor. 1. If AE and BF be produced to meet the circumference of the circumscribing circle again in e and f ; then will $ae = IE$, and $bf = IF$; this is evident enough from the preceding preparation and demonstration.

Cor. 2. Let the lines Af , Cf ; and ae , be be drawn. Then the area of the six sided figure $AfCeBGA$ is double the area of the triangle ACB .

For area $AGB = \text{area } AIB$

$AfC = AIC$

$CeB = CIB$,

and therefore by addition the area of the six sided figure $AfCeBGA = 2 \times \text{area of the triangle } ACB$.

It may here be remarked that each pair of the adjoining sides of the six sided figure $AfCeBGA$, are equal, that is, $AG = AI = Af$; $Cf = CI = Ce$: and $BG = BI = BE$.

Cor. 3. The perimeter of the six sided figure $AfCeBGA$, is $= 2AI + 2BI + 2CI$, which obviously extends round the periphery of the circumscribing circle, and therefore the lines AI , BI , and CI , being applied in succession, as chords in the circumscribing circle, will just extend a semicircle.

Cor. 4. If circles are described through the extremities of the sides, and the point of intersection of the perpendiculars of any plane triangle, each of these circles will be equal to the circle circumscribing the triangle.

For the triangle AIB , is similar and equal to the triangle AGB ; therefore the circle described through A , I , B , will be equal to the circle circumscribing the triangle ACB , and the same may be said of the others. *Q. E. D.*

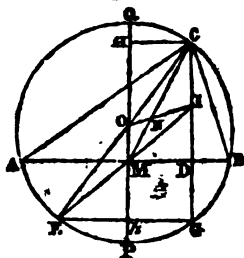
PROPOSITION II.

ACB is a plane triangle circumscribed by a circle, PQ a diameter bisecting the base AB in M ; through the centre O , draw the diameter CE ; draw the chord CG , at right angles to AB , and cutting it in D : through E and M draw EM , meet-

ing CD in I ; then I , is the point in which perpendiculars from the angles upon the opposite sides of the triangle ACB intersect each other.

Demon. By reason of the parallels OM, CD : $EO : EC = 2EO :: EM : EI = 2EM :: AM = DG : GI = 2DG$; wherefore $DI = DG$, and therefore from what has been demonstrated in the preceding part of this paper, I is the point of intersection of perpendiculars from the angles upon the opposite sides of the triangle ACB .

Fig. 2.



Cor. 1. $CI = 2OM$. For $EO : EC = 2EO :: OM : CI = 2OM$. That is, the distance of the point of intersection of perpendiculars from the angles upon the opposite sides, from any angle as C , is double the distance of the centre of the circumscribing circle from the middle of the side opposite to the same angle C .

Cor. 2. The line EI is bisected in M , as is evident enough, or may be thus proved, $EO : OC = EO :: EM : MI = EM$. From whence we gather the following property; that is to say, if diameters of the circumscribing circle are drawn through the angles, of any plane triangle, and straight lines drawn from the opposite ends of these diameters to the point where perpendiculars from the angles upon the opposite sides intersect each other; these lines will pass through the middle of the sides which are opposite to the angles, through which the respective diameters of the circumscribing circle are drawn; and these lines will also be bisected by the said middle points of the sides:

Cor. 3. Draw the lines OI , and MC intersecting in N : then N is the point in which lines from the angles bisecting the opposite sides of the triangle ACB intersect each other. The triangles MNO and CNI , are similar $OM : CI = 2OM :: MN : NC = 2MN$. Therefore N , is the point in which lines from the angle bisecting the opposite sides of the triangle ACB intersect each other: as $NC = 2MN$, is the well known property of that point.

Cor. 4. The line OI joining the centre of the circumscribing circle and the point where perpendiculars from the angles upon the opposite sides intersect, is bisected in N . For $OM : CI = 2OM :: ON : NI = 2ON$, that is $ON = \frac{1}{2}OI$.

I shall now subjoin a few problems, together with the geometrical construction of some of them to shew the use of the foregoing properties.

PROB. 1. Given the diameter of the circumscribing circle, and the two distances of the point of intersection of perpendiculars from the angles upon the opposite sides from the extremities of the base, to construct the plane triangle geometrically.

(Fig. 1.) Describe the circumscribing circle; and from any point, as G , in its periphery, apply the chords GA , GB , equal to the given distances of the point of intersection of the perpendiculars from the angles upon the opposite sides, from the two extremities of the base; join AB , and draw GD perpendicular thereto; produce GD till it meets the periphery of the circle again in C ; draw AC , BC , and ACB is the required triangle, as is plain from what has been demonstrated.

PROB. 2. Given the base, vertical angle, and the distance of the centre of the circumscribing circle, from the point where perpendiculars from the angles upon the opposite sides intersect each other, to construct the plane triangle geometrically.

The base and vertical angle of the triangle ACB being given, the circumscribing circle is given as is very well known: and it further appears from what has been demonstrated in this paper, that the locus of the point I , is the periphery of a circle passing through A and B , and equal to the circumscribing circle. Let therefore this circle be described, and with the centre O and given distance OI , describe an arc intersecting the periphery of the circle passing through A , I , B which determines the point I ; through I , draw CD perpendicular to AB ; join AC , BC , and ACB is the required triangle, as is plain enough from what has been demonstrated in the preceding properties.

PROB. 3. Given the circumscribing circle, the vertical angle, and the distance of the middle of the base from the point of intersection of perpendiculars from the angles upon the opposite sides, to construct the plane triangle geometrically.

(Fig. 2.) Describe the given circle and cut off the segment ACB , capable of containing the given vertical angle. From the centre O draw OM perpendicular to AB , then $AM = MB$. From M as a centre, with ME equal to the given distance of the middle of the base from the intersection of the perpendiculars as a radius, describe an arc intersecting the periphery of the circumscribing circle below the base AB in E ; draw the diameter zoc ; join AC , BC , and ACB is the required triangle. For draw CD perpendicular to AB , and produce EM to meet CD in I ; then by reason of the parallels OM , CD ; $EO : OC :: EO :: EM : MI = EM =$ the given distance by construction.

PROB. 4. Given the radius of the circumscribing circle, the distance of the middle of the base from the point where perpendiculars from the angles upon the opposite sides intersect, together with the difference of the angles at the base, to construct the plane triangle.

(Fig. 2.) Describe the circumscribing circle, and draw any radius OE ; make the right angled triangle EOH , having the angle EOH equal to the given difference of the angles at the base. From E to OH apply EM equal to the given distance of the intersection of the perpendiculars from the middle of the base; through M draw the chord AMB at right angles to OH ; draw the diameter EOC , and join AC , BC , and ACB is the required triangle.

PROB. 5. Given the circumscribing circle, and the distance of the point where perpendiculars from the angles, upon the opposite sides intersect from the point where lines from the angles bisecting the opposite sides intersect, together with the difference of the angles at the base to construct the plane triangle.

PROB. 6. Given the base, vertical angle, and the rectangle of the lines from the extremities of the base to the point where perpendiculars from the angles upon the opposite sides intersect; to construct the plane triangle.

PROB. 7. Given the base, vertical angle, and the ratio of two lines from the extremities of the base to the point where perpendiculars from the angles upon the opposite sides intersect; to construct the plane triangle.

The three foregoing problems may be easily constructed, by attentively considering the properties demonstrated in the preceding part of this paper.

PROB. 8. Given the lengths of the three lines from the middle of the sides to the point where perpendiculars from the angles upon the opposite sides intersect; to determine the plane triangle.

ARTICLE XV.

ON SPHERICAL GEOMETRY, *by Mr. DAVIES.*

Continued from page 80.

VI.

7th. The perpendiculars from the angles of a spherical triangle upon the opposite sides, all pass through the same point.

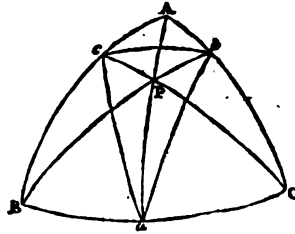
For

$$\cos BA : \cos CA :: \cos BA : \cos CA$$

$$\cos Cb : \cos Ab :: \cos CB : \cos AB$$

$$\cos AC : \cos BC :: \cos AC : \cos BC,$$

or compounding and equating the results, &c. we have



$$\cos BA \cos Cb \cos AC = \cos CA \cos Ab \cos BC \dots (12).$$

Whence (Conv. of 5th.) the three perpendiculars pass through the same point.

In the present case, we have also (since the arcs which are perpendicular to the sides, pass at the same time through one point and through the opposite angles) the simultaneous equations

$$\cos BA \cos Cb \cos AC = \cos AC \cos Ab \cos CB \dots (13)$$

$$\sin Ba \sin Cb \sin Ac = \sin AC \sin Ba \sin Cb \dots (14)$$

$$\tan Ba \tan Cb \tan Ac = \tan AC \tan Ba \tan CB \dots (15)$$

$$\cot Ba \cot Cb \cot Ac = \cot AC \cot Ba \cot CB \dots (16)$$

$$\sec Ba \sec Cb \sec Ac = \sec AC \sec Ba \sec CB \dots (17)$$

$$\operatorname{cosec} Ba \operatorname{cosec} Cb \operatorname{cosec} Ac = \operatorname{cosec} AC \operatorname{cosec} Ba \operatorname{cosec} CB \dots (18).$$

Also multiplying (12 and 13) and, also taking the reciprocals of these

$$\sin 2Ba \sin 2Cb \sin 2Ac = \sin 2Ac \sin 2Ba \sin 2Cb \dots (19)$$

$$\operatorname{cosec} 2Ba \operatorname{cosec} 2Cb \operatorname{cosec} 2Ac = \operatorname{cosec} 2Ac \operatorname{cosec} 2Ba \operatorname{cosec} 2Cb \dots (20).$$

But a corresponding set of properties also hold with respect to the segments into which the perpendiculars divide the angles A, B, C. For by right angled triangles,

$$\left. \begin{array}{l} \cos aAC = \cos AC \sin C \\ \cos cCB = \cos BC \sin B \\ \cos bBA = \cos BA \sin A \end{array} \right\} \begin{array}{l} \cos aAB = \cos AB \sin B \\ \cos cCA = \cos CA \sin A \\ \cos bBC = \cos BC \sin C \end{array} \dots (21).$$

Compounding these we shall have

$$\left. \begin{array}{l} \cos aAC \cos cCB \cos bBA = \cos AC \cos BC \cos BA \sin A \sin B \sin C \\ \cos aAB \cos cCA \cos bBC = \cos AB \cos AC \cos CB \sin A \sin B \sin C \end{array} \right\} (22).$$

But because the three arcs *aa*, *bb*, *cc* pass through the same point, and are perpendicular to the opposite sides, we have by (12. VI.)

$$\cos AC \cos BC \cos BA = \cos AB \cos AC \cos CB \dots (23)$$

hence the right hand members of equation marked (21) are equal, and hence the left hand too; or,

$$\cos aAC \cos cCB \cos bBA = \cos aAB \cdot \cos cCA \cos bBC \dots (24)$$

A set of results corresponding with (13-20) may hence be put down respecting the *angles* into which A, B, C are divided: and as they have frequently presented themselves in enquiries respecting the properties of spherical triangles, I shall write them out.

$$\cos BAA \cos CBb \cos ACC = \cos aAC \cos bBA \cos cCB \dots (25)$$

$$\sin BAA \sin CBb \sin ACC = \sin aAC \sin bBA \sin cCB \dots (26)$$

$$\tan BAA \tan CBb \tan ACC = \tan aAC \tan bBA \tan cCB \dots (27)$$

$$\cot BAA \cot CBb \cot ACC = \cot aAC \cot bBA \cot cCB \dots (28)$$

$$\sec BAA \sec CBb \sec ACC = \sec aAC \sec bBA \sec cCB \dots (29)$$

$$\operatorname{cosec} BAA \operatorname{cosec} CBb \operatorname{cosec} ACC = \operatorname{cosec} aAC \operatorname{cosec} bBA \operatorname{cosec} cCB \dots (30)$$

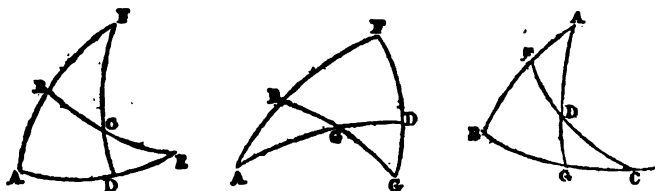
$$\sin 2BAA \sin 2CBb \sin 2ACC = \sin 2aAC \sin 2bBA \sin 2cCB \dots (31)$$

$$\operatorname{cosec} 2BAA \operatorname{cosec} 2CBb \operatorname{cosec} 2ACC = \operatorname{cosec} 2aAC \operatorname{cosec} 2bBA \operatorname{cosec} 2cCB \dots (32)$$

8^o. We may also remark the angles *cab*, *abc*, *bca* are bisected by the arcs *Aa*, *Bb*, *Cc*: as is evident from (VI. 4). the perpendicular from the intersection of two diagonals, as *cb*, *Aa*, coinciding with one of them, *Aa*, are hence *caa* = *baa*, In like manner we may shew that the others are bisected.

VII.

The properties of the complete quadrilateral given by Carnot in his *Géométrie de Position*, and in his *Essai sur la Théorie des Transversales*, so far as the *products of the segments* are concerned, have perfect analogies on the sphere, when the sines of the segments of the arcs are substituted for the products of the lines themselves in *plano*.



For let the annexed figures be the complete quadrilateral, then the triangle *FAD* cut by the transversal *BC* and the triangle *AUG* cut by the transversal *FD* give the following

$$\sin FQ \cdot \sin DG \cdot \sin AB = \sin CD \cdot \sin GA \cdot \sin BF$$

$$\sin CG \cdot \sin DA \cdot \sin BF = \sin GD \cdot \sin AF \cdot \sin BC.$$

Whence multiplying, cancelling DG , BF and giving the result the form of a proportion we obtain

$$\sin CF \cdot \sin CG : \sin FA \sin AG :: \sin BC \sin CD : \sin BA \sin AD \dots (1).$$

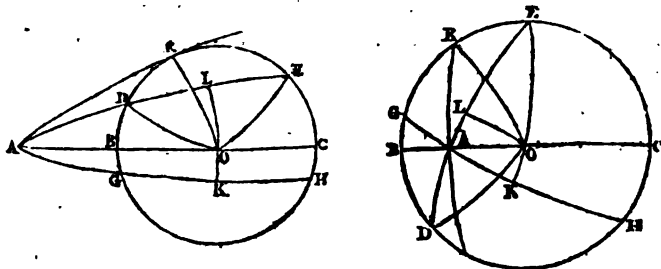
We might now, as Carnot has done, did space permit us, tabulate the results of all the combinations of the lines which arise from considering the seven arcs that constitute the complete quadrilateral and its diagonals under every possible variety of combination : but as it would occupy room which we require for other and far more interesting inquiries, I shall, as Carnot did the contents of the preceding pages, "*abandonné ces conséquences à la sagacité du lecteur.*" I do this, too, with the less reluctance, as after we have treated of *spherical loci*, we shall be able to take a more extended view of the general doctrine of transversals, and to investigate several curious properties respecting them which would be impossible by means such as we have hitherto employed.

SECTION THE SECOND.

ON LESS-CIRCLES OF THE SPHERE IN CONNECTION WITH GREAT-CIRCLE TRANSVERSALS.

VIII.

Let BEC be a less-circle of the sphere, and O its centre*.



Let A be any point on the sphere in its diametrical great-circle, and let a great-circle ADE cut the given less-circle in D

* I have used in my paper in the Edin. Trans. the term *centre*, or *spherical centre*, to express the point on the sphere which is usually called the *pole*, reserving the name of *plane-centre* for the centre of the same circle considered as a *plane figure*. The reasons which induced me to adopt those, are explained in that paper.

and E. What is the relation of the segments of that circle corresponding to Euc. III. 35, 36?

Let $OL \perp AE \therefore DL = LE$.

Then we have seen, (IV. Lem.) that

$$\sin^2 OA \searrow \sin^2 OB = \sin BA \sin AC \dots\dots\dots(1)$$

$$\sin^2 LA \searrow \sin^2 LD = \sin DA \sin AE \dots\dots\dots(2)$$

But by right angled triangles $\cos LA = \cos OA \sec LO \dots\dots(3)$

$$\cos LD = \cos OD \sec LO \dots\dots(4)$$

Hence

$$\cos^2 LA = \cos^2 OA \sec^2 LO, \text{ or } \sin^2 LA = 1 - \cos^2 OA \sec^2 LO \dots\dots(5)$$

$$\cos^2 LD = \cos^2 OD \sec^2 LO, \text{ or } \sin^2 LD = 1 - \cos^2 OD \sec^2 LO \dots\dots(6)$$

From (5, 6) we find

$$\begin{aligned} \sin^2 LA \searrow \sin^2 LD &= (\cos^2 OD \searrow \cos^2 OA) \sec^2 LO \\ &= (\sin^2 OA \searrow \sin^2 OD) \sec^2 LO \\ &= (\sin^2 OA \searrow \sin^2 OB) \sec^2 LO \dots\dots\dots(7) \end{aligned}$$

By comparing (2, and 7) we have at once the relation sought, viz.

$$\sin EA \sin AD = \sin CA \sin AB \sec^2 LO \dots\dots\dots(8)$$

Again, $\cos LO \cos LA = \cos OA$, or $\sec^2 LO = \frac{\cos^2 LA}{\cos^2 AO}$: and

hence we have

$$\sin EA \sin AD : \sin CA \sin AB :: \cos^2 LA : \cos^2 AO \dots\dots(9)$$

Let now another secant AGH cut the circle BDC in G and H, and draw the perpendicular OK upon it. Then applying (9) to both secants,

$$\sin HA \cdot \sin AG : \sin EA \cdot \sin AD :: \cos^2 AK : \cos^2 AL \dots\dots(10)$$

$$:: \cos^2 OL : \cos^2 OK \dots\dots(11)$$

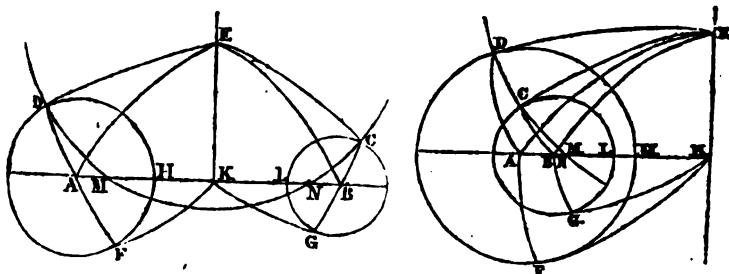
We might also remark that from (9) we get

$$\frac{\sin EA \sin AD}{\cos^2 LA} = \frac{\sin CA \cdot \sin AB}{\cos^2 OA} = \text{const.}$$

$$\text{Hence } \frac{\sin EA}{\cos LA} \cdot \frac{\sin AD}{\cos LA} = \text{constant also} \dots\dots\dots(12)$$

VIII.

1. Let DHF, and GLC be two less circles, whose centres are A, B.



Divide the arc AB joining their centres so that

$$\cos AH : \cos BL :: \cos AK : \cos KB \dots \dots \dots (1)$$

and raise the perpendicular great circle KE. Then tangents from any point E in KE to the given circles will be equal to one another.

Let ED, EC be two such tangents: draw the radii AD, BC to the points of contact. Draw also the tangents KF, KG, and join AF, BG.

Then in the first place, AFK, BGK are right angles: and hence

$$\cos KF \cos FA = \cos AK \dots \dots \dots (1)$$

$$\cos KG \cos GB = \cos BK \dots \dots \dots (2)$$

But by hypothesis, $\cos AH \cos BK = \cos BL \cos AK \dots \dots (3)$

If therefore we multiply equation (1) by $\cos BL$ and (2) by $\cos AH$, and in virtue of equation (3) we equate the results, we shall have

$$\cos KF \cos FA \cos BL = \cos KG \cos GB \cos AH \dots \dots (4)$$

But $GB = BL$ and $FA = FH$. Hence their cosines are equal.

This reduces (4) to $\cos KF = \cos KG$; and hence taking $\cos^{-1}$ of both sides we finally obtain, $KF = KG \dots \dots \dots (5)$

Next let any other point be taken, as E.

Here we have $\cos AK \cos KE = \cos AE = \cos AD \cos ED \dots \dots (6)$

$$\cos BK \cos KE = \cos BE = \cos BC \cos CE \dots \dots (7)$$

From (6, 7) we have,

$$\cos AD \cos DE \cos BK = \cos BC \cos CE \cos AK \dots \dots (8)$$

But, by hypothesis, $\cos AK \cos BL = \cos AH \cos BK$, or since

$BL = BC$ and $AH = AD$, the equation (8) reduces to $\cos ED = \cos EC$, or taking $\cos^{-1}$ it is

$$ED = EC \dots \dots \dots (9)$$

The proposition is therefore established: and it may be remarked that *it could not possibly have been established by any kind of projection.*

The converse of this also follows; that the locus of the points from which equal great-circle tangents can be drawn to two circles of the sphere, is a great-circle at right angles to the circle AB , and dividing it so that the cosines are as the cosines of the radii of the circles themselves. This will, as well as a good number of converse propositions be discussed regularly under the form of *loci*, in their proper place in this series.

2. Draw from E , a circle whose spherical radius is the tangent ED or EC , then, wherever on KE , the centre E is taken, the circle will cut the arc AB in the same two points M and N .

$$\text{By } \triangle EDA \text{ we have } \cos ED \cos DA = \cos EA \dots (10)$$

$$\text{By } \triangle EKA \text{ we have } \cos EK \cos KA = \cos EA \dots (11)$$

$$\begin{aligned} \text{By } \triangle EKM \text{ we have } \cos EK \cos KM &= \cos EM \\ &= \cos ED \dots (12) \end{aligned}$$

From (10, 11, 12) we have $\cos AK \sec AD = \cos KM \dots (13)$
or since AK, AD (or AL) are constant, the arc KM is constant, and M is therefore a constant point. In like manner N is a constant point.

3. It also follows that $KM = KN$. For by right angled $\triangle s$

$$\cos MK \cos KE = \cos EM$$

$$\cos NK \cos KE = \cos EN$$

$\therefore$ since $EM = EN$, we have $MK = NK$, &c.

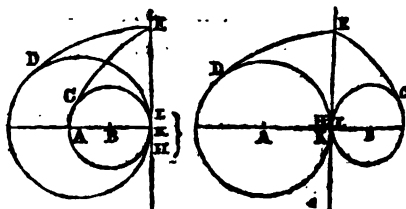
This might have been deduced from $KF = KG$, and the property just established. We also learn from this, that

$$\cos KM \text{ (or } KN) = \cos KF = \cos KA \sec AH = \cos KP \sec BL \dots (14)$$

Cor. 1. When the circles touch one another, then H, K, L (in both figures) coincide, and we have

$$\cos AK : \cos BK :: \cos AK : \cos BK,$$

and hence the great circle tangent at the point K of contact is the radical axis of the circles (A) and (B) . This indeed might have been derived independently.

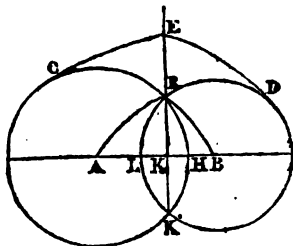


Cor. 2. Let the circles intersect in R and R'; join RR' by a great circle cutting AB in K. Then RR' is the radical axis.

For RR' is perpendicular to AB, and hence RK is the perpendicular from the vertical angle R of the triangle ARB upon the base. We have, therefore, the known property,

$$\cos AK : \cos BK :: \cos AR : \cos BR,$$

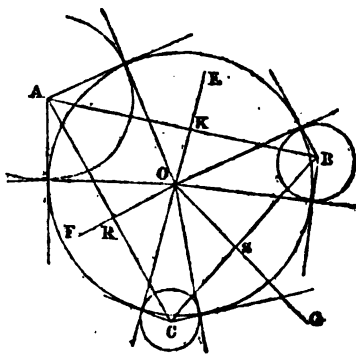
thus fulfilling the condition according to which the radical axis of the circles (A) and (B) was determined,



IX.

Let there be three circles (A), (B), (C), and let great circles be drawn to each of them, in the manner that EK was to (A) (B) in (VIII.). These circles correspond to the *axe-radical* of the French Geometers, when discussing circles in *plano*. Let these intersect the arcs AB, BC, CA in K, S, R.

Then EK, GS, FR will all pass through the same point O on the sphere. The method of proof is precisely the same as that used in the same enquiry in *plano*. In the same way it may be proved that a circle

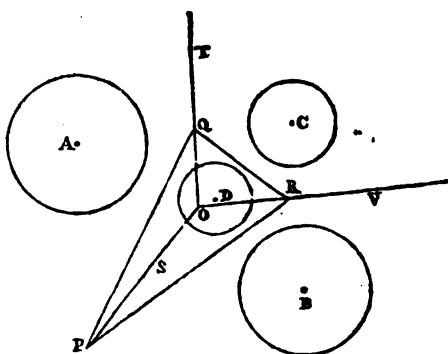


described from O, with the said equal tangent, will cut the three circles at right angles, or be orthogonal to them all.

We may obtain from this a number of curious properties perfectly analogous to those which M. Steiner has given in *tom. XVII. of the Annales des Mathématiques*, pp. 37—59; and they follow from what has been just established by reasoning to which the very same words apply throughout as M. Steiner has employed.

The corresponding properties of cones of revolution may be derived in this way: but perhaps a preferable method in respect to retaining the geometrical character of the enquiry is that which is employed in the *Repos*, Vol. V. p. 111—121.

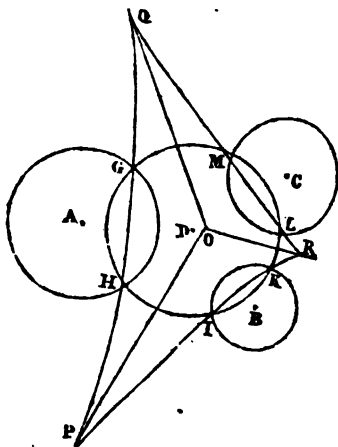
Let three circles (A), (B), (C) have the radical centre O, and radical axes OS, OV, OT. Let any other circle (D) be traced upon the same spherical surface; then this combined with each pair of the circles (A), (B), (C) will give three new radical axes.



Then the radical centre of (A), (B), (D) is R in OS the radical axis of (A), (B), radical centre of (A), (C), (D) is Q in OT the radical axis of (A), (C), radical centre of (B), (C), (D) is P in OV the radical axis of (B), (C).

It will follow then that whilst (A), (B), (C) remain the same, and (D) is a variable circle the three new radical centres P, Q, R, arising from each new circle (D) will always be situated upon OS, OR, OT.

Cor. Hence we readily derive the property on the sphere, analogous to that in Quest. 483. Vol. V, Repository, in plano, viz. that if three given circles on the sphere, (A), (B), (C) be cut by a fourth (D) in the points G, H, I, K, and L, M respectively: then the great-circle chords GH, IK, and LM being drawn will form a spherical triangle whose angle will always be posited upon the three great circles OR, OP, OQ.

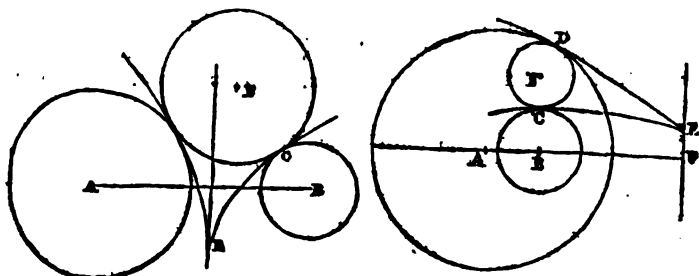


For these are respectively the radical axes of the fourth circle combined with each of the three given ones: and hence by what has been just established, the proposition is true.

By a similar process the plane theorem might have been established, as I was then well aware: but the process there employed was chosen on account of its giving me an opportunity of

laying down the very useful lemmata which were rendered subservient to that particular method of demonstration.

Another neat property belonging to the radical axis in the plane series of propositions should have its analogues noticed here. Let (A), (B) be two circles, and any third circle whatever be drawn to touch them in C, D; then the tangents at C, D will meet at E in the radical axis of (A), (B). For the tangents to (A), (E) and to (B), (E) at the points of contact are the respective radical axes of those pair of circles (VIII); and hence they as well as KE pass through the radical centre of all three; or in other words the variable radical centre (rendered variable by the variable magnitude and position of (E)) is always situated upon the radical axis of (A) and (B).



This property in plano has been often proved, and indeed is quite obvious. I have made it the basis of the elementary part of some speculations concerning the tactions of cones, spheres, and cylinders, in Vol. V. of the Repository. I shall continue that enquiry in the course of the present volume—probably in the next number. My reason, indeed, for dwelling rather more amply upon the properties of the radical axis and radical centre in the present case is—that as the method here employed may by a slight mutation of expression, to adapt it to the plane theorems, furnish a varied and more concise demonstration of them than is contained in that paper, at the same time that it applies directly to the sphere by means of principles purely spherical.

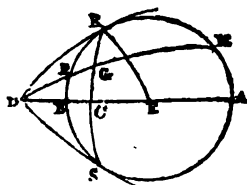
X.

Let tangents drawn from any point D to a circle, touch it in R, S. Let the chord of contact RS be described cutting AB in C: then

$$\sin AC : \sin CB :: \sin AD : \sin DB.$$

$$\text{For } \cos REC = \cot RE \tan EC \dots (1)$$

$$\cos REC = \tan RL \cot ED \dots (2)$$



By equating and reducing which we find

$$\tan EC \tan ED = \tan^2 ER = \tan^2 EB \dots \dots (3).$$

Hence (by III. 6.) we have

$$\sin AC : \sin CB :: \sin AD : \sin DE \dots \dots (4).$$

Hence, also, we readily infer that any other secant FH is harmonically divided in D and C*: and we should thence be able to derive the spherical analogues to Maclaurin's properties of the inscribed and circumscribed trapezia,—the *mystic hexagon* of Pascal,—the circumscribed hexagon of Brianchon—and the two inscribed triangles, which I first pointed out in the Philosophical Magazine, vol. 68, p. 338. As however, I could not do justice to subjects so ample, without prolonging this section beyond its prescribed limits, I shall defer them for a future one: and conclude the present by considering the spherical analogue to Monge's beautiful theorem concerning the mutual tangents of three circles. *Géom. Desc. Art. 39, 40:*

Let two circles (A), (B) in the sphere, have great circle tangents drawn to them, viz. EF, GH, IK, GM: and let them meet two and two in C, D. Then AB is harmonically divided in C, D. Put $AD \cdot E = AD \cdot G = D$.

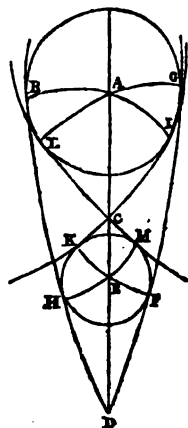
$$\left. \begin{array}{l} \text{For } \sin BF = \sin BD \sin D, \\ \sin AE = \sin AD \sin D \end{array} \right\}$$

$$\therefore \sin AE : \sin BF :: \sin AD : \sin BD \dots (1)$$

$$\sin y \sin AI : \sin BM :: \sin AC : \sin BC \dots (2)$$

But $AE = AI$, and $BF = BM$. Hence

$$\sin AC : \sin CB :: \sin AD : \sin DB \dots (3). \dagger$$

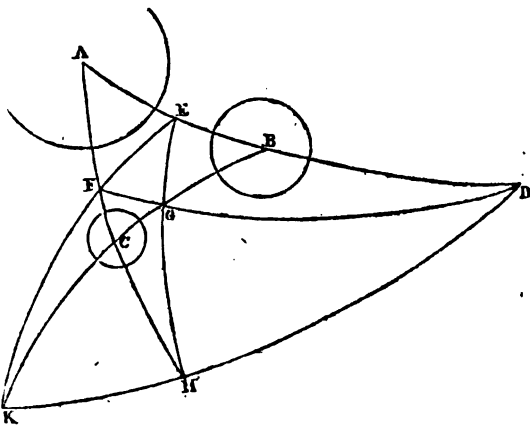


* I regret, though now too late to correct it, that this is not demonstrated at length instead of *inferred*. The paper is in type, and it would be inconvenient to take the form to pieces: but it shall be demonstrated in the subsequent sheet of this Article.—Bath, June 1, 1832.

† The whole series of properties that belong to *poles of similitude*, and to the harmonical divisions of lines drawn through them, in *plano*, belong also to the same poles and lines in *sphæro*. Some beautiful analogies to well known plane loci, might also, had we room, be stated and proved; as, to take a single instance, the 6th proposition of the first book of the Plane Loci of Apollonius.

Let three circles on the sphere, (A), (B), (C) and let the pairs of common tangents drawn as in the last case, meet in E, D; F, H; and G, K. Then will the sets of points K, H, D; K, F, E; H, G, E; and D, G, F; be situate in four great circles of the sphere.

For the spherical triangle ABC is cut so that (putting a, b, c for radii of A, B, C)



$$\sin AE : \sin EB :: \sin a : \sin b \dots \dots (4)$$

$$\sin FC : \sin FA :: \sin c : \sin a \dots \dots (5)$$

$$\sin BG : \sin GC :: \sin b : \sin c \dots \dots (6)$$

$$\sin AD : \sin DB :: \sin a : \sin b \dots \dots (7)$$

$$\sin HC : \sin AH :: \sin c : \sin a \dots \dots (8)$$

$$\sin KB : \sin KC :: \sin b : \sin c \dots \dots (9)$$

Compounding these in threes, we obtain the following equations,

$$(4, 5, 9) \text{ give, } \sin AE \cdot \sin FC \cdot \sin KB = \sin EB \sin FA \sin KC \dots (10)$$

$$(4, 6, 8) \text{ give, } \sin AE \sin BG \sin CH = \sin EB \sin GC \sin HA \dots (11)$$

$$(5, 6, 7) \text{ give, } \sin FC \cdot \sin BG \sin AD = \sin FA \sin GC \sin DB \dots (12)$$

$$(7, 8, 9) \text{ give, } \sin AD \sin HC \sin KB = \sin DB \sin AH \sin KC \dots (13)$$

Whence (by V. 2.) the equations 10, 11, 12, 13 respectively shew that E, F, K; E, G, H; F, G, D; and K, H, D are situated in great circles of the sphere.

It will now be quite evident that this class of enquiries is as simple as any that can occupy the geometer's attention. The only theorems used throughout this paper, are the analogies of Napier, and the proportionality between the sines of the sides and angles of spherical triangles; so that the whole process may be considered of the most elementary kind. Nor even when we proceed to a greater extent, so long as our enquiries have reference

to subjects of the same nature, shall we encounter any thing more difficult than we have yet done. It is not too much, then, to hope that this legitimate process will be substituted in all our enquiries for the foreign, confined, and inelegant one, of gnomonic projection, when we investigate the properties of spherical figures. The analogies are often remarkably close between the plane and spherical figure; and this will suggest during the study of plane properties the existence of the spherical ones. Yet we must be cautious in such inferences, and examine each individual case before we admit its truth. An instance of this necessity is furnished by the figure we have just been discussing: and in every case when angles in segments of circles are concerned or *implied* in the plane theorem, that theorem will want its analogue on the sphere, the locus of a spherical angle on a given base of the sphere, *not being a circular arc*.* I particularly allude here to the property established in the Solution of §12 of the present number of the Repository,—that the locus of the points from which two circles subtend equal angles, is a circle on the diameter of similitude. It is not the case here, as the circles ED , FH , GK do not contain *right angles*, a circumstance implied in the assumed truth of the property on the sphere.

I would suggest, in conclusion of this section, *that in our solutions of questions in periodical works (as the Repository and Diaries) we should adopt the practice of ascertaining in all cases when we investigate a plane theorem, whether its analogue on the sphere be true or not.* The labour would commonly be inconsiderable, as in many cases the process is even simpler than in *plano*; and very rarely is it in reality more difficult. We should thus, whilst we were adding to our knowledge, and increasing our astonishment in contemplating the beautiful and varied relation of figures, acquire a facility in the use of the trigonometrical calculus which would be of the utmost value to us in almost every department of physical enquiry as well as in purely mathematical researches.

- * This locus will be investigated hereafter. Indeed it is *partially* discussed in my paper in the Ed. Trans. Art. XXXVII. I must not, however, be understood to say that all loci which are circles in *plano*, become the *same curve* in *sphæro*. The locus of the points from which two given circles subtend equal angles, and the locus of the vertex of a triangle whose base and vertical angle are given, though both circles in *plano*, are different curves in *sphæro*. And so of others.

ARTICLE XVI.

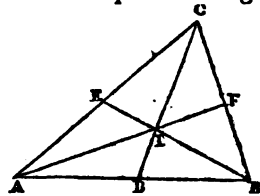
Two Indeterminate Problems.

By Mr. JAMES CUNLIFFE, London.

PROBLEM I.

To find numerical values for the sides of a plain triangle such, that the lengths of the three right lines from the angles to the middle of the opposite sides, may all be expressed by rational numbers.

Solution. ACB is a triangle, CD, AF and BE, lines from the angles to the middle of the opposite sides.



Put $AC = x$, $BC = y$, and $AB = z = x + y - d$.

Then by a known property, $4 \times CD^2 = 2 \times (AC^2 + BC^2) - AB^2$
 $= 2x^2 + 2y^2 - (x + y - d)^2 = x^2 - 2xy + y^2 + 2d \times (x + y) - d^2 =$
 a square,

$4 \times AF^2 = 2 \times (AC^2 + AB^2) - BC^2 = 4x^2 + 4xy + y^2 - 4d \times (x + y) + 2d^2 =$
 a square,

$4 \times BE^2 = 2 \times (BC^2 + AB^2) - AC^2 = x^2 + 4xy + 4y^2 - 4d \times (x + y) + 2d^2 =$
 a square.

Put $4x^2 + 4xy + y^2 - 4d \times (x + y) + 2d^2 =$
 $(2x + y - m)^2 = 4x^2 + 4xy + y^2 - 2m \times (2x + y) + m^2$
 whence $x = \frac{2d^2 - 4dy + 2my - m^2}{4d - 4m}$; and $y =$

$$\frac{2d^2 - 4dx + 4mx - m^2}{4d - 2m}.$$

Again put $x^2 + 4xy + 4y^2 - 4d \times (x + y) + 2d^2 =$
 $(x + 2y - n)^2 = x^2 + 4xy + 4y^2 - 2n \times (x + 2y) + n^2$
 whence $x = \frac{2d^2 - 4dy + 4ny - n^2}{4d - 2n}$; and $y =$

$$\frac{2d^2 - 4dx + 2nx - n^2}{4d - 4n}. \text{ Therefore}$$

$$2x = \frac{2d^2 - 4dy + 2my - m^2}{2d - 2m} = \frac{2d^2 - 4dy + 4ny - n^2}{2d - n}; \text{ whence } y =$$

$$\frac{d^2 \times (4m - 2n) - 2d \times (m^2 - n^2) - mn \times (2n - m)}{4d \times (m + n) - 6mn}$$

$$\text{Also } 2y = \frac{2d^2 - 4dx + 4mx - m^2}{2d - m} = \frac{2d^2 - 4dx + 2nx - n^2}{2d - 2n};$$

$$\text{whence } x = \frac{d^2 \times (4n - 2m) + 2d \times (m^2 - n^2) - mn \times (2m - n)}{4d \times (m + n) - 6mn}$$

$$x - y = \frac{-6d^2 \times (m - n) + 4d \times (m^2 - n^2) - 3mn(m - n)}{4d \times (m + n) - 6mn},$$

$$x + y = \frac{2d^2 \times (m + n) - mn \times (m + n)}{4d \times (m + n) - 6mn},$$

$$z = x + y - d = \frac{2d^2 \times (m + n) - 6mnd + mn \times (m + n)}{4d \times (m + n) - 6mn}.$$

Omitting the common denominator

$4d \times (m + n) - 6mn$; we may take $x = d^2 \times (4n - 2m) + 2d \times (m^2 - n^2) - mn \times (2m - n)$, $y = d^2 \times (4m - 2n) - 2d \times (m^2 - n^2) - mn \times (2m - n)$, and $z = 2d^2 \times (m + n) - 6mnd + mn \times (m + n)$.

These values will make the two expressions $2x^2 + 2y^2 - z^2$; and $2y^2 + 2z^2 - 2x^2$ squares. There remains therefore only the expression $2x^2 + 2y^2 - z^2 = 36d^4 \times (m - n)^2 - 24d^3 \times (m + n) \times (2m^2 - 5mn + 2n^2) + 4d^2 \times (4m^4 + 7m^3n - 39m^2n^2 + 7mn^3 + 4n^4) - 12dmn \times (m + n) \times (2m^2 - 5mn + 2n^2) + 9m^2n^2 \times (m - n)^2$, to be made a square.

$$\begin{aligned} \text{Assume } 6d^2 \times (m - n) - \frac{2d \times (m + n) \times (2m^2 - 5mn + 2n^2)}{m - n} \\ 3mn + (m - n) \text{ for the root, that is, put } 36d^4 \times (m - n)^2 - 24d^3 \times \\ (m + n) \times (2m^2 - 5mn + 2n^2) + 4d^2 \times (4m^4 + 7m^3n - 39m^2n^2 + \\ 7mn^3 + 4n^4) - 12dmn \times (m + n) \times (2m^2 - 5mn + 2n^2) + 9m^2n^2 \times \\ (m - n)^2 = \left\{ 6d^2 \times (m - n) - \frac{2d \times (m + n) \times (2m^2 - 5mn + 2n^2)}{m - n} \right. \\ \left. 3mn \times (m - n) \right\}^2 = 36d^4 \times (m - n)^2 - 24d^3 \times (m + n) \times \\ (2m^2 - 5mn + 2n^2) + \frac{4d^2 \times (m + n)^2 \times (2m^2 - 5mn + 2n^2)^2}{m - n} \\ 36d^2mn \times (m - n)^2 + 12dmn \times (m + n) \times (2m^2 - 5mn + 2n^2) \\ + 9m^2n^2 \times (m - n)^2. \text{ Which being properly reduced gives} \\ d = \frac{3 \times (m - n) \times (m^2 - n^2) \times (2m^2 - 5mn + 2n^2)}{10m^4 - 4m^3n + 60m^2n^2 - 41mn^3 + 10n^4} = \\ 3 \times \frac{(m - n) \times (m^2 - n^2) \times (2m^2 - 5mn + 2n^2)}{10 \times (m - n)^4 - mn \times (m^2 + n^2)}; \text{ where } m \text{ and} \end{aligned}$$

n , may be expounded by any rational numbers at pleasure.

Example 1. Take $m = 3$, and $n = 2$; then $d = \frac{15}{17}$, $x =$

$$2d^2 + 10d - 24 = 2 \times (d^2 + 5d - 12) = -\frac{3236}{289} =$$

$$-\frac{12}{289} \times 328, y = 8d^2 - 10d - 6 = 2 \times (4d^2 - 5d - 3)$$

$$= -\frac{2484}{289} = -\frac{12}{289} \times 207, \text{ and } z = 10d^2 - 36d + 30 =$$

$$2 \times (5d^2 - 18d + 15) = \frac{1740}{289} = \frac{12}{289} \times 145. \text{ Wherefore we may take 328, 207 and 145 for the sides of a triangle that will answer.}$$

If the sides, and lengths of the lines from the angles to the middle of the opposite sides, are required all to be integers, then the sides must obviously be all even numbers: wherefore if there happens to be odd numbers amongst those expressing the sides, let them all be multiplied by 2: thus each of the preceding numbers multiplied by 2 gives 656, 414 and 290, for the sides of a triangle, in which the lengths of the lines from the angles to the middle of the opposite sides, will all be expressed by integers.

Example 2. Take $m = 3$, and $n = 1$; then $d = \frac{24}{13}$; $x =$

$$2d^2 + 16d - 15 = \frac{1805}{169} = \frac{15}{169} \times 87, y = 10d^2 - 16d + 3 =$$

$$\frac{1275}{169} = \frac{15}{169} \times 85, \text{ and } z = 8d^2 - 18d + 12 = \frac{1020}{169} = \frac{15}{169} \times 68,$$

and therefore we may take 87, 85 and 68 for the sides of a triangle that will answer. Multiplying each of the foregoing numbers by 2, we shall have 174, 170 and 136 for the sides of a triangle in which the lengths of the lines from the angles to the middle of the opposite sides will all be whole numbers.

Example 3. Take $m = 2$ and $n = 1$; then $d = \frac{27}{41}$; $x =$

$$-8d^2 + 6d + 10 = 2(-4d^2 + 3d + 5) = \frac{17620}{1681} = \frac{20}{1681} \times 881, y$$

$$= 10d^2 - 6d - 8 = 2 \times (5d^2 - 3d - 4) = -\frac{12800}{1681} =$$

$$-\frac{20}{1681} \times 640, z = 2d^2 + 12d - 2 = 2 \times (d^2 + 6d - 1) =$$

$$\frac{11380}{1681} = \frac{20}{1681} \times 569. \text{ Wherefore we may take 881, 640 and 569 for the sides of a triangle that will answer.}$$

Example 4. Take $m=3$, and $n=-1$; then $d=\frac{48}{37}$, $x=10d^2+16d+21=\frac{34125}{1369}=\frac{105}{1369}\times 325$, $y=14d^2-16d-15=-\frac{16695}{1369}=-\frac{105}{1369}\times 159$, $z=4d^2+18d-6=\frac{32970}{1369}=\frac{105}{1369}\times 314$. Therefore we may take 325, 314 and 159 for the sides of another triangle that will answer.

Example 5. Take $m=5$ and $n=4$: then $d=\frac{3}{5}$: $x=6d^2+18d-120=6\times(d^2+3d-20)=-\frac{6}{25}\times 446$, $y=12d^2-18d-60=6\times(2d^2-3d-10)=-\frac{6}{25}\times 277$, and $z=18d^2-120+180d=6\times(3d^2-20d+30)=\frac{6}{25}\times 477$, and therefore 477, 446 and 277 are three numbers that will answer.

We have now found five different sets of numbers of the kind required, and all of them moderately low.

It is evident from a bare inspection of the properties upon which the foregoing solution is founded, that the proposed problem reduces to the finding of three rational square numbers, such, that if from double the sum of any two, the remaining square is subtracted, the result shall be a square.

The sum of any two of the numbers expressing the sides, must be greater than the third, or no triangle can be formed with them.

Another formula for the value of d , may be obtained in the following manner. The general expression to be made a square is $36d^4\times(m-n)^2-24d^3\times(m+n)\times(2m^2-5mn+2n^2)+4d^2\times(4m^4+7m^3n-39m^2n^2+7mn^3+4n^4)-12dmn\times(m+n)\times(2m^2-5mn+2n^2)+9m^2n^2\times(m-n)^2$. Assume $6d^2\times(m-n)+2d\times\frac{m+n}{m-n}\times(2m^2-5mn+2n^2)-3mn\times(m-n)$ for the root, that is, put $\{6d^2\times(m-n)+2d\times\frac{m+n}{m-n}\times(2m^2-5mn+2n^2)-3mn\times(m-n)\}^2$

$= 36d^4 \times (m-n)^2 + 24d^3 \times (m+n) \times (2m^2 - 5mn + 2n^2) + 4d^2 \times (m+n^2) \times (2m^2 - 5mn + 2n^2)^2$
 $\frac{(m-n)^2}{(m-n)^2} - 36d^2mn \times (m-n)^2 - 12dmn \times (m+n) \times (2m^2 - 5mn + 2n^2) + 9m^2n^2 \times (m-n)^2$, which after proper reduction gives

$$d = \frac{10m^3n - 41m^2n^2 + 60m^3n^3 - 41m^2n^4 + 10mn^5}{6 \times (m+n) \times (m-n)^2 \times (2m^2 - 5mn + 2n^2)}$$

or $d = mn \times \frac{\{10 \times (m-n)^4 - mn \times (m^2 + n^2)\}}{6 \times (m-n) \times (m^2 - n^2) \times (2m^2 - 5mn + 2n^2)}$: where m , and n , may be taken at pleasure.

Example 1. Take $m=3$, and $n=2$; then $d = \frac{17}{5}$, $x =$

$$2 \times (d^2 + 5d - 12) = \frac{828}{25} = \frac{4}{25} \times 207, y = 2 \times (4d^2 - 5d - 3)$$

$$= \frac{1312}{25} = \frac{4}{25} \times 328, \text{ and } z = 2 \times (5d^2 - 18d + 15) = \frac{580}{25} = \frac{4}{25}$$

$\times 145$. Therefore 328, 207 and 145, will answer being exactly the same numbers, as deduced in the first Example by the other formula for the value of d .

Example 2. Take $m=3$, and $n=1$; then $d = \frac{13}{16}$.

$$x = -2d^2 + 16d - 15 = -\frac{850}{256} = -\frac{10}{256} \times 85, y = 10d^2 - 16d + 3$$

$$= -\frac{870}{256} = -\frac{10}{256} \times 87, z = 8d^2 - 18d + 12 = \frac{680}{256} = \frac{10}{256} \times 68.$$

Therefore 87, 85 and 68, are three numbers that will answer; and are the same with those found in the second Example by the other formula for d .

Example 3. Take $m=2$, and $n=-1$; then $d = -\frac{41}{27}$.

$$x = 2 \times (-4d^2 + 3d + 5) = -\frac{12800}{729} = -\frac{20}{729} \times 640, y = 2 \times$$

$$(5d^2 - 3d - 4) = \frac{17620}{729} = \frac{20}{729} \times 881, z = 2 \times (d^2 + 6d - 1)$$

$$= -\frac{11380}{729} = -\frac{20}{729} \times 569. \text{ Therefore } 569, 640 \text{ and } 881, \text{ are}$$

three numbers that will answer, and these are the same, as the

numbers found in the third example of the first formula for the value of d .

Example 4. Take $m = 3$, and $n = -1$; then $d = -\frac{7}{3^2}$,
 $x = -10d^2 + 16d + 21 = -\frac{11130}{1024} = -\frac{70}{1024} \times 159$, $y = 14d^2 -$
 $16d - 15 = \frac{22750}{1024} = \frac{70}{1024} \times 325$, $z = 4d^2 + 18d - 6 = -\frac{21980}{1024} =$
 $= -\frac{70}{1024} \times 314$. Therefore 159, 314 and 325 are three num-
 bers that will answer, and are the same as the numbers found in
 the fourth Example by the other formula for d .

Example 5. Take $m = 4$ and $n = 3$: then $d = \frac{58}{7}$, $x =$
 $4d^2 + 14d - 60 = \frac{16200}{49} = \frac{20}{49} \times 810$, $y = 10d^2 - 14d - 24 =$
 $\frac{26780}{49} = \frac{20}{49} + 1339$, $z = 14d^2 - 72d + 84 = \frac{21980}{49} = \frac{20}{49} \times 1099$.
 Therefore 810, 1099 and 1339 are three numbers that will
 answer.

In the foregoing solution, I have put down expressions for $x - y$, and $x + y$: this at first sight may be thought unnecessary; but in my opinion they will be found to facilitate the calculation. The best expression to be made a square is $2x^2 + 2y^2 - z^2 = 2x^2 + 2y^2 - (x + y - d)^2 = x^2 - 2xy + y^2 + 2d \times (x + y) - d^2 = (x - y)^2 + 2d \times (x + y) - d^2 =$ a square. This explains the reason of putting down those expressions.

Example 6. Take $m = 5$ and $n = 4$; then $d = \frac{50}{3}$, $x = 6d^2 +$
 $18d - 120 = \frac{16620}{9} = \frac{60}{9} \times 277$, $y = 12d^2 - 18d - 60 =$
 $\frac{26760}{9} = \frac{60}{9} \times 446$, $z = 18d^2 - 120d + 180 = \frac{28620}{9} = \frac{60}{9} \times 477$.

Therefore 277, 446 and 477 are three numbers that will answer. The same numbers were found at Example 5 of the other formula for the value of d .

PROBLEM II.

Find three such square numbers, that the difference between the sum of any two of them, and the third may be a square number.

Solution. Let the three required squares be denoted by x^2 , y^2 and z^2 ,

$$\text{And by the question put } \begin{cases} x^2 + y^2 - z^2 = a^2 \\ x^2 + z^2 - y^2 = b^2 \\ y^2 + z^2 - x^2 = c^2 \end{cases}$$

The sum of the three is $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$, and from this sum subtracting each of the preceding equations in succession, we shall have $2z^2 = b^2 + c^2$ } To those who are a
 $2y^2 = a^2 + c^2$ } little conversant in
 $2x^2 = a^2 + b^2$ } these matters, it will

be easy to perceive, that we may take $z = m^2 + n^2$;
 $b = m^2 + 2mn - n^2$, and $c = m^2 - 2mn - n^2$.

Also $y = p^2 + q^2$; $a = p^2 + 2pq - q^2$; and $c = p^2 - 2pq - q^2$;
 therefore $c = m^2 - 2mn - n^2 = p^2 - 2pq - q^2$. For b^2 , z^2 and c^2 ,
 are obviously three squares in arithmetical progression, of which
 z^2 , is the mean or middle term. Also a^2 , y^2 , and c^2 are squares
 in arithmetical progression, and y^2 the middle term. * $m^2 -$
 $2mn - n^2 = p^2 - 2pq - q^2$, completing the square of the first
 side, $m^2 - 2mn + n^2 = p^2 - 2pq + 2n^2 - q^2$. Now the first side of
 the preceding expression is a square, therefore the other side
 must be so too; and this will manifestly be the case, if $n = q$ for
 then $m^2 - 2mn + n^2 = p^2 - 2pq + q^2$, that is, $(m - n)^2 =$
 $(q - p)^2$, or $m - n = q - p$, whence $p = q + n - m = 2n - m$.
 Hence $a = p^2 + 2pq - q^2 = (2n - m)^2 + 2n \times (2n - m) - n^2$
 $= m^2 - 6mn + 7n^2$; and hence again $2x^2 = a^2 + b^2 =$
 $(m^2 - 6mn + 7n^2)^2 + (m^2 + 2mn - n^2)^2 = 2 \times (m^4 - 4m^3n +$
 $26m^2n^2 - 44mn^3 + 25n^4)$ or $x^2 = m^4 - 4m^3n + 26m^2n^2 - 44mn^3 +$

$$25n^4 = \text{a square} = (m^2 - \frac{22mn}{5} + 5n^2)^2 = m^4 - \frac{44m^3n}{5} +$$

$$\frac{22^2}{25} \times m^2n^2 + 10m^2n^2 - 44mn^3 + 25n^4, \text{ which reduces to } \frac{m}{n} =$$

$$\frac{7}{10}; \text{ therefore we may take } m = 7, \text{ and } n = 10; \text{ whence } z =$$

$$m^2 + n^2 = 149, y = p^2 + q^2 = 5n^2 - 4mn + m^2 = 269, \text{ and}$$

$$x = m^2 - \frac{22mn}{5} + 5n^2 = 241.$$

Therefore 149, 241 and 269 are three numbers that will
 t 2

answer the question, and they are the least integral numbers, that I have been able to find that will answer, all being different.

Or we may proceed more generally as follows.

We have seen that $2z^2 = b^2 + c^2$; $2y^2 = a^2 + c^2$; and $2x^2 = a^2 + b^2$. From the two former of these $c^2 = 2z^2 - b^2 = 2y^2 - a^2$. Put $y = pv - z$, and $a = qv + b$, then $2z^2 - b^2 = 2y^2 - a^2 = 2 \times (pv - z)^2 - (qv + b)^2 = 2z^2 - b^2 - 4pvz - 2qvb + v^2 \times (2p^2 - q^2)$, whence $v = \frac{4pz + 2qb}{2p^2 - q^2}$; and hence $y = pv - z = \frac{2 \times (2p^2 + q^2) + 2pqb}{2p^2 - q^2}$; and $q = qv + b = \frac{b \times (2p^2 + q^2) + 4pqz}{2p^2 - q^2}$.

Now $2z^2 = b^2 + c^2$, that is, b^2 , z^2 and c^2 are three squares in arithmetical progression, as before remarked, z^2 being the middle term: we may therefore take $z = m^2 + n^2$; $b = m^2 + 2mn - n^2$, and $c = m^2 - 2mn - n^2$; whence $y = \frac{z \times (2p^2 + q^2) + 2pqb}{2p^2 - q^2} = \frac{(2p^2 + q^2) \times (m^2 + n^2) + 2pq \times (m^2 + 2mn - n^2)}{2p^2 - q^2}$
 $= \frac{m^2 \times (2p^2 + 2pq + q^2) + 4mnpq + n^2 \times (2p^2 - 2pq + q^2)}{2p^2 - q^2}$
 $a = \frac{b \times (2p^2 + q^2) + 4pqz}{2p^2 - q^2} = \frac{(2p^2 + q^2) \times (m^2 + 2mn - n^2) + 4pq \times (m^2 + n^2)}{2p^2 - q^2}$
 $= \frac{m^2 \times (2p^2 + 4pq + q^2) + 2mn \times (2p^2 + q^2) - n^2 \times (2p^2 - 4pq + q^2)}{2p^2 - q^2}$

We may therefore take $z = (m^2 + n^2) \times (2p^2 - 4pq + q^2)$
 $y = (2p^2 + q^2) \times (m^2 + n^2) + 2pq \times (m^2 + 2mn - n^2)$;
 $a = (2p^2 + q^2) \times (m^2 + 2mn - n^2) + 4pq \times (m^2 + n^2)$
 $b = (2p^2 - q^2) \times (m^2 + 2mn - n^2)$
 $a^2 = (2p^2 + q^2)^2 \times (m^2 + 2mn - n^2)^2 + 8pq \times (2p^2 + q^2) \times (m^2 + n^2) \times (m^2 + 2mn - n^2) + 16p^2q^2 \times (m^2 + n^2)^2$
 $b^2 = (2p^2 - q^2)^2 \times (m^2 + 2mn - n^2)^2$
 $2z^2 = a^2 + b^2 = 2 \times (4p^4 + q^4) \times (m^2 + 2mn - n^2)^2 + 8pq \times (2p^2 + q^2) \times (m^2 + n^2) \times (m^2 + 2mn - n^2) + 16p^2q^2 \times (m^2 + n^2)^2$
 $* x^2 = \frac{a^2 + b^2}{2} = (4p^4 + q^4) \times (m^2 + 2mn - n^2)^2 + 4pq \times (2p^2 + q^2) \times (m^2 + n^2) \times (m^2 + 2mn - n^2) + 8p^2q^2 \times (m^2 + n^2)^2$
 $= \text{a square; dividing by } (m^2 + 2mn - n^2)^2; 4p^4 + q^4 + 4pq \times$

$$(2p^2 + q^2) \times \frac{(m^2 + n^2)}{(m^2 + 2mn - n^2)} + \frac{8p^2q^2 \times (m^2 + n^2)^2}{(m^2 + 2mn - n^2)^2} = \text{a square}$$

$$= 4p^4 + 4pqd \times (2p^2 + q^2) + 8p^2q^2d^2 + q^4 = \text{a square,}$$

putting $d = \frac{m^2 + n^2}{m^2 + 2mn - n^2}$.

The preceding expression when properly arranged will be,
 $4p^4 + 8p^2qd + 8p^2q^2d^2 + 4pq^3d + q^4 = \text{a square}$; assume
 $2p^2 + 2pqd + d^2q^2$, for the root, that is, put $4p^4 + 8p^2qd +$
 $8p^2q^2d^2 + 4pq^3d + q^4 = (2p^2 + 2pqd + d^2q^2)^2 = 4p^4 + 8p^2qd +$
 $8p^2q^2d^2 + 4pq^3d^2 + q^4d^2$, $4pq^3d + q^4 = 4pq^3d^2 + q^4d^2$, dividing
 by q^2 , $4pd + q = 4pd^2 + qd^2$, $\frac{p}{q} = -\frac{(1 + d^2)}{4d} = -$

$\frac{\{(m^2 + n^2)^2 + 2mn \times (m^2 - n^2)\}}{2 \times (m^2 + n^2) \times (m^2 + 2mn - n^2)}$ where m and n , may be ex-
 pounded by any whole numbers at pleasure.

Example 1. Take $m = 2$, and $n = 1$; then $\frac{p}{q} = -\frac{37}{70}$

therefore we may take $p = -37$, and $q = 70$; whence $z =$
 $(2p^2 - q^2) \times (m^2 + n^2) = -5 \times 2162 = -10810 = -2 \times 5405$,
 $y = (2p^2 + q^2) \times (m^2 + n^2) + 2pq \times (m^2 + 2mn - n^2) = 1930 = 2 \times 965$,
 $x = (m^2 + 2mn - n^2) \times (2p^2 + 2pq + d^2q^2) = 7 \times 1538 = 10766 =$
 2×5383 , therefore 5405, 5383, and 965 are the roots of three
 squares that will answer.

Example (2). Take $m = 1$, and $n = 2$; then $d = \frac{m^2 + n^2}{m^2 + 2mn - n^2}$
 $= \frac{5}{1} = \frac{p}{q} = -\frac{(1 + d^2)}{4d} = -\frac{13}{10}$: therefore we may take $p = -13$,

and $q = 10$, and hence $z = (2p^2 - q^2) \times (m^2 + n^2) = 1190 =$
 2×595 , $y = (2p^2 + q^2) \times (m^2 + n^2) + 2pq \times (m^2 + 2mn - n^2)$
 $= 1930 = 2 \times 965$, $x = (m^2 + 2mn - n^2) \times (2p^2 + 2pqd + d^2q^2) =$
 $1538 = 2 \times 769$.

Therefore we may take $z = 595$, $y = 965$, and $x = 769$. These
 are the roots of three squares that will answer.

Another expression for $\frac{p}{q}$ may be obtained as follows. The
 general expression to be made a square is $4p^4 + 8p^2qd +$
 $8p^2q^2d^2 + 4pq^3d + q^4$; assume $2p^2d^2 + 2pqd + q^2$, for its root,
 that is, put $4p^4 + 8p^2qd + 8p^2q^2d^2 + 4pq^3d + q^4 = (2p^2d^2 +$

$2pqd + q^2 = 4p^4d + 8p^3qd^2 + 8p^2q^2d^2 + 4pq^3d + q^4$; which being properly reduced gives $\frac{p}{q} = -\frac{2d}{(1+d^2)}$, where m and n , as before, may be taken at pleasure.

Example 1. Take $m=2$, and $n=1$; then $d = \frac{m^2 + n^2}{m^2 + 2mn - n^2}$
 $= 5 \frac{p}{q} = -\frac{2d}{(1+d^2)} = -\frac{5}{19}.$

We may therefore take $p = -5$, and $q = 19$, whence $z = (2p^2 - q^2) \times (m^2 + n^2) = -119 \times 5 = -595$, $y = (2p^2 + q^2) \times (m^2 + n^2) + 2pq = 965$, $x = (m^2 + 2mn - n^2) \times (2p^2d^2 + 2pqd + q^2) = 769$, which are the same numbers as were found in the second example by the other formula for

$\frac{p}{q}$. We shall be enabled to obtain more simple and convenient

expressions for effecting a solution to the foregoing question by adopting a somewhat different mode of proceeding.* We have seen that the general expression to be made a square is $(4p^4 + q^4) \times (m^2 + 2mn - n^2)^2 + 4pq \times (2p^2 + q^2) \times (m^2 + n^2) \times (m^2 + 2mn - n^2) + 8p^2q^2 \times (m^2 + n^2)^2$ which being arranged according to the several powers or dimensions of m and n , becomes $m^4 \times (2p^2 + 2pq + q^2)^2 + 4m^3n \times (4p^4 + 4p^3q + 2pq^3 + q^4) + 2m^2n^2 \times (4p^4 + 8p^2q^2 + q^4) - 4mn^3 \times (4p^4 - 4p^3q - 2pq^3 + q^4) + n^4 \times (2p^2 - 2pq + q^2)^2 =$ a square. Assume $m^2 \times (2p^2 + 2pq + q^2) + 2mn \times \frac{(4p^4 + 4p^3q + 2pq^3 + q^4)}{2p^2 + 2pq + q^2} + n^2 \times (2p^2 - 2pq + q^2)$ for

its root, that is, put $m^2 \times (2p^2 + 2pq + q^2)^2 + 4m^3n \times (4p^4 + 4p^3q + 2pq^3 + q^4) + 2m^2n^2 \times (4p^4 + 8p^2q^2 + q^4) - 4mn^3 \times (4p^4 - 4p^3q - 2pq^3 + q^4) + n^4 \times (2p^2 - 2pq + q^2)^2 =$
 $\{m^2 \times (2p^2 + 2pq + q^2) + 2mn \times \frac{(4p^4 + 4p^3q + 2pq^3 + q^4)}{2p^2 + 2pq + q^2} + n^2 \times (2p^2 - 2pq + q^2)\}^2 = m^4 \times (2p^2 + 2pq + q^2)^2 + 4m^3n \times (4p^4 + 4p^3q + 2pq^3 + q^4) + 4m^2n^2 \times \frac{(4p^4 + 4p^3q + 2pq^3 + q^4)^2}{(2p^2 + 2pq + q^2)^2} +$
 $2m^2n^2 \times (4p^4 + q^4) + 4mn^3 \times \frac{(2p^2 - 2pq + q^2)}{2p^2 + 2pq + q^2} \times$

$(4p^4 + 4p^3q + 2pq^3 + q^4) + n^4 \times (2p^2 - 2pq + q^2)^2$ whence
 $\frac{m}{n} = -2 \times \frac{(16p^5 + 16p^7q - 8p^5q^3 - 8p^4q^4 + 4p^3q^5 + 2pq^7 + q^8)}{(2p^2 - q^2) \times (4p^4 - q^4) \times (2p^2 + 4pq + q^2)}$
 $= -\frac{2 \times (2p^2 + 2pq + q^2)}{2p^2 + 4pq + q^2}$. where p , and q , may be expounded
 by any whole numbers at pleasure.

Example 1. Take p and q , each $= 1$: then $\frac{m}{n} = -\frac{10}{7}$: we
 may therefore take $m = -10$, and $n = 7$; whence $z = (2p^3 - q^3)$
 $\times (m^2 + n^2) = 149$, $y = (2p^3 + q^3) \times (m^2 + n^2) + 2pq \times$
 $(m^2 + 2mn - n^2) = 269$, and $x = m^2 \times (2p^2 + 2pq + q^2) + 2mn \times$
 $\frac{(4p^4 + 4p^3q + 2pq^3 + q^4)}{2p^2 + 2pq + q^2} + n^2 \times (2p^2 - 2pq + q^2) = 241$.

That is 149, 241 and 269 are the root of three squares that will answer; being the very same as found in the first solution.

Example 2. Take $p = 2$, and $q = -1$: then $\frac{m}{n} = -\frac{10}{1}$;
 we may therefore take $m = -10$, and $n = 1$: whence $z =$
 $(2p^3 - q^3) \times (m^2 + n^2) = 707$,
 $y = (2p^3 + q^3) \times (m^2 + n^2) + 2pq \times (m^2 + 2mn - n^2) = 593$,
 and $x = m^2 \times (2p^2 + 2pq + q^2) + 2mn \times \frac{(4p^4 + 4p^3q + 2pq^3 + q^4)}{2p^2 + 2pq + q^2}$
 $+ n^2 \times (p^2 - 2pq + q^2) = 397$.

Therefore 397, 593 and 707 are the roots of three squares that will answer the question.

It may be remarked that every set of numbers that answers the proposed question will also furnish an answer to the following question, viz. To find three integer squares, half the sum of any two of which shall be a square number. This is manifest from a bare inspection of the expressions $2x^2 = b^2 + c^2$, $2y^2 = a^2 + c^2$, and $2z^2 = a^2 + b^2$.

ARTICLE XVII.

By the Rev. Mr. BRICE BRONWIN.

SUPPLEMENT TO THE LUNAR THEORY.

PROP. I.

To find the mean distance, eccentricity, and progression of the apogee from the differential equations of the moon's motion.

We have found $r^2 \frac{dv}{dt} = h + \int^1 D r dt$; or $\frac{1}{dt} d(r^2 \frac{dv}{dt}) = \frac{1}{D} r$;
 or $r^2 \frac{dv}{dt} d(r^2 \frac{dv}{dt}) = \frac{1}{D} r^2 \frac{dv}{dt}$. Integrating $(r^2 \frac{dv}{dt})^2 = h^2 +$
 $2 \int^1 D r^2 dv$. Therefore $dt = \frac{r^2 dv}{\sqrt{h^2 + 2 \int^1 D r^2 dv}} =$
 $\frac{dv}{u^2 \sqrt{h^2 + 2 \int^1 \frac{D}{u^3} dv}}$, by making $r = \frac{1}{u}$.

Again we have found

$$\frac{d^2 r}{dt^2} - r \frac{dv^2}{dt^2} + \frac{\mu}{r^3} + D = 0.$$

This, when dt is variable and dv invariable, becomes

$$d^2 r - r \frac{d^2 t}{dt^2} - r dv^2 + \frac{\mu}{r^3} dt^2 + D dt^2 = 0.$$

Substituting for dt , $d^2 t$, and $\frac{1}{u}$ for r ; this equation, after all reductions, will be

$$\frac{d^2 u}{dv^2} + u + \frac{\frac{1}{D} \frac{du}{dv} - \mu u^3 - D u}{u^3 (h^2 + 2 \int^1 \frac{D}{u^3} dv)} = 0; \text{ or}$$

$$\frac{d^2 u}{dv^2} + u - \frac{\mu}{h^2} - \frac{D}{h^2 u^2} + \frac{1}{h^2 u^3} \cdot \frac{du}{dv} + \frac{2}{h^2} \left(\frac{d^2 u}{dv^2} + u \right) \int^1 \frac{D}{u^3} dv = 0$$

If we expand the disturbing forces, we shall have a series of terms depending upon angles that vary but little during half

a revolution; we will call this function — Λ . We shall moreover have a series of terms depending upon angles whose arguments differ little from v or cv the anomaly; these may be put under the form

$$bae \cos cv + cae \sin cv,$$

where b and c will each contain a series of terms depending on angles that vary slowly. There will also be other terms in the development, but they are not to our present purpose. Our equation therefore now stands thus:

$$\frac{d^2u}{dv^2} + u - \frac{\mu}{h^2} - \Lambda + bae \cos cv + cae \sin cv = 0.$$

Let $u = b + k \cos (cv - \pi)$, b , k , and π being functions of v that vary slowly. Then

$$\frac{d^2u}{dv^2} = \frac{d^2b}{dv^2} + \frac{d^2k}{dv^2} \cos (cv - \pi) - 2\left(c - \frac{d\pi}{dv}\right) \frac{dk}{dv} \sin (cv - \pi) + k \frac{d^2\pi}{dv^2} \sin (cv - \pi) - k \left(c - \frac{d\pi}{dv}\right)^2 \cos (cv - \pi).$$

Substituting these values, and comparing separately the mean terms and the co-efficients of $\sin (cv - \pi)$ and of $\cos (cv - \pi)$; we shall find these equations

$$\frac{d^2b}{dv^2} + b - \frac{\mu}{h^2} - \Lambda = 0 \dots \dots \dots (1)$$

$$\frac{d^2k}{dv^2} - \left(c - \frac{d\pi}{dv}\right)^2 k + k + bk = 0 \dots \dots \dots (2)$$

$$k \frac{d^2\pi}{dv^2} - 2\left(c - \frac{d\pi}{dv}\right) \frac{dk}{dv} + ck = 0 \dots \dots \dots (3)$$

If we consider that a first differentiation will introduce factors of the order of p , and a second of the order of p^2 , into the quantities b , k , π ; we shall easily perceive that (1) will give

$$b = \frac{\mu}{h^2} + \Lambda \text{ nearly; (2) will give } \frac{d\pi}{dv} = c - 1 - \frac{1}{2} B \text{ nearly; and (3)}$$

$$\frac{dk}{dv} = \frac{1}{2} ck \text{ nearly. And hence again from (2) } \frac{d\pi}{dv} = c - 1 -$$

$$\frac{1}{2} B - \frac{1}{2} \frac{dc}{dv} \text{ very nearly; and from (3) } \frac{dk}{dv} = \frac{1}{2} ck - \frac{1}{2} k \frac{ds}{dv} \text{ very nearly.}$$

Now $\frac{k}{b} = e$ the eccentricity, and $a = \frac{1 + e^2}{b}$ the mean distance. These last will therefore be known.

The method above employed is an extension of one used by La Place to determine the secular equations of these elements, We shall now use a method a little different to effect the same purpose.

PROP. II.

To find as in the last Prop. by another method.

Resuming the same equation, and the same expression of u ; we have $\frac{du}{dv} = \frac{db}{dv} + \frac{dk}{dv} \cos (cv - \pi) + k \frac{d\pi}{dv} \sin (cv - \pi) - kc \sin (cv - \pi)$.

Now let $\frac{dk}{dv} \cos cv + k \frac{d\pi}{dv} \sin cv = 0 \dots \dots \dots (1)$.

Then $\frac{du}{dv} = \frac{db}{dv} - kc \sin (cv - \pi)$, and $\frac{d^2u}{dv^2} = \frac{d^2b}{dv^2} - \frac{dk}{dv} c \sin (cv - \pi) + kc \frac{d\pi}{dv} \cos (cv - \pi) - kc^2 \cos (cv - \pi)$.

If we suppose $c = 1$, and $\frac{d^2b}{dv^2} = 0$ on account of its minuteness; our differential equation will become, when for $\frac{d^2u}{dv^2}$ and u we put their values,

$$-\frac{dk}{dv} \sin cv + k \frac{d\pi}{dv} \cos cv + b - \frac{\mu}{h^2} - A + Bae \cos cv + Cae \sin cv = 0 \dots (2)$$

Here it is obvious that we may make another assumption, and that we must have $b = \frac{\mu}{h^2} + A$ as before. Then (2) becomes

$$-\frac{dk}{dv} \sin cv + k \frac{d\pi}{dv} \cos cv + Bae \cos cv + Cae \sin cv = 0.$$

Multiply (1) by $\cos cv$ and this last by $-\sin cv$ and adding the products together, there arises $\frac{dk}{dv} - \frac{1}{2} cae = 0$, or $\frac{dk}{dv} = \frac{1}{2} cae$, reserving only mean terms.

Similarly we find $k \frac{d\pi}{dv} + \frac{1}{2} Bae = 0$, or $\frac{d\pi}{dv} = -B \frac{ae}{k} = -\frac{1}{2} B$ nearly.

If we put k for ae , from which it does not differ than by a quantity of the second order; then $\frac{dk}{k} = \frac{1}{2} c dv$, and $k = k_0 + \frac{1}{2} k_0 \int c dv$ nearly, k_0 being an arbitrary constant, or the constant part of k .

These results differ from those of the last Prop. only by very small quantities.

It must be observed that more ellipses than one will solve the Problem of three Bodies.

PROP. III.

To find as in the two last Propositions by methods still different.

Make $u = \frac{\mu}{h^2} + A + k \cos cv$, and suppose A and k constant as they are nearly so. Then $\frac{d^2 u}{dv^2} = -kc^2 \cos cv$; and by substitution the equation $\frac{d^2 u}{dv^2} + u - \frac{\mu}{h^2} - A + Bae \cos cv + Cae \sin cv = 0$ becomes

$$(1 - c^2)k \cos cv + Bae \cos cv + Cae \sin cv = 0.$$

Or, rejecting the last term, which alters the eccentricity; and dividing by $\cos cv$,

$$(1 - c^2)k + Bae = 0, \text{ and } c = \sqrt{1 + \frac{Bae}{k}} = 1 + \frac{1}{2} B - \frac{1}{8} B^2.$$

Therefore the anomaly $= \int c dv = cv + \frac{1}{2} \int B dv$ nearly. This result agrees with that of the last proposition.

Again, let $u = \frac{\mu}{h^2} + A + x$; and to abridge put $M = Bae \cos cv + Cae \sin cv$. Then we have $\frac{d^2 x}{dv^2} + x + M = 0$, or $\frac{d^2 x}{dv^2} + (1 + \frac{M}{x})x = 0$. Make $c^2 = 1 + \frac{M}{x}$; and we have $\frac{d^2 x}{dv^2} + c^2 x = 0$. The integral of this is $x = k \cos cv$, k being an arbitrary, and another arbitrary being understood. Hence $c = \sqrt{1 + \frac{M}{k \cos cv}}$

$$= \sqrt{1+B} = 1 + \frac{1}{2}B \text{ as before.}$$

Or again, putting $u = \frac{\mu}{h^2} + A + ae \cos cv$, we have $ae \cos cv$
 $= u - \frac{\mu}{h^2} - A$. Therefore $rae \cos cv = Bu - \left(\frac{\mu}{h^2} + A\right)E$;
 and our equation becomes $\frac{d^2u}{dv^2} + (1+B)u - \left(\frac{\mu}{h^2} + A\right)(1+B)$
 $= 0$.

Multiply this by du , and integrate

$$\frac{du^2}{dv^2} - \overline{1+B} \{u^2 - 2\left(\frac{\mu}{h^2} + A\right)u + D\} = 0.$$

If E be the value of u at an apse, we find $D = 2\left(\frac{\mu}{h^2} + A\right)E - E^2$,
 and putting $u = E + x$, we have

$$\frac{dx^2}{dv^2} + \overline{1+B} \{x^2 - 2\left(\frac{\mu}{h^2} + A - E\right)x\} = 0,$$

$$\text{or } dv = \frac{dx}{\sqrt{1+B} \sqrt{2\left(\frac{\mu}{h^2} + A - E\right)x - x^2}}.$$

The integral taken from $x = 0$ to $2\left(\frac{\mu}{h^2} + A - E\right)x - x^2 = 0$,
 is the angle between the apsides; which therefore is $v = \frac{180^\circ}{\sqrt{1+B}}$.

Consequently $c = \sqrt{1+B} = 1 + \frac{1}{2}B$.

This last method is that of the ingenious Thomas Simpson, given in his Vol. of Essays.

From the preceding propositions it appears how easily the mean parts of the elements a , e , π , including equations that vary slowly, may be determined. The results also are the same, whether found by methods which suppose those quantities which vary slowly constant or variable.

PROP. IV.

To find the regression of the nodes, and variation of the inclination.

In Prob. IX. of the Lunar Theory we have already investigated the equation

$$\frac{d^2\sigma}{dv^2} + \sigma + Hs \sin gv + Ks \cos gv = 0;$$

H and K being quantities that vary slowly.

Let $\sigma = s \sin(gv - \theta)$, and let s and θ vary.

$$\frac{d^2\sigma}{dv^2} = \frac{d^2s}{dv^2} \sin gv + 2\left(g - \frac{d\theta}{dv}\right) \frac{ds}{dv} \cos gv - s \frac{d^2\theta}{dv^2} \cos gv - s\left(g - \frac{d\theta}{dv}\right)$$

$\sin gv$. Substituting these values, we shall find as in Prop. I.

$$\frac{d^2s}{dv^2} - s\left(g - \frac{d\theta}{dv}\right)^2 + s + Hs = 0;$$

$$\frac{d^2\theta}{dv^2} - 2\left(g - \frac{d\theta}{dv}\right) \frac{ds}{dv} - Ks = 0.$$

And from these equations, precisely as in the Prop. just referred to, we shall find

$$\frac{d\theta}{dv} = g - 1 - \frac{1}{2}H + \frac{1}{2}\frac{dK}{dv}, \quad \frac{ds}{dv} = -\frac{1}{2}Ks - \frac{1}{2}s\frac{dH}{dv}.$$

The same things may be determined otherwise.

Let the variation of σ arising from the variation of s and θ be nothing; or

$$\frac{ds}{dv} \sin gv - s \frac{d\theta}{dv} \cos gv = 0 \dots \dots (1),$$

$$\text{Then } \frac{d^2\sigma}{dv^2} = -s \sin gv + \frac{ds}{dv} \cos gv + s \frac{d\theta}{dv} \sin gv.$$

The values of σ and $\frac{d^2\sigma}{dv^2}$ substituted, we have

$$\frac{ds}{dv} \cos gv + s \frac{d\theta}{dv} \sin gv + Hs \sin gv + Ks \cos gv = 0 \dots (2).$$

From (1) and (2) exactly as in Prop. II. we find

$$\frac{ds}{dv} + \frac{1}{2}Ks = 0, \quad s \frac{d\theta}{dv} + \frac{1}{2}Hs = 0;$$

$$\text{or } \frac{ds}{dv} = -\frac{1}{2}Ks, \quad \frac{d\theta}{dv} = -\frac{1}{2}H.$$

Again, suppose $\sigma = s \sin gv$, and s and g constant. By substitution in our fundamental equation, it will become

$$(1 - g^2)s \sin gv + Hs \sin gv + Ks \cos gv = 0,$$

Or, leaving out the term which does not pertain to the regression, and dividing by $s \sin gv$;

$$1 - g^2 + H = 0, \quad g = \sqrt{1 + H} = 1 + \frac{1}{2}H \text{ nearly.}$$

Conseq. $\int g dv = gv + \frac{1}{2} \int H dv$; or $\theta = -\frac{1}{2} \int H dv$.

We may put our equation under this from :

$$\frac{d^2 \sigma}{dv^2} + \left(1 + \frac{P}{\sigma}\right) \sigma = 0, \text{ or } \frac{d^2 \sigma}{dv^2} + g^2 \sigma = 0.$$

The integral of which is $\sigma = s \sin gv$, where s is an arbitrary, and another is understood after gv . Hence $g = \sqrt{1 + \frac{P}{\sigma}} = \sqrt{1 + H} = 1 + \frac{1}{2} H$.

In the regression of the node the term $ks \cos gv$ produce, no effect. And since $\sigma = s \sin gv$, the term $hs \sin gv = H\sigma$. Our general equation therefore becomes

$$\frac{d^2 \sigma}{dv^2} + (1 + H) \sigma = 0.$$

Whence by integration $\frac{d\sigma^2}{dv^2} + 1 + H \cdot (\sigma^2 - \gamma^2) = 0$,

$$\text{or } dv = \frac{d\sigma}{\sqrt{1+H} \sqrt{\gamma^2 - \sigma^2}}. \text{ Therefore integrating}$$

from $\sigma = 0$ to $\sigma = \gamma$, we have $v = \frac{180^\circ}{\sqrt{1+H}}$ the angle between node and node; and $g = \sqrt{1+H} = 1 + \frac{1}{2} H$.

This method has some unnecessary steps, and has been introduced, as well as some others; to show how easily it will apply, when we are in quest of such equations only as vary slowly, and which we have denominated mean quantities.

ARTICLE XVIII.

ON THE COMPUTATION OF SUPERFICIES AND SOLIDS.

Bounded with Lines and Planes passing through Points by means of the Co-ordinates to each Point.

. By Mr. W. S. B. WOOLHOUSE.

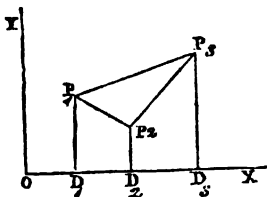
SUPERFICIES.

1. To compute the surface of a TRIANGLE by means of rectangular co-ordinates to each of its angular points.

Let $P_1 P_2 P_3$ represent the triangle; OX the axis and O the origin of the ordinates which are denoted thus:

$OD_1 = x_1$, $P_1 D_1 = y_1$; $OD_2 = x_2$, $P_2 D_2 = y_2$; $OD_3 = x_3$ and $P_3 D_3 = y_3$.

Then



$$\text{trapezoid } P_1 D_2 = D_1 D_2 \times \frac{P_1 D_1 + P_2 D_2}{2} = (x_2 - x_1) \cdot \frac{y_1 + y_2}{2};$$

$$\text{trapezoid } P_2 D_3 = D_2 D_3 \cdot \frac{P_2 D_2 + P_3 D_3}{2} = (x_3 - x_2) \cdot \frac{y_2 + y_3}{2}.$$

The sum of which gives

$$\text{area } D_1 P_1 P_2 P_3 D_3 = (x_2 - x_1) \cdot \frac{y_1 + y_2}{2} + (x_3 - x_2) \cdot \frac{y_2 + y_3}{2}.$$

Deducting this area from

$$\text{trapezoid } P_1 D_3 = D_1 D_3 \cdot \frac{P_1 D_1 + P_3 D_3}{2} = (x_3 - x_1) \cdot \frac{y_1 + y_3}{2},$$

we have *triangle* $P_1 P_2 P_3 =$

$$(x_1 - x_2) \cdot \frac{y_1 + y_2}{2} + (x_2 - x_3) \cdot \frac{y_2 + y_3}{2} + (x_3 - x_1) \cdot \frac{y_3 + y_1}{2}.$$

which may be more simply expressed by two different forms, viz.

$$\Delta P_1 P_2 P_3 = \pm \frac{1}{2} \left\{ (x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2 \right\} \quad \left. \begin{array}{l} \text{Or} \\ \Delta P_1 P_2 P_3 = \mp \frac{1}{2} \left\{ (y_2 - y_1)x_3 + (y_3 - y_2)x_1 + (y_1 - y_3)x_2 \right\} \end{array} \right\} \quad (1)$$

The signs taking place according as P_2 is below or above the right line $P_1 P_3$. But in actual numerical calculation a method

of ascertaining which particular sign ought to be used is unnecessary, since the area of the triangle must itself be positive.

If the middle ordinates OD_2 and PD_2 be considered as the first the equations will still hold. For substituting x_1, y_1 instead of x_2, y_2 and x_3, y_3 for x_1, y_1 , the result becomes exactly the same with a contrary sign; and hence the above expressions are perfectly general, answering to any arrangement of the ordinates, provided the result be always taken as positive.

For the sake of brevity we assume

$$\frac{1}{2} \{ (x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2 \} = \Delta(1, 2, 3)$$

where 1, 2, 3 indicates the arrangement of the ordinates.

Then according to the above $\Delta(1, 2, 3) = \Delta(2, 3, 1) = \Delta(3, 1, 2) = -\Delta(3, 2, 1) = -\Delta(2, 1, 3) = -\Delta(1, 3, 2)$, a change of sign taking place whenever the points go round the triangle in a contrary direction, as will be found by substitution in the above equation; and this it will be necessary to bear in mind in order to preserve the proper signs in the resolution of the problems hereafter given.

The following exhibits a scheme of the three distinct operations which are here intended to be adopted in the computation of the areas of triangles by this method.

Arrange the
ordinates thus : $\begin{vmatrix} x_1 \cdot y_1 \\ x_2 \cdot y_2 \\ x_3 \cdot y_3 \end{vmatrix}$, then

$$\begin{array}{ll} \text{1st } \begin{vmatrix} x_1 \cdot y_1 \\ x_2 \cdot y_2 \\ x_3 \cdot y_3 \end{vmatrix} & \begin{vmatrix} x_1 \cdot y_1 \\ x_2 \cdot y_2 \\ x_3 \cdot y_3 \end{vmatrix} \begin{vmatrix} (x_3 - x_2)y_1 \\ (x_2 - x_1)y_2 \\ (x_1 - x_3)y_3 \end{vmatrix} \\ \text{and 3rd } \begin{vmatrix} x_1 \cdot y_1 \\ x_2 \cdot y_2 \\ x_3 \cdot y_3 \end{vmatrix} & \begin{vmatrix} (x_3 - x_2)y_1 \\ (x_1 - x_3)y_2 \\ (x_2 - x_1)y_3 \end{vmatrix} \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{the sum of} \\ \text{which gives} \\ 2 \Delta(1, 2, 3). \end{array}$$

Thus for example, suppose $x_1 = 2, x_2 = 5, x_3 = 7; y_1 = 6, y_2 = 4, \text{ and } y_3 = 8$.

$$\text{Then } \begin{vmatrix} 2 \cdot 6 \\ 5 \cdot 4 \\ 7 \cdot 8 \end{vmatrix}, \text{ 1st } \begin{vmatrix} 2 \cdot 6 \\ 5 \cdot 4 \\ 7 \cdot 8 \end{vmatrix} \begin{vmatrix} 12 \\ 24 \\ 24 \end{vmatrix} \text{ 2nd } \begin{vmatrix} 2 \cdot 6 \\ 5 \cdot 4 \\ 7 \cdot 8 \end{vmatrix} \begin{vmatrix} 12 \\ 24 \\ 24 \end{vmatrix} \text{ and 3rd } \begin{vmatrix} 2 \cdot 6 \\ 5 \cdot 4 \\ 7 \cdot 8 \end{vmatrix} \begin{vmatrix} 12 \\ -20 \\ 24 \end{vmatrix} \left. \begin{array}{l} \\ \\ \end{array} \right\} 16$$

$$\therefore \Delta(1, 2, 3) = 8.$$

It will be observed that the whole calculation is comprehended in the 3rd, the 1st and second being given merely to shew *distinctly* each operation; and consequently that the computation is simply thus :

$$\begin{vmatrix} 2 \cdot 6 \\ 5 \cdot 4 \\ 7 \cdot 8 \end{vmatrix} \begin{vmatrix} 12 \\ -20 \\ 24 \end{vmatrix} \left. \begin{array}{l} \\ \\ \end{array} \right\} 16 = 2 \Delta(1, 2, 3)$$

$$\therefore \Delta(1, 2, 3) = 8.$$

Should the result vanish, the points are in the same straight line.

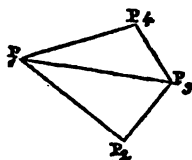
Note. In this case when the triangle vanishes and the points are in a straight line, we shall therefore have

$$(x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2 = 0,$$

which, if we suppose x_1, y_1, x_2, y_2 to be fixed and x_3, y_3 variable, will represent the analytical equation of a straight line passing through these two fixed points or the condition to be fulfilled by x_3, y_3 to be in a line with the other two points.

2. To find an expression for the surface of a quadrilateral by means of rectangular co-ordinates to each of its angular points.

P_1, P_2, P_3, P_4 representing the quadrilateral, and $x_1, y_1; x_2, y_2; x_3, y_3; x_4, y_4$ denoting the co-ordinates of the points P_1, P_2, P_3 , and P_4 , we have, by the last problem, the $\Delta P_1 P_2 P_3 = \Delta(1, 2, 3)$; and the $\Delta P_1 P_4 P_3 = -\Delta(1, 4, 3) = \Delta(1, 3, 4)$.



But $\Delta(1, 2, 3) = \frac{1}{2} \{ (x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2 \}$
and similarly $\Delta(1, 3, 4) = \frac{1}{2} \{ (x_3 - x_1)y_4 + (x_4 - x_3)y_1 + (x_1 - x_4)y_3 \}$
The sum of which gives the area of the quadrilateral $P_1 P_2 P_3 P_4$
 $= \frac{1}{2} \{ (x_2 - x_1)y_3 + (x_1 - x_3)y_2 + (x_3 - x_4)y_1 + (x_4 - x_1)y_3 \}$
 $= \frac{1}{2} \{ (x_3 - x_1)(y_4 - y_2) - (y_3 - y_1)(x_4 - x_2) \}.$

3. To find an expression for the area or surface of any polygon, regular or irregular, by means of rectangular co-ordinates to each of its angular points.

By dividing the polygon into triangles with diagonal lines from the 1st point and summing them together we obviously have the surface $= \Delta(1, 2, 3) + \Delta(1, 3, 4) + \Delta(1, 4, 5) + \&c. \dots + \Delta(1, n-1, n) =$

$$\begin{aligned} & \frac{1}{2} \{ (x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2 \} \\ & + \frac{1}{2} \{ (x_3 - x_1)y_4 + (x_4 - x_3)y_1 + (x_1 - x_4)y_3 \} \\ & + \frac{1}{2} \{ (x_4 - x_1)y_5 + (x_5 - x_4)y_1 + (x_1 - x_5)y_4 \} \\ & \quad \&c. \quad \quad \quad \&c. \quad \quad \quad \&c. \\ & + \frac{1}{2} \{ (x_{n-1} - x_1)y_n + (x_n - x_{n-1})y_1 + (x_1 - x_n)y_{n-1} \} \\ & = \frac{1}{2} \{ (x_n - x_2)y_1 + (x_1 - x_3)y_2 + (x_2 - x_4)y_3 + (x_3 - x_5)y_4 + \&c. \dots \} \end{aligned}$$

$$+ (x_{n-1} - x_n)y_{n-1} + (x_{n-1} - x_1)y_n \}$$

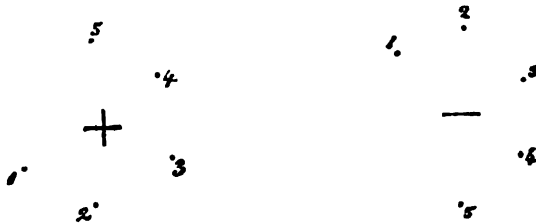
or, which is the same thing,

$$= \frac{1}{2} \{ (y_2 - y_n)x_1 + (y_3 - y_1)x_2 + (y_4 - y_2)x_3 + (y_5 - y_3)x_4 \dots \\ + (y_1 - y_{n-1})x_n \}.$$

Hence the following Theorem.

The area or surface of any polygon is equal to the sum of the products of the ordinates to each point into the difference of the abscissas to the two adjacent points, or equal to the sum of the products of the abscissas to each point into the difference of the ordinates to the two adjacent points, the differences being taken round the polygon in the same direction, and the result considered positive.

4. In using the above formulæ the result will always be positive or negative accordingly as the arrangement of the points is taken in the directions hereunder represented

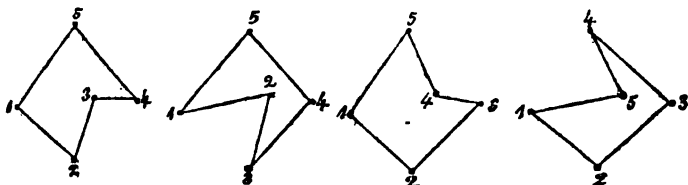


because the triangles $\Delta (1, 2, 3)$, $\Delta (1, 3, 4)$, $\Delta (1, 4, 5)$ &c. which compose them are all positive in the former and negative in the latter.

That the expressions will give different signs for the two directions is very evident by considering the polygon to be seen from the other side of its plane (xy), for then the arrangement will be inverted and the abscissas (x) will all appear with contrary and the ordinates (y) with the same signs as before.

For the triangle we must in the same manner observe the arrangements for the positive and for the negative values of $\Delta (1, 2, 3)$. Hence to preserve the proper signs in analytic operations we must always take the positive arrangement, viz. *to move on to the right with the lower and retreat to the left with the higher points*; and it will always be necessary to take them in the order in which they are to be joined to form the polygon —: this latter observation however cannot be required for the triangle since there is only one way of joining its points. As an illustration of this last remark the following diagrams exhibit four

different pentagons formed by joining the *same* five points taken with different arrangements, and whose separate surfaces will be properly represented by the above general formulæ taking the same succession of ordinates.



When none of the points can be encircled by a triangle or quadrilateral, &c. formed by joining three or more of the others, only one polygon can be formed with them.

5. To determine the area or surface of a triangle by the connection of three points in space by means of three rectangular co-ordinates to each point.

Let s_1, s_2, s_3 denote the sides which are respectively opposed to the points P_1, P_2, P_3 (see fig. 1) and

$$2s_2s_3 \cos P_1 = s_2^2 + s_3^2 - s_1^2$$

$$\therefore s_2^2 s_3^2 \cos^2 P_1 = \frac{1}{4}(s_2^2 + s_3^2 - s_1^2)^2,$$

each of which equal values deducted from $s_2^2 s_3^2$ we get $s_2^2 s_3^2 \sin^2 P_1$ or 4 times the square of the area

$$= s_2^2 s_3^2 - \frac{1}{4}(s_2^2 + s_3^2 - s_1^2)^2.$$

But

$$s_1^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2,$$

$$s_2^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2,$$

$$s_3^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

Consequently 4 times the square of the area

$$\begin{aligned} &= \{(x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) - 2(x_1x_2 + y_1y_2 + z_1z_2)\}^2 \\ &\cdot \{(x_1^2 + y_1^2 + z_1^2) + (x_3^2 + y_3^2 + z_3^2) - 2(x_1x_3 + y_1y_3 + z_1z_3)\}^2 \\ &- \{x_1^2 + y_1^2 + z_1^2 - (x_1x_2 + y_1y_2 + z_1z_2) - (x_1x_3 + y_1y_3 + z_1z_3) \\ &\quad + (x_2x_3 + y_2y_3 + z_2z_3)\}^2. \end{aligned}$$

Which expanded will be found equivalent to

$$\begin{aligned} &\{(x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2\}^2 \\ &+ \{(y_2 - y_1)z_3 + (y_3 - y_2)z_1 + (y_1 - y_3)z_2\}^2 \\ &+ \{(z_2 - z_1)x_3 + (z_3 - z_2)x_1 + (z_1 - z_3)x_2\}^2. \end{aligned}$$

It thus appears that the square of the required area =

$$\Delta^2(1, 2, 3) \text{ for } xy + \Delta^2(1, 2, 3) \text{ for } yz + \Delta^2(1, 2, 3) \text{ for } zx.$$

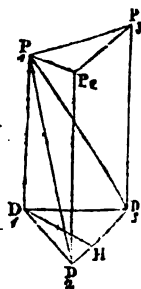
We therefore see by a method purely analytical that the square of any surface is equal to the sum of the squares of its orthographic projections on any three rectangular co-ordinate planes. For $\Delta(1, 2, 3)$ for x and y is the projection on any plane parallel to the plane xy ; $\Delta(1, 2, 3)$ for y and z is that on a plane parallel to the plane of yz , and $\Delta(1, 2, 3)$ for z and x the projection on a plane parallel to the plane zx .

This elegant property, which has before been established by mathematicians on principles materially different from the above, may be very advantageously employed in computing the areas of plane polygonal surfaces by means of the three co-ordinates to each angular point, since it requires merely the application of the foregoing problems. It will be observed, however, that the ordinates must have as many dependencies among them as the excess of the number of sides or points above 3,* and consequently that the three ordinates to three of the points being given, only two of the ordinates to the residue will be necessary.

SOLIDS.

6. To determine the solidity of a triangular prism when the plane of one end is not perpendicular but oblique to its edges.

In the annexed figure let the upper end $P_1 P_2 P_3$ be that which is oblique to the edges of the prism $P_1 D_1$, $P_2 D_2$, and $P_3 D_3$. Draw $D_1 H$ perp. to $D_2 D_3$ and join $P_1 D_2$ and $P_1 D_3$. Then the triangular base $D_1 D_2 D_3$, being at right angles with the edges $P_1 D_1$, $P_2 D_2$, and $P_3 D_3$, it is perp. to the plane $P_2 D_3 D_3 P_3$, and consequently $D_1 H$ is also perp. to the plane $P_2 D_3$; and therefore $P_1 D_1$ being parallel to the plane $P_2 D_3$, it follows that $D_1 H$ is equal to the perp. altitude of the pyramid whose vertex is P_1 and base the trapezoid $P_2 D_3$. Consequently the solidity of this pyramid =



$$\begin{aligned} \frac{D_1 H}{3} \times \text{trapezoid } P_2 D_3 &= \frac{D_1 H}{3} \times D_2 D_3 \cdot \frac{P_2 D_2 + P_3 D_3}{2} \\ &= \frac{P_2 D_2 + P_3 D_3}{3} \cdot \frac{D_1 H \cdot D_2 D_3}{2} = \frac{P_2 D_2 + P_3 D_3}{3} \cdot \Delta D_1 D_2 D_3. \end{aligned}$$

* This will be evident from a consideration of the note to the problem following.

Also the solid content of the pyramid whose vertex is P_1 and base the triangle $D_1 D_2 D_3 = \frac{P_1 D_1}{3} \cdot \Delta D_1 D_2 D_3$.

These pyramids added together evidently compose the given prism, whose solidity is therefore =

$$\frac{P_1 D_1 + P_2 D_2 + P_3 D_3}{3} \cdot \Delta D_1 D_2 D_3.$$

Or if $P_1 D_1 = y_1$ and $P_2 D_2 = y_2$ and $P_3 D_3 = y_3$,

$$\text{Solidity} = \frac{y_1 + y_2 + y_3}{3} \Delta (1, 2, 3).$$

It hence appears that the solidity of any triangular prism is equal to the product of one-third the sum of its edges into the area of the triangular section perp. to the said edges,

For if a triangular section be drawn through any point perp. to the edges, the above result will apply equally to both portions on each side of the section.

7. By means of the above the solidity of an oblique prism whose section is a polygon of any number of sides may easily be deduced by dividing it into triangular ones with diagonal planes through each edge and the first one. Thus, for instance, when the prism has four edges, we deduce its solidity =

$$\frac{z_1 + z_2 + z_3}{3} \cdot \Delta (1, 2, 3) + \frac{z_3 + z_4 + z_1}{3} \cdot \Delta (3, 4, 1)$$

$$\text{or} = \frac{z_2 + z_3 + z_4}{3} \cdot \Delta (2, 3, 4) + \frac{z_4 + z_1 + z_2}{3} \cdot \Delta (4, 1, 2).$$

Half the sum of which gives the solidity symmetrically

$$= \frac{z_1 + z_2 + z_3}{6} \cdot \Delta (1, 2, 3) + \frac{z_2 + z_3 + z_4}{6} \cdot \Delta (2, 3, 4) + \frac{z_3 + z_4 + z_1}{6} \cdot \Delta (3, 4, 1) + \frac{z_4 + z_1 + z_2}{6} \cdot \Delta (4, 1, 2).$$

8. As the four points are in the same plane

$$\Delta (1, 2, 3) + \Delta (3, 4, 1) = \Delta (2, 3, 4) + \Delta (4, 1, 2).$$

If from each of these, multiplied by $\frac{z_1 + z_2 + z_3 + z_4}{3}$, the

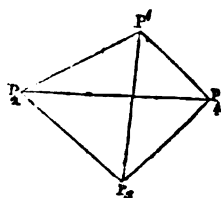
two preceding values of solidity be respectively taken away we get

$$\frac{z_4}{3} \cdot \Delta (1, 2, 3) + \frac{z_2}{3} \cdot \Delta (3, 4, 1) = \frac{z_1}{3} \cdot \Delta (2, 3, 4) + \frac{z_3}{3} \cdot \Delta (4, 1, 2)$$

$\therefore x_1 \cdot \Delta(2, 3, 4) - x_2 \cdot \Delta(3, 4, 1) + x_3 \cdot \Delta(4, 1, 2) - x_4 \cdot \Delta(1, 2, 3) = 0$,
which may be regarded as a condition of four points being in the same plane with each other.

9. To find the solidity bounded by planes passing through four points in space, or to find the solidity of any pyramid by means of three rectangular co-ordinates to each point.

Let P_1, P_2, P_3 and P_4 represent an orthographic projection of the four points upon the plane in which the two first ordinates x and y are taken, and let the ordinates be denoted by $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; x_4, y_4, z_4$. Then by the preceding together with the first problem we have the oblique prism above



the triangle $P_1 P_2 P_3 = \frac{x_1 + x_2 + x_3}{3} \cdot \Delta(1, 2, 3)$; the prism

above the triangle $P_3 P_4 P_1 = \frac{z_3 + z_4 + z_1}{3} \cdot \Delta(3, 4, 1)$; the

prism above the triangle $P_2 P_3 P_4 = \frac{z_2 + z_3 + z_4}{3} \cdot \Delta(2, 3, 4)$

and that above the triangle $P_4 P_1 P_2 = \frac{z_1 + z_1 + z_2}{3} \cdot \Delta(4, 1, 2)$.

Now the difference between the sum of the prisms above $P_1 P_2 P_3$ and $P_3 P_4 P_1$ and the sum of those above $P_2 P_3 P_4$ and $P_4 P_1 P_2$ will evidently give the required solidity of the pyramid, that is *the required solidity* =

the oblique prism above $P_1 P_2 P_3$ + the oblique prism above $P_3 P_4 P_1$
— the oblique prism above $P_2 P_3 P_4$ — the oblique prism above

$$P_4 P_1 P_2 = \frac{z_1 + z_2 + z_3}{3} \cdot \Delta(1, 2, 3) - \frac{z_2 + z_4 + z_4}{3} \cdot \Delta(2, 3, 4)$$

$$+ \frac{z_3 + z_4 + z_1}{3} \cdot \Delta(3, 4, 1) - \frac{z_1 + z_1 + z_2}{3} \cdot \Delta(4, 1, 2) =$$

$$\frac{z_1}{3} \left\{ \Delta(1, 2, 3) + \Delta(3, 4, 1) - \Delta(4, 1, 2) \right\}$$

$$- \frac{z_2}{3} \left\{ \Delta(2, 3, 4) + \Delta(4, 1, 2) - \Delta(1, 2, 3) \right\}$$

$$+ \frac{z_3}{3} \left\{ \Delta(3, 4, 1) + \Delta(1, 2, 3) - \Delta(2, 3, 4) \right\}$$

$$- \frac{z_1}{3} \left\{ \Delta(4, 1, 2) + \Delta(2, 3, 4) - \Delta(3, 4, 1) \right\} \\ = \frac{z_1}{3} \Delta(2, 3, 4) - \frac{z_2}{3} \Delta(3, 4, 1) + \frac{z_3}{3} \Delta(4, 1, 2) - \frac{z_4}{3} \Delta(1, 2, 3).$$

Or by Prob. 1st =

$$\frac{z_1}{6} \left\{ (x_3 - x_2)y_4 + (x_4 - x_3)y_2 + (x_2 - x_4)y_3 \right\} \\ - \frac{z_2}{6} \left\{ (x_4 - x_3)y_1 + (x_1 - x_4)y_3 + (x_3 - x_1)y_4 \right\} \\ + \frac{z_3}{6} \left\{ (x_1 - x_4)y_2 + (x_2 - x_1)y_4 + (x_4 - x_2)y_1 \right\} \\ - \frac{z_4}{6} \left\{ (x_2 - x_1)y_3 + (x_3 - x_2)y_1 + (x_1 - x_3)y_2 \right\}$$

an expression entirely in terms of the ordinates.

We have now determined two formula for the required solidity, viz.

$$\frac{z_1 + z_2 + z_3}{3} \cdot \Delta(1, 2, 3) - \frac{z_2 + z_3 + z_4}{3} \cdot \Delta(2, 3, 4) + \frac{z_3 + z_4 + z_1}{3} \\ \cdot \Delta(3, 4, 1) - \frac{z_4 + z_1 + z_2}{3} \Delta(4, 1, 2)$$

$$\text{and } \frac{z_1}{3} \cdot \Delta(2, 3, 4) - \frac{z_2}{3} \cdot \Delta(3, 4, 1) + \frac{z_3}{3} \cdot \Delta(4, 1, 2) - \frac{z_4}{3} \Delta(1, 2, 3).$$

But notwithstanding the particular symmetry and simplicity of both, the latter is certainly preferable to the former, and, it is presumed, will be found better adapted for actual computation when expressed in the following form.

$$\text{Solidity} = \frac{1}{6} \left\{ 2\Delta(2, 3, 4) \cdot z_1 - 2\Delta(3, 4, 1) \cdot z_2 + 2\Delta(4, 1, 2) \cdot z_3 - \right. \\ \left. 2\Delta(1, 2, 3) \cdot z_4 \right\}.$$

For example. Suppose the ordinates to the 1st point = 2, 5 and 9; the ordinates to the 2nd point = 4, 3, and 1; those to the 3rd point = 7, 1, and 5; and those to the 4th = 10, 8, and 6. In this case the arrangement stands thus:

Ordinates.	1st point	= 2, 5, and 9
————	2nd	= 4, 3, and 1
————	3rd	= 7, 1, and 5
————	4th	= 10, 8, and 6.

And the following exhibits the whole calculation :

$$\begin{array}{ll}
 \text{2nd point } 4.3 \mid 9 \} & \text{3rd point } 7.1 \mid -8 \} \\
 \text{3rd } 7.1 \mid -6 \} 27 = & \text{4th } 10.8 \mid 40 \} 47 = \\
 \text{4th } 10.8 \mid 24 \} 2\Delta(2, 3, 4) & \text{1st } 2.5 \mid 15 \} 2\Delta(3, 4, 1) \\
 \\
 \text{4th point } 10.8 \mid 16 \} & \text{1st point } 2.5 \mid 15 \} \\
 \text{1st } 2.5 \mid 30 \} 22 = & \text{2nd } 4.3 \mid -15 \} 2 = \\
 \text{2nd } 4.3 \mid -24 \} 2\Delta(4, 1, 2) & \text{3rd } 7.1 \mid 2 \} 2\Delta(1, 2, 3)
 \end{array}$$

$$\begin{aligned}
 \therefore 2\Delta(2, 3, 4) \cdot z_1 &= + 27.9 = + 243 \\
 - 2\Delta(3, 4, 1) \cdot z_2 &= - 47.1 = - 47 \\
 + 2\Delta(4, 1, 2) \cdot z_3 &= + 22.5 = + 110 \\
 - 2\Delta(1, 2, 3) \cdot z_4 &= - 2.6 = - 12
 \end{aligned}$$

$$\begin{array}{r}
 6)294 \\
 \hline
 \end{array}$$

49 Solidity.

In all cases it will be observed that

$$2\Delta(2, 3, 4) + 2\Delta(4, 1, 2) = 2\Delta(3, 4, 1) + 2\Delta(1, 2, 3),$$

As the extension of the Problem to any number of points may be easily accomplished by methods entirely similar to the foregoing, I shall not at present proceed further into the subject. It is necessary however to observe, that when there are more than four points the solid may be differently formed, and consequently that a description of its formation must be given in the Problem.

ARTICLE XIX.

NEW RESEARCHES IN SPHERICAL TRIGONOMETRY.

By Mr. T. S. DAVIES, Bath.

The radii of the circles described in and about a spherical triangle have long since been assigned as functions of the sides and angles respectively of that triangle by Lagrange and others; but the circles described in and about the three other triangles made by producing the sides of the original one till they two and two intersected a second time, had never been discussed till the

subject was taken up by Mr. Lowry in the *Mathematical Repository* about fourteen years ago.*

That elegant and inventive geometer there gave expressions for the several inscribed radii, and to my own knowledge he was in possession of a number of other curious properties which he did not there put down. The long suspension of the *Repository* (his usual organ of publication) which took place about that time, and the occupation of his mind in the interval between its suspension and renewal, by other subjects, prevented his publication of any of the remaining results he had obtained.

In 1825, Mr. (now Dr.) Luby, of Dublin, published a thin octavo on *Trigonometry*, a work of considerable merit in several points of view. From the *Repository* he transferred Mr. Lowry's values of the radii inscribed in the supplemental associated triangles;† and it was probably by this work, (Luby's) that Dr. Lardner's attention was called to these expressions, and by which he was led to the values of π and N in terms of the inscribed and circumscribing radii respectively of the four associated triangles.‡

Nothing more has, I am pretty sure, been done by English writers, as the only two subsequent authors, (Messrs. Hind§ and Wilson) the former has merely given the properties amongst

* *Math. Repos. No. 16. p. 1—8.* The following errata, pointed out to me by Mr. Lowry himself occur in that solution, and which this may be the proper place to notice.

Page 6.

In equation (14) for $\tan \frac{a}{2}$ read $\sin \frac{a}{2}$

———— (15) — $\tan R$ — $\tan r$.

———— (16) — $\tan \frac{a}{2}$ — $\sin \frac{a}{2}$.

———— (20) — $\sin \frac{A}{4}$ — $\sin \frac{A}{2}$.

(In three places.)

† Vide *Young's Trigonometry, Supplement*, for this designation.

‡ Lardner's *Trigonometry*, p. 150 and 153.

§ This gentleman must excuse me if I intimate to him that a little more specific acknowledgment of the sources of the propositions which he deserves great praise for being so indefatigable in collecting, would be much more satisfactory to the authors of the problems and theorems with which his books are enriched. One would think the *Ladies' and Gentleman's Diaries*, and such other works were unknown to such compilers did we not see good evidence of the fact in the number of propositions which are annually extracted from them.

his exercises upon the last page of his work, and the latter has given a demonstration little if at all differing from the original one by Dr. Lardner. With respect to foreign works I could speak with less certainty, though still in a style far from extreme diffidence. It is not in Cagnoli, nor in any one of the systematic writers on trigonometry that I have seen: neither have I found in such of the foreign journals as I have happened to consult, the slightest indication of an *implied* knowledge of these values. There is one author, however, whose work I have never been able to procure a sight of,—*viz.* Lhuillier—who, it is possible might enter upon these topics: and I am the more inclined to suspect that he had from a question in the Mathematical Companion, No. XXII. p. 809, that seems to be taken from him, having a relation not very remote from these.* This I think is the only place where it is at all probable that any anticipation of these results may be found.

The origin of all these researches amongst English writers is very clearly the paper of Mr. Lowry, both on account of the proof furnished by Dr. Luby's reference to the Repository, and by the following incident. Mr. Lowry had employed a figure in which the angle and side related to the circle in the supplemental triangle which he examined were B and b . He marked the radius r' , and took the others related to A , a and B , b respectively, he marked r'' and r''' : that is, in reference to symmetrical arrangement of the letters denoting the angles and the accents upon the symbols of the radii, he had been accidentally led by his figure to interchange r' and r'' . All the authors who have given demonstrations of the properties have made the same transposition. This interchange could scarcely have been accidental, inasmuch as in that case instead of one single mode of interchange we should have expected as many varieties as there were authors who made them.

The following speculations are founded upon the values of the inscribed radii given by Mr. Lowry, and the circumscribed by Dr. Lardner. They form, however, but a very small portion of the results to which I have been led. A few others, altogether different from these (one single result excepted) are given in a supplement to Young's Trigonometry: and a collection still more ample will probably be published in another form in the course of the present year.

Bath, Oct. 25, 1832.

* The title of the work is *Elémens d'Analyse Géométrique et d'Analyse Algébrique*, etc.

Let r, r_1, r_2, r_3 be the radii of the circles inscribed in the four associated triangles, that of the fundamental one being r : and r_1, r_2, r_3 touching the sides a, b, c , in order. Then,

$$\tan r \tan r_1 = \sin \frac{s-b}{2} \sin \frac{s-a}{2} = \sin \frac{c+a-b}{2} \sin \frac{c-a-b}{2} = \sin^2 \frac{c}{2} - \sin^2 \frac{a-b}{2} \dots \dots \dots (1)$$

$$\tan r \tan r_2 = \sin \frac{s-a}{2} \sin \frac{s-c}{2} = \sin \frac{b+c-a}{2} \sin \frac{c-a-b}{2} = \sin^2 \frac{b}{2} - \sin^2 \frac{c-a}{2} \dots \dots \dots (2)$$

$$\tan r \tan r_3 = \sin \frac{s-b}{2} \sin \frac{s-c}{2} = \sin \frac{a+b-c}{2} \sin \frac{a-b-c}{2} = \sin^2 \frac{a}{2} - \sin^2 \frac{b-c}{2} \dots \dots \dots (3)$$

$$\tan r_1 \tan r_2 = \sin s \sin \frac{s-a}{2} = \sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} = \sin^2 \frac{a+b}{2} - \sin^2 \frac{c}{2} \dots \dots \dots (4)$$

$$\tan r_1 \tan r_3 = \sin s \sin \frac{s-b}{2} = \sin \frac{a+c+b}{2} \sin \frac{a+c-b}{2} = \sin^2 \frac{a+c}{2} - \sin^2 \frac{b}{2} \dots \dots \dots (5)$$

$$\tan r_2 \tan r_3 = \sin s \sin \frac{s-a}{2} = \sin \frac{b+c+a}{2} \sin \frac{b+c-a}{2} = \sin^2 \frac{b+c}{2} - \sin^2 \frac{a}{2} \dots \dots \dots (6)$$

By adding (1, 4), (2, 5), and 3, 6) we get

$$\tan r \tan r_1 + \tan r_1 \tan r_2 = \sin^2 \frac{a+b}{2} - \sin^2 \frac{a-b}{2} = \sin a \sin b \dots (7)$$

$$\tan r \tan r_2 + \tan r_2 \tan r_3 = \sin^2 \frac{c+a}{2} - \sin^2 \frac{c-a}{2} = \sin a \sin c \dots (8)$$

$$\tan r \tan r_3 + \tan r_3 \tan r_1 = \sin^2 \frac{b+c}{2} - \sin^2 \frac{b-c}{2} = \sin b \sin c \dots (9)$$

Whence by adding (7, 8, 9) we get a remarkable formula, viz,

$$\tan r \tan r' + \tan r \tan r'' + \tan r \tan r''' + \tan r' \tan r'' + \tan r' \tan r''' + \tan r'' \tan r''' = \sin a \sin b + \sin a \sin c + \sin b \sin c \dots\dots\dots(10)$$

or the sum of all the rectangles of the tangents of the four radii is equal to the sum of all the rectangles of sines of the sides.

The plane theorem corresponding to this has been already proved by different methods in the last number of the Mathematical Repository (XXIII). In the discovery of that property itself, however, I find I had been preceded by Carl Jacobi of Königsberg, in a curious dissertation on the properties of plane triangles, published at Leipsig in 1825. Of this work I shall add only that a translation will appear in the present and subsequent number of the Repository. It did not, however, originate with Jacobi, for he refers to a little work by Furbach, published at Nuremberg, which I have not seen, entitled "*Eigenschaften einiger merkwürdigen Puncti des Geraden Dreiecks*, &c. 1822." It is only by accident that such books find their way to this country, and they can scarcely ever be obtained to order, so that it would be almost hopeless to search for this tract,—though doubtless it contains a series of very curious and interesting properties of plane triangles. I can, therefore, hold out no promise of being able to give a fuller account of it.

By the property of the polar triangles, that $R' + r = \frac{\pi}{2}$. etc.

and $\cot. R' = \tan r$, $\sin A' = \sin a$, etc.

(Vid. Young's Trigonometry. Suppt.)

$$\cot R' \cot R'_{1''} + \cot R'_{1'} \cot R'_{2''} = \sin A' \sin B' \dots\dots(11)$$

$$\cot R' \cot R'_{2''} + \cot R'_{2'} \cot R'_{1''} = \sin A' \sin C' \dots\dots(12)$$

$$\cot R' \cot R'_{1'} + \cot R'_{1''} \cot R'_{2''} = \sin B' \sin C' \dots\dots(13)$$

or by adding these, we have an equation analogous to (10), and equally curious:—

$$\left. \begin{aligned} &\cot R' \cot R'_{1''} + \cot R'_{1'} \cot R'_{2''} \\ &+ \cot R' \cot R'_{2''} + \cot R'_{2'} \cot R'_{1''} \\ &+ \cot R'_{1'} \cot R'_{2''} + \cot R'_{2''} \cot R'_{1''} \end{aligned} \right\} = \sin A' \sin B' + \sin A' \sin C' + \sin B' \sin C' \dots\dots(14)$$

Put (7), (8), (9) equal respectively to $t_{1''}$, $t_{1'}$ and t_1 ; then we have by division

$$\left. \begin{aligned} \frac{t_{1''}}{t_{1'}} &= \frac{\sin b}{\sin c} \\ \frac{t_{1''}}{t_1} &= \frac{\sin a}{\sin c} \end{aligned} \right\} \dots\dots\dots(15)$$

$$\begin{aligned} \therefore \sin b &= \sin c \cdot \frac{t_{III}}{t_{II}} \left\{ \right. \\ \sin a &= \sin c \cdot \frac{t_{II}}{t_I} \left. \right\} \dots \dots \dots (16) \end{aligned}$$

which inserted in (7) give

$$\sin a \sin b = t_{II} = \sin^2 c \cdot \frac{t_{III}^2}{t_I t_I}, \text{ or}$$

$$\sin^2 c = \frac{t_{II} t_{II}}{t_{III}^2} = \frac{(\tan r \tan r_{III} + \tan r \tan r_{II})(\tan r_{III} + \tan r_{II} \tan r_{III})}{\tan r \tan r_{III} + \tan r \tan r_{II}} \quad (17)$$

and, in like manner,

$$\sin^2 b = \frac{(\tan r \tan r_{III} + \tan r \tan r_{II})(\tan r \tan r_I + \tan r_{II} \tan r_{III})}{\tan r \tan r_{II} + \tan r \tan r_{III}} \dots (18)$$

$$\sin^2 a = \frac{(\tan r \tan r_{II} + \tan r \tan r_{III})(\tan r \tan r_{III} + \tan r \tan r_{II})}{\tan r \tan r_I + \tan r_{II} \tan r_{III}} \dots (19)$$

We might instead of writing the polar triangles as before change the $\tan r$, &c. into $\cot A'$ etc.: and change the sides a, b, c , into A', B', C' , and a corresponding set of formulæ will result, which need not, however, be set down in this place.

To proceed:—

$$\sin^2 A = \frac{4 \sin s \sin s-a \sin s-b \sin s-c}{\sin^2 b \sin^2 c},$$

$$= \frac{4 \tan r \tan r_I \tan r_{II} \tan r_{III}}{\sin^2 b \sin^2 c},$$

$$= \frac{4 \tan r \tan r_I \tan r_{II} \tan r_{III}}{t_I^2},$$

$$= \frac{4 \tan r \tan r_I \tan r_{II} \tan r_{III}}{(\tan r \tan r_I + \tan r_{II} \tan r_{III})^2}.$$

$$\therefore \cos^2 A = 1 - \sin^2 A = \frac{(\tan r \tan r_I + \tan r_{II} \tan r_{III})^2 - 4 \tan r \tan r_I \tan r_{II} \tan r_{III}}{(\tan r \tan r_I + \tan r_{II} \tan r_{III})^2}$$

$$\cos A = \pm \frac{\tan r \tan r_I - \tan r_{II} \tan r_{III}}{\tan r \tan r_I + \tan r_{II} \tan r_{III}} \left\{ \right.$$

In like manner

$$\cos B = \pm \frac{\tan r \tan r_{II} - \tan r_I \tan r_{III}}{\tan r \tan r_{II} + \tan r_I \tan r_{III}} \left\{ \dots \dots (20). \right.$$

$$\cos C = \pm \frac{\tan r \tan r_{III} - \tan r_I \tan r_{II}}{\tan r \tan r_{III} + \tan r_I \tan r_{II}} \left. \right\}$$

Now there does not at first sight seem to be any reason why we should select one of these signs rather than the other, or in other words why they should not both be applicable. The truth is, as in many other algebraic processes, that the problem we have solved is more general than the one proposed, inasmuch as we have obtained not only the vertical angle in equation (20), but also its supplement. To ascertain which of the signs belongs to the angle itself we may proceed thus:—

By (1, 4) we have

$$\tan r \tan r_{,,,} - \tan r, \tan r_{,,} = \sin^2 \frac{c}{2} - \sin^2 \frac{a-b}{2} - \sin^2 \frac{a+b}{2} + \sin^2 \frac{c}{2} \dots (21)$$

$$\left. \begin{aligned} \text{Now } \sin^2 \frac{a-b}{2} &= \frac{1}{2}(1 - \cos a - b) = \frac{1}{2} - \frac{1}{2} \cos a \cos b - \frac{1}{2} \sin a \sin b \\ \sin^2 \frac{a+b}{2} &= \frac{1}{2}(1 - \cos a - b) = \frac{1}{2} - \frac{1}{2} \cos a \cos b + \frac{1}{2} \sin a \sin b \\ 2 \cos^2 \frac{c}{2} &= 1 - \cos c. \end{aligned} \right\} \dots (22)$$

Or inserting (22) of (21) we get

$$\cos c = \pm \frac{1 - \cos c - 1 + \cos a \cos b}{\sin a \sin b} = \pm \frac{-\cos c + \cos a \cos b}{\sin a \sin b} \dots (23)$$

$$\begin{aligned} \text{But we know that } \cos c &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} \\ &= - \frac{-\cos c + \cos a \cos b}{\sin a \sin b}, \end{aligned}$$

or the sign — is to be taken in equations (20): that is we have

$$\left. \begin{aligned} \cos A &= - \frac{\tan r \tan r, - \tan r_{,,} \tan r_{,,,}}{\tan r \tan r, + \tan r_{,,} \tan r_{,,,}} \\ \cos B &= - \frac{\tan r \tan r_{,,} - \tan r, \tan r_{,,,}}{\tan r \tan r_{,,} + \tan r, \tan r_{,,,}} \\ \cos C &= - \frac{\tan r \tan r_{,,,} - \tan r, \tan r_{,,}}{\tan r \tan r_{,,,} + \tan r, \tan r_{,,}} \end{aligned} \right\} \dots \dots (24)$$

From these we have

$$\left. \begin{aligned} \frac{1 - \cos A}{2} &= \sin^2 \frac{1}{2} A = \frac{\tan r \tan r,}{\tan r \tan r, + \tan r_{,,} \tan r_{,,,}} \\ \frac{1 + \cos A}{2} &= \cos^2 \frac{1}{2} A = \frac{\tan r_{,,} \tan r_{,,,}}{\tan r \tan r, + \tan r_{,,} \tan r_{,,,}} \end{aligned} \right\} \dots \dots (25)$$

By reason of which

$$\tan^2 \frac{1}{2} A = \frac{\tan r \tan r,}{\tan r_{,,} \tan r_{,,,}}$$

Similarly we obtain

$$\left. \begin{aligned} \sin^2 \frac{1}{2} B &= \frac{\tan r \tan r''}{\tan r \tan r'' + \tan r', \tan r'''} \\ \cos^2 \frac{1}{2} B &= \frac{\tan r', \tan r'''}{\tan r \tan r'' + \tan r', \tan r'''} \\ \tan \frac{1}{2} B &= \frac{\tan r \tan r''}{\tan r', \tan r'''} \end{aligned} \right\} \dots\dots\dots (26)$$

$$\left. \begin{aligned} \sin^2 \frac{1}{2} C &= \frac{\tan r \tan r'''}{\tan r \tan r''' + \tan r', \tan r''} \\ \cos^2 \frac{1}{2} C &= \frac{\tan r', \tan r''}{\tan r \tan r''' + \tan r', \tan r''} \\ \tan^2 \frac{1}{2} C &= \frac{\tan r \tan r'}{\tan r', \tan r''} \end{aligned} \right\} \dots\dots\dots (27)$$

If we multiply the third equations of (25, 26, 27) together we shall have

$$\tan^2 \frac{1}{2} A \tan^2 \frac{1}{2} B \tan^2 \frac{1}{2} C = \frac{\tan r}{\tan r'} \cdot \frac{\tan r}{\tan r''} \cdot \frac{\tan r}{\tan r'''} \dots\dots (28)$$

Hence also from the same equations, we get by division

$$\left. \begin{aligned} \frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} B} &= \frac{\tan r}{\tan r''} \\ \frac{\tan \frac{1}{2} A}{\tan \frac{1}{2} C} &= \frac{\tan r'}{\tan r'''} \\ \frac{\tan \frac{1}{2} B}{\tan \frac{1}{2} C} &= \frac{\tan r''}{\tan r'''} \end{aligned} \right\} \dots\dots\dots (29)$$

Leaving these equations and the consequences which may be deduced from them for awhile, we may obtain certain other quantities frequently occurring in trigometrical researches, in functions of the radii above mentioned.

By (15) we have

$$\frac{\sin b}{\sin c} = \frac{t'''}{t''} \text{ or } \frac{\sin b + \sin c}{\sin b - \sin c} = \frac{t''' + t''}{t''' - t''}, \text{ or}$$

$$\begin{aligned} \frac{\tan \frac{1}{2}(b+c)}{\tan \frac{1}{2}(b-c)} &= \frac{(\tan r \tan r''' + \tan r'' \tan r') + (\tan r \tan r'' + \tan r' \tan r''')}{(\tan r \tan r''' + \tan r'' \tan r') - (\tan r \tan r'' + \tan r' \tan r''')} \\ &= \frac{\tan r (\tan r''' + \tan r'') + \tan r' (\tan r''' + \tan r'')}{\tan r (\tan r''' + \tan r'') - \tan r' (\tan r''' - \tan r'')} \\ &= \frac{(\tan r + \tan r') (\tan r''' + \tan r'')}{(\tan r - \tan r') (\tan r''' - \tan r'')} \\ &= \frac{\sin (r + r') \sin (r''' + r'')}{\sin (r - r') \sin (r''' - r'')}, \text{ or finally} \end{aligned}$$

$$\left. \begin{aligned} \tan \frac{1}{2} \overline{b+c} &= \frac{\cos(r+r_1-r_{II}-r_{III})-\cos(r+r_1+r_{II}+r_{III})}{\cos(-r+r_1-r_{II}+r_{III})-\cos(-r+r_1+r_{II}-r_{III})} \\ \tan \frac{1}{2} \overline{b-c} &= \frac{\cos(r-r_1-r_{II}+r_{III})-\cos(r+r_1+r_{II}+r_{III})}{\cos(-r+r_1-r_{II}+r_{III})-\cos(-r-r_1+r_{II}+r_{III})} \\ \tan \frac{1}{2} \overline{a+b} &= \frac{\cos(r-r_1-r_{II}+r_{III})-\cos(r+r_1+r_{II}+r_{III})}{\cos(-r+r_1-r_{II}+r_{III})-\cos(-r-r_1+r_{II}+r_{III})} \\ \tan \frac{1}{2} \overline{a-b} &= \frac{\cos(r-r_1+r_{II}-r_{III})-\cos(r+r_1+r_{II}+r_{III})}{\cos(-r+r_1-r_{II}+r_{III})-\cos(-r-r_1+r_{II}+r_{III})} \\ \tan \frac{1}{2} \overline{a+c} &= \frac{\cos(r-r_1+r_{II}-r_{III})-\cos(r+r_1+r_{II}+r_{III})}{\cos(-r+r_1-r_{II}+r_{III})-\cos(-r-r_1+r_{II}+r_{III})} \\ \tan \frac{1}{2} \overline{a-c} &= \frac{\cos(r-r_1+r_{II}-r_{III})-\cos(r+r_1+r_{II}+r_{III})}{\cos(-r+r_1-r_{II}+r_{III})-\cos(-r-r_1+r_{II}+r_{III})} \end{aligned} \right\} (30)$$

We shall defer the further consideration of the very varied classes of formulæ where trigonometrical functions of the componentsof the triangles are expressed in trigonometrical functions of the sums and differences of the radii, till a future part of this paper, and confine our attention for the present chiefly if not wholly to those expressions in which only by the sum, differences, products, &c. of the tangents of the radii themselves are concerned. The enquiry is rich in curious and interesting results, which ever way it is viewed.

Again, by the common trigonometrical formulæ we have

$$\begin{aligned} \sin \frac{A+B+C}{2} &= \\ &+ \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} + \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ \sin \frac{A+B-C}{2} &= \\ &+ \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} - \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ \sin \frac{A-B+C}{2} &= \\ &+ \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} + \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ \sin \frac{-A+B+C}{2} &= \\ &- \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} + \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \end{aligned} \quad (31)$$

In like manner for the sines we have the following results, first putting

$$\sqrt{(\tan r \tan r_1 + \tan r_1 \tan r_{11})(\tan r \tan r_{11} + \tan r_{11} \tan r_1)(\tan r \tan r_{111} + \tan r_{111} \tan r_1)} =$$

$$\sin \frac{A+B+C}{2} = \frac{-\tan r + \tan r_1 + \tan r_{11} + \tan r_{111}}{D'} \cdot \left\{ \tan r \tan r_1 \tan r_{11} \tan r_{111} \right\}^{\frac{1}{4}}$$

$$\sin \frac{A+B-C}{2} = \frac{\tan r + \tan r_1 + \tan r_{11} - \tan r_{111}}{D'} \cdot \left\{ \tan r \tan r_1 \tan r_{11} \tan r_{111} \right\}^{\frac{1}{4}}$$

$$\sin \frac{A-B+C}{2} = \frac{\tan r + \tan r_1 - \tan r_{11} + \tan r_{111}}{D'} \cdot \left\{ \tan r \tan r_1 \tan r_{11} \tan r_{111} \right\}^{\frac{1}{4}} \quad (34)$$

$$\sin \frac{-A+B+C}{2} = \frac{\tan r - \tan r_1 + \tan r_{11} + \tan r_{111}}{D'} \cdot \left\{ \tan r \tan r_1 \tan r_{11} \tan r_{111} \right\}^{\frac{1}{4}}$$

By means of (32, 33, 34) we may obtain a considerable number of curious theorems. I shall here select from my notes only a very few of those which I have deduced.

By a common formula

$$\cot^2 \frac{1}{2} a = - \frac{\cos s \cos s - A}{\cos s - B \cos s - C} = - \frac{\cos \frac{A+B+C}{2} \cos \frac{-A+B+C}{2}}{\cos \frac{A-B+C}{2} \cos \frac{A+B-C}{2}}, \text{ that is by (35)}$$

$$\cot^2 \frac{1}{2} a = \frac{(-\cot r + \cot r_1 + \cot r_{11} + \cot r_{111})(\cot r - \cot r_1 + \cot r_{11} + \cot r_{111})}{(\cot r + \cot r_1 - \cot r_{11} + \cot r_{111})(\cot r + \cot r_1 + \cot r_{11} - \cot r_{111})}$$

$$\cot^2 \frac{1}{2} b = \frac{(-\cot r + \cot r_1 + \cot r_{11} + \cot r_{111})(\cot r + \cot r_1 - \cot r_{11} + \cot r_{111})}{(\cot r - \cot r_1 + \cot r_{11} + \cot r_{111})(\cot r + \cot r_1 + \cot r_{11} - \cot r_{111})} \quad (36)$$

$$\cot^2 \frac{1}{2} c = \frac{(-\cot r + \cot r_1 + \cot r_{11} + \cot r_{111})(\cot r + \cot r_1 + \cot r_{11} - \cot r_{111})}{(\cot r - \cot r_1 + \cot r_{11} + \cot r_{111})(\cot r + \cot r_1 - \cot r_{11} + \cot r_{111})}$$

Denote now the several cotangents employed in these equations, thus

$$\left. \begin{aligned} \cot r &= \frac{-\cot \rho + \cot \rho_1 + \rho_{11} + \cot \rho_{111}}{4} \\ \cot r_1 &= \frac{\cot \rho - \cot \rho_1 + \cot \rho_{11} + \cot \rho_{111}}{4} \\ \cot r_{11} &= \frac{\cot \rho + \cot \rho_1 - \cot \rho_{11} + \cot \rho_{111}}{4} \\ \cot r_{111} &= \frac{\cot \rho + \cot \rho_1 + \cot \rho_{11} - \cot \rho_{111}}{4} \end{aligned} \right\} \dots (36)$$

or, which comes to the same thing,

$$\left. \begin{aligned} \cot p &= -\cot r + \cot r_1 + \cot r_{II} + \cot r_{III} \\ \cot q &= \cot r - \cot r_1 + \cot r_{II} + \cot r_{III} \\ \cot p_{II} &= \cot r + \cot r_1 - \cot r_{II} + \cot r_{III} \\ \cot p_{III} &= \cot r + \cot r_1 + \cot r_{II} - \cot r_{III} \end{aligned} \right\} \dots (37)$$

Hence also

$$\left. \begin{aligned} \cot p \cot p_1 + \cot p_{II} \cot p_{III} &= \\ &= (-\cot r + \cot r_1 + \cot r_{II} + \cot r_{III})(\cot r - \cot r_1 + \cot r_{II} + \cot r_{III}) \\ &+ (\cot r + \cot r_1 - \cot r_{II} + \cot r_{III})(\cot r + \cot r_1 + \cot r_{II} + \cot r_{III}) \\ &= \{(\cot r_{II} + \cot r_{III}) + (\cot r - \cot r_1)\} \{(\cot r_{II} + \cot r_{III}) - (\cot r - \cot r_1)\} \\ &+ \{(\cot r_1 + \cot r_{II}) + \cot r_{III} - \cot r_{III}\} \{(\cot r_1 + \cot r_{II}) - (\cot r_{II} - \cot r_{III})\} \\ \cot p \cot p' + \cot p_{II} \cot p_{III} &= (\cot r + \cot r_{III})^2 - (\cot r - \cot r_1)^2 \\ &+ (\cot r + \cot r_1)^2 - (\cot r_{II} - \cot r_{III})^2 \\ &= 4 \cot r_{II} \cot r_{III} + 4 \cot r \cot r_1 \\ &= 4(\cot r \cot r_1 + \cot r_{II} \cot r_{III}) \end{aligned} \right\} (38)$$

Then we have

$$\left. \begin{aligned} \cot^2 \frac{c}{2} &= \frac{\cot p_1 \cot p_{II}}{\cot p \cot p_{III}} \\ \cos^2 \frac{c}{2} &= \frac{\cot p_1 \cot p_{II}}{\cot p \cot p_{III} + \cot p_1 \cot p_{II}} \\ \sin^2 \frac{c}{2} &= \frac{\cot p \cot p_{III}}{\cot p \cot p_{III} + \cot p_1 \cot p_{II}} \end{aligned} \right\} \dots \dots \dots (39)$$

$$\left. \begin{aligned} \cot^2 \frac{b}{2} &= \frac{\cot p_1 \cot p_{III}}{\cot p \cot p_{II}} \\ \cos^2 \frac{b}{2} &= \frac{\cot p_1 \cot p_{III}}{\cot p \cot p_{II} + \cot p_1 \cot p_{III}} \\ \sin^2 \frac{b}{2} &= \frac{\cot p \cot p_{II}}{\cot p \cot p_{II} + \cot p_1 \cot p_{III}} \end{aligned} \right\} \dots \dots \dots (40)$$

$$\left. \begin{aligned} \cot^2 \frac{a}{2} &= \frac{\cot q_{II} \cot q_{III}}{\cot q \cot q_1} \\ \cos^2 \frac{a}{2} &= \frac{\cot q_{II} \cot q_{III}}{\cot q \cot q_1 + \cot q_{II} \cot q_{III}} \\ \sin^2 \frac{a}{2} &= \frac{\cot q \cot q_1}{\cot q \cot q_1 + \cot q_{II} \cot q_{III}} \end{aligned} \right\} \dots \dots \dots (41)$$

From these equations, compared with (25, 26, 27) we learn the remarkable fact, *that whatever function* $\frac{A}{2}, \frac{B}{2}, \frac{C}{2}$ *are of*

the tangents of r, r', r'', r''' the same functions will be the complements of $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ be of the cotangents of e, e', e'', e'''

The following equations flow at once from those just established :—

$$\cot^2 \frac{c}{2} \cot^2 \frac{b}{2} \cot^2 \frac{a}{2} = \frac{\tan e}{\tan e'} \cdot \frac{\tan e}{\tan e''} \cdot \frac{\tan e}{\tan e'''} \dots (42)$$

which may be compared with (28).

$$\left. \begin{aligned} \cos c &= -\frac{\cot e \cot e''' - \cot e'' \cot e}{\cot e \cot e''' + \cot e'' \cot e} \\ \cos b &= -\frac{\cot e \cot e''' - \cot e'' \cot e}{\cot e \cot e''' + \cot e'' \cot e} \\ \cos a &= -\frac{\cot e \cot e' - \cot e'' \cot e'''}{\cot e \cot e' + \cot e'' \cot e'''} \end{aligned} \right\} \dots \dots (43)$$

$$\left. \begin{aligned} \sin^2 c &= \frac{4 \cot e \cot e' \cot e'' \cot e'''}{(\cot e \cot e''' + \cot e' \cot e'')^2} \\ \sin^2 b &= \frac{4 \cot e \cot e' \cot e'' \cot e'''}{(\cot e \cot e''' + \cot e' \cot e'')^2} \\ \sin^2 a &= \frac{4 \cot e \cot e' \cot e'' \cot e'''}{(\cot e \cot e' + \cot e'' \cot e''')^2} \end{aligned} \right\} \dots \dots (44)$$

$$\left. \begin{aligned} \tan^2 c &= \frac{4 \cot e \cot e' \cot e'' \cot e'''}{(\cot e \cot e''' - \cot e' \cot e'')^2} \\ \tan^2 b &= \frac{4 \cot e \cot e' \cot e'' \cot e'''}{(\cot e \cot e''' - \cot e' \cot e'')^2} \\ \tan^2 a &= \frac{4 \cot e \cot e' \cot e'' \cot e'''}{(\cot e \cot e' - \cot e'' \cot e''')^2} \end{aligned} \right\} \dots \dots (45)$$

Again, by (33, 34) we have

$$\left. \begin{aligned} \sin \frac{A+B+C}{2} \\ \cos \frac{A+B+C}{2} \end{aligned} \right\} = \frac{\frac{-\tan r + \tan r' + \tan r'' + \tan r'''}{4} \{\tan r \tan r', \tan r'', \tan r'''\}^{\frac{1}{2}}}{\frac{-\cot r + \cot r' + \cot r'' + \cot r'''}{4} \{\tan r \tan r', \tan r'', \tan r'''\}^{\frac{1}{2}}} \dots (46)$$

But the area of the spherical triangle, being E , we have

$$\cot \frac{E}{2} = \frac{\cos \left(\frac{A+B+C}{2} - \frac{\pi}{2} \right)}{\sin \frac{A+B+C}{2}} = - \frac{\sin \frac{A+B+C}{2}}{\cos \frac{A+B+C}{2}} \dots (47).$$

And from (46, 47) we have the spherical excess of the fundamental triangle in terms of the four radii, viz.

$$\cot \frac{E}{2} = \frac{\tan r_{///} + \tan r_{//} + \tan r - \tan r}{\cot r_{///} + \cot r_{//} + \cot r - \cot r} \cdot \{ \tan r, \tan r, \tan r_{//}, \tan r_{///} \}^{\frac{1}{2}} \quad (48).$$

From this also we may at once derive a new expression in terms of the sides of the triangle, for the spherical excess, viz.

$$\cot \frac{E}{2} = \left\{ \frac{-\operatorname{cosec} s + \operatorname{cosec} s-a + \operatorname{cosec} s-b + \operatorname{cosec} s-c}{-\sin s + \sin s-a + \sin s-b + \sin s-c} \right\}^{\frac{1}{2}} \cdot \{ \sin s \sin s-a \sin s-b \sin s-c \}^{\frac{1}{2}} \quad (49)$$

which is obtained by putting for $\tan r$, $\cot r$, etc. their values

$$\frac{\pi}{\sin s}, \frac{\sin s}{\pi}, \text{ etc.}$$

With respect also to the supplemental triangles, we shall have similar results. For denoting them by E' , E'' , E''' , we have by simple reductions

$$\left. \begin{aligned} \cot \frac{E'}{2} &= \tan \frac{-A+B+C}{2} \\ \cot \frac{E''}{2} &= \tan \frac{A-B+C}{2} \\ \cot \frac{E'''}{2} &= \tan \frac{A+B-C}{2} \end{aligned} \right\} \dots (50).$$

Putting in the second sides of (50) the values furnished by (31, 32) we get the cotangents of the semi-areas of the three triangles. We have then

$$\begin{aligned}
 \cot \frac{E}{2} &= \frac{-\tan r + \tan r_1 + \tan r_{II} + \tan r_{III}}{-\cot r + \cot r_1 + \cot r_{II} + \cot r_{III}} \cdot \{\tan r \tan r_1 \tan r_{II} \tan r_{III}\}^{\frac{1}{2}} \\
 \cot \frac{E'}{2} &= \frac{\tan r - \tan r_1 + \tan r_{II} + \tan r_{III}}{\cot r - \cot r_1 + \cot r_{II} + \cot r_{III}} \cdot \{\tan r \tan r_1 \tan r_{II} \tan r_{III}\}^{\frac{1}{2}} \\
 \cot \frac{E''}{2} &= \frac{\tan r + \tan r_1 - \tan r_{II} + \tan r_{III}}{\cot r + \cot r_1 - \cot r_{II} + \cot r_{III}} \cdot \{\tan r \tan r_1 \tan r_{II} \tan r_{III}\}^{\frac{1}{2}} \\
 \cot \frac{E'''}{2} &= \frac{\tan r + \tan r_1 + \tan r_{II} - \tan r_{III}}{\cot r + \cot r_1 + \cot r_{II} - \cot r_{III}} \cdot \{\tan r \tan r_1 \tan r_{II} \tan r_{III}\}^{\frac{1}{2}}
 \end{aligned}
 \tag{51}$$

These cotangents then have the same ratios as the fractional terms of their values have.

We have already given an expression for E in terms of R, R_1, R_{II}, R_{III} (vid. I. 33.) and we shall hereafter give other remarkable expressions having a close alliance with those already introduced into this paper. We shall for the present moment only set down the following equations relating to the inscribed radii, and then pass to the expressions in terms of the circumscribing radii corresponding with those already laid down relative to the inscribed ones.

From (25, 26, 27) we have

$$\left. \begin{aligned}
 \cot \frac{1}{2}A \cot \frac{1}{2}B &= \tan r_{III} \cot r \\
 \cot \frac{1}{2}A \cot \frac{1}{2}C &= \tan r_{II} \cot r \\
 \cot \frac{1}{2}B \cot \frac{1}{2}C &= \tan r_1 \cot r
 \end{aligned} \right\} \dots\dots\dots (52)$$

From which

$$\left. \begin{aligned}
 \cot \frac{1}{2}A \cot \frac{1}{2}B + \cot \frac{1}{2}A \cot \frac{1}{2}C + \cot \frac{1}{2}B \cot \frac{1}{2}C &= \\
 (\tan r_{III} + \tan r_{II} + \tan r_1) \cot r & \\
 \cot^2 \frac{1}{2}A \cot^2 \frac{1}{2}B + \cot^2 \frac{1}{2}A \cot^2 \frac{1}{2}C + \cot^2 \frac{1}{2}B \cot^2 \frac{1}{2}C &= \\
 (\tan^2 r_{III} + \tan^2 r_{II} + \tan^2 r_1) \cot^2 r & \\
 \dots \dots \dots \dots \dots \dots \dots \dots &= \\
 \dots \dots \dots \dots \dots \dots \dots \dots & \\
 \cot^n \frac{1}{2}A \cot^n \frac{1}{2}B + \cot^n \frac{1}{2}A \cot^n \frac{1}{2}C + \cot^n \frac{1}{2}B \cot^n \frac{1}{2}C &= \\
 (\tan^n r_{III} + \tan^n r_{II} + \tan^n r_1) \cot^n r &
 \end{aligned} \right\} (53)$$

In like manner taking the reciprocals of (52) we shall have a similar series

$$\left. \begin{array}{l} \dots \dots \dots \dots \dots \dots = \\ \dots \dots \dots \dots \dots \dots = \\ \dots \dots \dots \dots \dots \dots = \\ \tan^{\frac{1}{2}} A \tan^{\frac{1}{2}} B + \tan^{\frac{1}{2}} A \tan^{\frac{1}{2}} C + \tan^{\frac{1}{2}} B \tan^{\frac{1}{2}} C = \\ (\cot^{\frac{1}{2}} r_{\dots} + \cot^{\frac{1}{2}} r_{\dots} + \cot^{\frac{1}{2}} r_{\dots}) \tan^{\frac{1}{2}} r \end{array} \right\} (54).$$

The use to which the *polar triangle* has been applied is invariably that of enabling us to infer from any established properties of the sides and angles of a spherical triangle, the respective analogous properties when the sides and angles are interchanged. The property of the radii of inscribed and circumscribing circles established in works on Trigonometry, will enable us to make similar transformations in the formula which involve their radii: and we have already in (11, 12, 13, 14) given one specimen of this class of changes. We might proceed in the same manner to write out all the analogies in $R, R_{\dots}, R_{\dots}, R_{\dots}$ and $P, P_{\dots}, P_{\dots}, P_{\dots}$ corresponding to those already given in $r, r_{\dots}, r_{\dots}, r_{\dots}$ and $\rho, \rho_{\dots}, \rho_{\dots}, \rho_{\dots}$. As, however, it is of importance to attend to the sign in the case of the value of $\cos a$, (in the same manner that it was with the case of the value of $\cos A$ in equation 20), I shall proceed through a few of the steps by a course analogous to that already employed instead of deriving the result by means of the polar property.

By the common formula

$$\begin{aligned} \sin^2 a &= \frac{4(-\cos S \cos S - A \cos S - B \cos S - C}{\sin^2 B \sin^2 C} \\ &= \frac{4 \cot R \cot R_{\dots} \cot R_{\dots} \cot R_{\dots}}{(\cot R \cot R_{\dots} + \cot R_{\dots} \cot R_{\dots})^2}. \end{aligned}$$

$$\text{Hence } \cos a = (1 - \sin^2 a) = \pm \frac{\cot R \cot R_{\dots} - \cot R_{\dots} \cot R_{\dots}}{\cot R \cot R_{\dots} + \cot R_{\dots} \cot R_{\dots}},$$

Here the same difficulty occurs as in (20) already considered; and by a comparison with the value of $\cos a$ in terms of A, B, C , we shall find that $+$ is the proper sign. We might, indeed, have inferred this also from the polar triangles, as a in one triangle is the supplement of A in the other. We thus get the several values of the following quantities as calculated below.

$$\begin{aligned}
 \sin^2 a &= \frac{4 \cot R \cot R, \cot R_{//} \cot R_{///}}{(\cot R \cot R, + \cot R_{//} \cot R_{///})^2} \\
 \cos a &= \frac{\cot R \cot R, - \cot R_{//} \cot R_{///}}{\cot R \cot R, + \cot R_{//} \cot R_{///}} \\
 \sin^2 b &= \frac{4 \cot R \cot R, \cot R_{//} \cot R_{///}}{(\cot R \cot R_{//} + \cot R, \cot R_{///})^2} \\
 \cos b &= \frac{\cot R \cot R_{//} - \cot R, \cot R_{///}}{\cot R \cot R_{//} + \cot R, \cot R_{///}} \\
 \sin^2 c &= \frac{4 \cot R \cot R, \cot R_{//} \cot R_{///}}{(\cot R \cot R_{///} + \cot R, \cot R_{//})^2} \\
 \cos c &= \frac{\cot R \cot R_{///} - \cot R, \cot R_{//}}{\cot R \cot R_{///} + \cot R, \cot R_{//}}
 \end{aligned}
 \quad \dots\dots(55)$$

And from these we immediately obtain

$$\begin{aligned}
 \sin^2 \frac{1}{2} a &= \frac{\cot R_{//} \cot R_{///}}{\cot R \cot R, + \cot R_{//} \cot R_{///}} \\
 \cos^2 \frac{1}{2} a &= \frac{\cot R \cot R_{//}}{\cot R \cot R, + \cot R_{//} \cot R_{///}} \\
 \tan^2 \frac{1}{2} a &= \frac{\cot R_{//} \cot R_{///}}{\cot R, \cot R}
 \end{aligned}
 \quad \dots\dots(56)$$

$$\begin{aligned}
 \sin^2 \frac{1}{2} b &= \frac{\cot R, \cot R_{///}}{\cot R \cot R_{//} + \cot R, \cot R_{///}} \\
 \cos^2 \frac{1}{2} b &= \frac{\cot R \cot R_{//}}{\cot R \cot R_{//} + \cot R, \cot R_{///}} \\
 \tan^2 \frac{1}{2} b &= \frac{\cot R, \cot R_{///}}{\cot R \cot R_{//}}
 \end{aligned}
 \quad \dots\dots(57)$$

$$\begin{aligned}
 \sin^2 \frac{1}{2} c &= \frac{\cot R, \cot R_{//}}{\cot R \cot R_{///} + \cot R, \cot R_{//}} \\
 \cos^2 \frac{1}{2} c &= \frac{\cot R \cot R_{///}}{\cot R \cot R_{///} + \cot R, \cot R_{//}} \\
 \tan^2 \frac{1}{2} c &= \frac{\cot R \cot R_{//}}{\cot R, \cot R_{///}}
 \end{aligned}
 \quad \dots\dots(58)$$

It is almost unnecessary to remark that the values of $\cos a$, $\cos b$, $\cos c$, and all the functions which depend in any way upon these, as we have derived them, may be changed into the same forms as those of $\cos A$, $\cos B$, $\cos c$ and their dependent expression. For

$$\cos a = + \frac{\cot R \cot R, - \cot R_{//} \cot R_{///}}{\cot R \cot R, + \cot R_{//} \cot R_{///}}$$

which divided by $\cot R \cot R, \cot R, \cot R,$ becomes

$$\cos a = - \frac{\tan R \tan R, - \tan R, \tan R,}{\tan R \tan R, + \tan R, \tan R,}.$$

And similarly,

$$\left. \begin{aligned} \tan^2 \frac{1}{2} a &= \frac{\cot R, \cot R,}{\cot R \cot R, + \cot R, \cot R,} = \frac{\tan R \tan R,}{\tan R \tan R, + \tan R, \tan R,} \\ \tan^2 \frac{1}{2} a &= \frac{\cot R \cot R,}{\cot R \cot R, + \cot R, \cot R,} = \frac{\tan R, \tan R,}{\tan R \tan R, + \cot R, \cot R,} \\ \tan^2 \frac{1}{2} a &= \frac{\cot R, \cot R,}{\cot R \cot R,} = \frac{\tan R \tan R,}{\tan R, \tan R,} \end{aligned} \right\} (59)$$

and similarly for the corresponding functions of b and c , which we may consider as belonging to this table without actually writing them down.

$$\text{Again, } \tan^2 \frac{1}{2} a \tan^2 \frac{1}{2} b \tan^2 \frac{1}{2} c = \frac{\tan R}{\tan R,} \cdot \frac{\tan R}{\tan R,} \cdot \frac{\tan R}{\tan R,} \quad (60)$$

$$\left. \begin{aligned} \frac{\tan \frac{1}{2} a}{\tan \frac{1}{2} b} &= \frac{\tan R,}{\tan R,} \\ \frac{\tan \frac{1}{2} a}{\tan \frac{1}{2} c} &= \frac{\tan R,}{\tan R,} \\ \frac{\tan \frac{1}{2} b}{\tan \frac{1}{2} c} &= \frac{\tan R,}{\tan R,} \end{aligned} \right\} \dots \dots \dots (61),$$

Also, as in (50) putting $\sin A \sin T = D,$ etc.

$$\frac{\sin B}{\sin C} = \frac{T,}{T,}; \text{ hence } \frac{\sin B + \sin C}{\sin B - \sin C} = \frac{T, + T,}{T, - T,}; \text{ that is}$$

$$\begin{aligned} \frac{\tan \frac{1}{2} (B + C)}{\tan \frac{1}{2} (B - C)} &= \frac{(\cot R + \cot R,)(\cot R, + \cot R,)}{(\cot R - \cot R,)(\cot R, - \cot R,)} \\ &= \frac{\sin (R + R,) \sin (R, + R,)}{\sin (R - R,) \sin (R, - R,)}, \text{ that is} \end{aligned}$$

$$\left. \begin{aligned} \frac{\tan \frac{1}{2} (B + C)}{\tan \frac{1}{2} (B - C)} &= \frac{\cos (R + R, - R, - R,) - \cos (R + R, + R, + R,)}{\cos (R - R, + R, - R,) - \cos (R - R, - R, + R,)} \\ \frac{\tan \frac{1}{2} (C + A)}{\tan \frac{1}{2} (C - A)} &= \frac{\cos (R + R, - R, - R,) - \cos (R + R, + R, + R,)}{\cos (R - R, + R, - R,) - \cos (R - R, - R, + R,)} \\ \frac{\tan \frac{1}{2} (A + B)}{\tan \frac{1}{2} (A - B)} &= \frac{\cos (R + R, - R, - R,) - \cos (R + R, + R, + R,)}{\cos (R - R, + R, - R,) - \cos (R - R, - R, + R,)} \end{aligned} \right\} (62)$$

By the expansions of $\sin \frac{a+b+c}{2}$, $\cos \frac{a+b+c}{2}$, etc.

as in (31, 32) we did of $\sin \frac{A+B+C}{2}$, etc. and inserting those

expressions the values of $\sin \frac{1}{2}a$, $\sin \frac{1}{2}b$, etc. we should find the values of those functions of the sides in terms of the four circumscribing radii: but in virtue of the polar property already employed we obtain them by simple inspection as fast as we can write them out. We shall therefore consider it unnecessary to put down the work by the process above described, but merely employ the polar property in the derivation of the following formulæ. The rules for this change are:—

1. In all cases of half sides or angles or half sums of sides or angles interchange the functions, sine and cosine in passing from the angles to the sides or the contrary.

2. Interchange the tangents and cotangents in passing between the inscribed and circumscribed circle.

$$\tan^2 \frac{1}{2}A =$$

$$\frac{(-\tan R + \tan R_1 + \tan R_{II} + \tan R_{III})(\tan R - \tan R_1 + \tan R_{II} + \tan R_{III})}{(\tan R + \tan R_1 - \tan R_{II} + \tan R_{III})(\tan R + \tan R_1 + \tan R_{II} - \tan R_{III})}$$

$$\tan^2 \frac{1}{2}B =$$

$$\frac{(-\tan R + \tan R_1 + \tan R_{II} + \tan R_{III})(\tan R + \tan R_1 - \tan R_{II} + \tan R_{III})}{(\tan R - \tan R_1 + \tan R_{II} + \tan R_{III})(\tan R + \tan R_1 + \tan R_{II} - \tan R_{III})} \quad \dots (63).$$

$$\tan^2 \frac{1}{2}C =$$

$$\frac{(-\tan R + \tan R_1 + \tan R_{II} + \tan R_{III})(\tan R + \tan R_1 + \tan R_{II} - \tan R_{III})}{(\tan R - \tan R_1 + \tan R_{II} + \tan R_{III})(\tan R + \tan R_1 - \tan R_{II} + \tan R_{III})}$$

Putting

$$\left. \begin{aligned} \tan P &= \frac{-\tan R + \tan R_1 + \tan R_{II} + \tan R_{III}}{4} \\ \tan P_1 &= \frac{\tan R - \tan R_1 + \tan R_{II} + \tan R_{III}}{4} \\ \tan P_{II} &= \frac{\tan R + \tan R_1 - \tan R_{II} + \tan R_{III}}{4} \\ \tan P_{III} &= \frac{\tan R + \tan R_1 + \tan R_{II} - \tan R_{III}}{4} \end{aligned} \right\} \dots (64).$$

We have

$$\left. \begin{aligned} \tan R &= -\tan P + \tan P' + \tan P'' + \tan P''' \\ \tan R' &= \tan P - \tan P' + \tan P'' + \tan P''' \\ \tan R'' &= \tan P + \tan P' - \tan P'' + \tan P''' \\ \tan R''' &= \tan P + \tan P' + \tan P'' + \tan P''' \end{aligned} \right\} \dots (65).$$

Also,

$$\tan P \tan P' + \tan P'' \tan P''' = 4(\tan R \tan R' + \tan R'' \tan R''') \dots (66)$$

Then we get

$$\left. \begin{aligned} \tan^2 \frac{C}{2} &= \frac{\tan P' \tan P''}{\tan P \tan P'''} \\ \sin^2 \frac{C}{2} &= \frac{\tan P' \tan P''}{\tan P \tan P''' + \tan P' \tan P'''} \\ \cos^2 \frac{C}{2} &= \frac{\tan P \tan P'''}{\tan P \tan P''' + \tan P' \tan P'''} \end{aligned} \right\} \dots \dots (67)$$

$$\left. \begin{aligned} \tan^2 \frac{B}{2} &= \frac{\tan P' \tan P'''}{\tan P \tan P''} \\ \sin^2 \frac{B}{2} &= \frac{\tan P' \tan P'''}{\tan P \tan P'' + \tan P' \tan P'''} \\ \cos^2 \frac{B}{2} &= \frac{\tan P \tan P''}{\tan P \tan P'' + \tan P' \tan P'''} \end{aligned} \right\} \dots \dots (68)$$

$$\left. \begin{aligned} \tan^2 \frac{A}{2} &= \frac{\tan P'' \tan P'''}{\tan P \tan P'} \\ \sin^2 \frac{A}{2} &= \frac{\tan P'' \tan P'''}{\tan P \tan P' + \tan P'' \tan P'''} \\ \cos^2 \frac{A}{2} &= \frac{\tan P \tan P'}{\tan P \tan P' + \tan P'' \tan P'''} \end{aligned} \right\} \dots \dots (69).$$

$$\tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} = \frac{\cot P}{\cot P'} \cdot \frac{\cot P}{\cot P''} \cdot \frac{\cot P}{\cot P'''} \dots \dots (70),$$

$$\left. \begin{aligned} \cos C &= \frac{\tan P \tan P''' - \tan P' \tan P''}{\tan P \tan P''' + \tan P' \tan P''} \\ \cos B &= \frac{\tan P \tan P'' - \tan P' \tan P'''}{\tan P \tan P'' + \tan P' \tan P'''} \\ \cos A &= \frac{\tan P \tan P' - \tan P'' \tan P'''}{\tan P \tan P' + \tan P'' \tan P'''} \end{aligned} \right\} \dots \dots (71),$$

$$\begin{aligned}
 \sin^2 C &= \frac{4 \tan P \tan P, \tan P,, \tan P,,,}{(\tan P \tan P,,, + \tan P, \tan P,,)^2} \\
 \sin^2 B &= \frac{4 \tan P \tan P, \tan P,, \tan P,,,}{(\tan P \tan P,, + \tan P, \tan P,,,)^2} \dots (72), \\
 \sin^2 A &= \frac{4 \tan P \tan P, \tan P,, \tan P,,,}{(\tan P \tan P, + \tan P,, \tan P,,,)^2} \\
 \tan^2 C &= \frac{4 \tan P \tan P, \tan P,, \tan P,,,}{(\tan P \tan P,,, - \tan P, \tan P,,)^2} \\
 \tan^2 B &= \frac{4 \tan P \tan P, \tan P,, \tan P,,,}{(\tan P \tan P,, + \tan P, \tan P,,,)^2} \dots (73), \\
 \tan^2 A &= \frac{4 \tan P \tan P, \tan P,, \tan P,,,}{(\tan P \tan P, - \tan P,, \tan P,,,)^2}
 \end{aligned}$$

Perhaps in conformity with the order we have followed in the part of the paper which has formed the foundation of these last few formulæ, we should proceed to the spherical excess in terms of $R, R, R,, R,,,$; but this is given very neatly in the work of Mr. Young, by a method, as there stated, for which I am indebted to my learned friend the Rev. H. F. C. Logan, Professor of Mathematics in the Catholic College of Prior-Park.

It must be apparent that amongst the four quantities $r, r, r,, r,,,$ as well as amongst the four $R, R, R,, R,,,$ some equation of condition must exist, inasmuch as three independent quantities *in spherico* as well as *in plano*, are sufficient to determine the triangle. We now proceed to investigate that relation, after which we shall shew the changes which the foregoing equations sustain from giving to r its value in terms of $r, r,, r,,,$, and to R its value in terms of $R, R,, R,,,$.

*To be continued in our next.**

* I take the opportunity afforded [me by reading the proofs to annex one single remark:—

I have since obtained a perusal of the work of Lhuillier alluded to in the note upon the first page of this paper; and find that he has not in any way anticipated these results. The property in the Companion and others of like kind *in plano*, form the extent of the researches in this direction: vid. Chapter 6th of his work. The whole work is well worthy of perusal by every geometer, and more especially as one of its objects is a comparison between the geometrical and algebraical methods of investigating the properties of figures.

I may also add that Dr. Lardner's value of (iv) had been anticipated

ARTICLE XX.

ON THE DEVIATION BETWEEN TWO CURVES.

(In Answer to a Note of Mr. Woolhouse inserted in the last Repository.)

By MR. DAVIES, Bath.

MY DEAR SIR,

Bath, Nov. 7th, 1832.

I had thought of letting Mr. Woolhouse's note concerning my solution pass without any remark from me, as I am sure no one who gives the slightest attention to the subject can fail to perceive that it is "a tissue of errors," such as rarely, if ever before, proceeded from the pen of a mathematician whose talents and attainments had the least claim to respectability. Had Mr. Woolhouse himself, indeed, condescended to "scan" a second time, before he published those remarks, I am sure he would have saved himself the mortification which his nung scan must have created *after* he had thus committed himself. If he has looked over his note at p. 122 of the last Repository, with any degree of attention and candor, he cannot fail to have discovered his mistake, and of course you will ere now have been furnished with his own correction; and in that case I beg you to cancel this. If, however, he should not have done so, perhaps you will be so good as to insert the following brief remarks. They are not intended to provoke further discussion, as I conceive the subject cannot admit of any: nor are they intended to give pain to Mr. Woolhouse, as I cannot conceive him to possess that littleness of spirit which can be pained by the correction of an oversight, nor can he judge it improper that I should notice it under the circumstances of the present case. *Errare est humanum*—and such mistakes as these, to which we are all alike liable, should teach us to set a humbler estimate upon our own powers than man in the pride of his wisdom is apt to do!

by M. Sorlin, in the 15th volume of Gergonne's *Annales des Mathématiques*, which I had overlooked in my enquiring into the history of this subject; as I had likewise the fact that the distance between the inscribed and circumscribed centres in terms of the radii themselves had been given by Sarrus and Sorlin in *Tomes XIII. and XV.* of that work. The final remark, page 261, Young's *Trigonometry*, is hence erroneous.

Bath, Feb. 19, 1833.

To bring the question fairly into view, I shall recapitulate the circumstances of the problem.

1. Two curves are formed, referred to the same co-ordinate axes, according to specified but different laws.

2. The equations of these curves involve, of course, certain variables and certain constants.

3. We are required to assign the "*greatest deviation*" of these curves from one another.

4. *No particular mode of determining the points in one curve which are related to any specified points in the other is given.*

5. Hence the unavoidable conclusion is, *that the problem is indeterminate and in its present form does not admit of a definite solution.*

The completing datum that I attached to this imperfect problem to render it capable of solution, was, that the related points may be determined by their having either equal abscissas, equal ordinates, or the third variable upon which both these were dependent, equal, in the two curves. These are indeed only three of the infinite number of conditions that might have been imposed upon the problem to render it determinate; but surely these were as *legitimate* as any that could be proposed. Mr. Woolhouse's error lay in supposing that the problem, as proposed, was complete, and this has led him to suppose that additional data would be superfluous. They would *then* indeed in all probability have been more than superfluous, as superfluous data taken at hazard must almost of necessity be contradictory. Had any correlation between the points of the two curves been given my datum would have been erroneous, inasmuch as it could not have been consistent with that, except, indeed, they had been identical: but this correlation was not *given*, and therefore I assumed one particular law of correlation, the simplest both in a geometrical and analytical point of view that I could devise, and the problem was still difficult enough of actual solution.

But it is singular that so obvious a fact should have escaped Mr. Woolhouse's observation, as, that he himself has been also fixing a law of correlation between the two curves, —not less so than I had done before. If, therefore, I was wrong, could he be right? The specific objection which he brings against my law is thus expressed. "*Now these observations are without foundation; for when two independent variables are thus equated a relation is at once fixed between any point in one curve and a corresponding point in the other.*" I grant there is that relation fixed. But if he fixes upon such a relation as takes place when a line joining a point

in each curve is perpendicular to both, does he not depart from his own dogma of fixing no relation, as much as I had done? Why a line *oblique* to both curves cannot represent the deviation of *correlative points* in the two curves I have got to learn—to me it is far from “obvious.” It is very far from “appearing to me very clearly” that the *only* correct way of applying analysis in such a case as this, is, to find the *shortest normal to the two curves*. Even if it were, it still violates Mr. Woolhouse’s canon, that *there should be no law of correlation fixed*, and, therefore, *he* could not be justified in offering this solution.

The introduction of the doctrine of infinitesimal deviations, is altogether uncalled for. The problem made no remark upon the greatest or least of all the “minute deviations,” but required the greatest deviation of all, supposing the law of correlation given. Had the whole of the deviations according to one or other assumed law of correlation been actually *minute*, it was our business to *prove* it, not to *assume* it. But Mr. Woolhouse cannot need to be informed that the *normal deviations between two curves, if extended to all their component elements, are, though individually arbitrary, essentially constant between any two assigned curves where it takes place*. In all other cases there is only a specific number of such normal deviations possible, and this always at finite intervals. It would be extremely easy to shew this by means of the general differential expressions for the quantities involved: but it might serve our present purpose to take the still more popular view that follows.

The normal to a curve is a tangent to its involute. If, therefore, a line be normal to two curves, it must be a common tangent to both these involutes. In this case, then, where two curves have throughout their whole course a common normal, that normal must be tangent to both their involutes in all possible positions. *This can only take place when they have the same involute*: and then, it needs no “extraordinary perspicacity” to discover that the *deviations must be constant*, and hence admit neither a maximum nor a minimum, in the proper mathematical sense of those terms. If the two involutes be not identical, then there will be only as many cases of common normals to the curves as there can be of common tangents to their involutes. If one involute should lie wholly on the concave side of the other, then of course no common tangent could be drawn to them, nor therefore could any common normal to the curves exist. Should the involutes be so situated that they touch in one

or more points without intersecting, then common normals to the curves may be drawn, as many in number as there were of those points of contact. The magnitude of the deviations would of course here depend upon the equations of the curves and the points of the normals which were originally in contact with the involutes themselves. Except the lengths of the evolutes from such origins, to all the successive points of common normalism be equal, there will be a deviation between the curves themselves: but if this should be the case, there will be none. But it is unnecessary to proceed further in these details. It is sufficient for our present purpose to remark that the only point in the curves before us where the normals will be common to both curves is at T (fig. p. 28), *and this point is precisely that where the deviation is least of all, or zero!* This is unfortunate for Mr. Woolhouse's doctrine.

I should add, that had Mr. Woolhouse attended to the details of the solution that I have given, he would have found a mistake, which I should have felt obliged by his correcting. It is the assumption of two lines coinciding, which are only parallel. The correction does not appear to materially alter the character of the resulting formula, and as Mr. Woolhouse has justly remarked, "the solution in numbers if not impracticable would be excessively tedious."

Believe me, my dear Sir,

Most faithfully your's,

THOMAS STEPHENS DAVIES.

P. S. I intended to remark that in my law of correlation I have taken the only quantity that can be considered identical in the two curves for the bond of correlation—the angle evolved in the Geneses of the two cycloids. In some points of view perhaps, this may therefore, be considered the *most appropriate* one to adopt. I had not, however, any such consideration in view in adopting it. Convenience of calculation was my only guide—the coincidence was accidental.

ARTICLE XXI.

ON THE SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION.

By WILLIAM SUTCLIFFE, Esq. A.M.

(Communicated by T. S. DAVIES, F.R.S.S. L. & E. F.R.A.S. &c.)

The equation is $ax^m \cdot \frac{d^n z}{dx^m} + by^n \cdot \frac{d^n z}{dy^n} = Cz + P$

when P is a known function of x and y .

If z be expanded in powers of y having constants or functions of x for their co-efficients, and if these co-efficients be in their turn expanded in powers of x , we shall obtain

$$\begin{aligned} z &= \dots + {}_p A_q \cdot x^p y^q + \dots \\ ax^m \frac{d^n z}{dx^m} &= \dots + p \cdot \overline{p-1} \dots \overline{p-m+1} \cdot a \cdot {}_p A_q x^p y^q + \dots \\ by^n \frac{d^n z}{dy^n} &= \dots + q \cdot \overline{q-1} \dots \overline{q-n+1} \cdot b \cdot {}_p A_q x^p y^q + \dots \end{aligned}$$

Reasoning on the general terms we have this conclusion

${}_p A_q \{ p \cdot \overline{p-1} \dots \overline{p-n+1} \cdot a + q \cdot \overline{q-1} \dots \overline{q-n+1} \cdot b - c \}$
is either equal to zero, or to the co-efficient of some of the terms of P , or its expansion.

First. Let it be equal to zero :—We may pass over the supposition that ${}_p A_q = 0$; and hence we have

$p \cdot \overline{p-1} \dots \overline{p-m+1} \cdot a + q \cdot \overline{q-1} \dots \overline{q-n+1} \cdot b = c$
which establishes a relation between p and q in all the terms entering into the composition of z .

For the sake of elucidation we will take a particular case, in which the relations of p and q can be so expressed as to effect a complete solution of the problem. Let $m = 1$, $n = 1$, and $P = 3x + x^2 y$. First take the case of the terms not containing P .

In this case, we have $pa + qb = c \therefore q = \frac{c - pa}{b}$.

Therefore the general form of these terms is,

$$Ax^p y^{\frac{c - pa}{b}} \quad \text{or} \quad \left(\frac{x}{y^{\frac{1}{b}}} \right)^p \cdot y^{\frac{c}{b}}.$$

Hence the form of the sum of any number of these terms may be represented thus:—

$$y^c \int \frac{x^a}{y^b}, \text{ where } f \text{ is arbitrary.}$$

Next, let the general term be considered as representing one of the two terms which form the value of p . Here as before $\Lambda(pa + qb - c)$ is the co-efficient, but we shall have particular values to assign to p and q instead of mere relations, and therefore the values resulting will not include arbitrary functions.*

Now $p = 3x + x^2y$. We will consider these terms separately.

1. Let the general term be considered as representing that term from which proceeds the occurrence of $3x$ in the proposed equation: then as x and y preserve their index, we must put in this general term $p = 1, q = 0, \therefore {}_1\Lambda_0(a - c)$ is the co-efficient of that term in the proposed equation where the powers of x and y are x^1 and y^0 ; and hence ${}_1\Lambda_0(a - c) = 3$ and ${}_1\Lambda_0 = \frac{3}{a - c}$.

2. Let the general term be considered as representing that from which x^2y is derived,

$\therefore p = 2$ and $q = 1$; and hence ${}_2\Lambda_1(2a + b - c) = 1$ and ${}_2\Lambda_1 = \frac{1}{2a + b - c}$

$$\therefore z = y^c \int \frac{x^a}{y^b} + \frac{3}{a - c} x + \frac{1}{2a + b - c} \cdot x^2y.$$

It will be more convenient to write the result thus:—

$$z = y^c \int \frac{x^b}{y^a} + \frac{3}{a - c} \cdot x + \frac{1}{2a + b - c} \cdot x^2y.$$

On trial it will be found that this answer is correct.

If p had not been composed of terms of the form $K \Lambda^a y^b$; it might have been reduced to that shape by expansion; or a quantity Q found by trial, such that $ax \frac{dQ}{dx} + by \frac{dQ}{dy} - cQ = p$,

in which case $z = y^c \int \frac{x^b}{y^a} + Q$.

Bath, March 10, 1833.

* It would be, perhaps, better, to say that these values need not be supposed to include arbitrary functions, the arbitrary function being already introduced. This, however, is a difference rather in language than in fact. W. S.

ARTICLE XXII.

On the figure of a homogeneous mass of fluid revolving about an axis with a given uniform angular velocity.

By the Rev. Mr. BRICE BRONWIN.

Let a, b, c be the rectangular co-ordinates of a point attracted by the mass; x, y, z those of a point attracting in the mass; m the mass; v the sum of the particles divided by their respective distances from the origin.

Then $v = \int \frac{dx dy dz}{\sqrt{(a-x)^2 + (b-y)^2 + (c-z)^2}}$. And it is

easy to see that $\frac{d^2 v}{da^2} + \frac{d^2 v}{db^2} + \frac{d^2 v}{dc^2} = 0$,

Let now r be the distance of the point attracted from the origin, θ the angle which r makes with the axis a of revolution, ω the angle which the plane of r and a makes with the plane of a and b . Or $a = r \cos \theta$, $b = r \sin \theta \cos \omega$, $c = r \sin \theta \sin \omega$. Let r', θ', ω' be the same as r, θ, ω , only that they refer to the point attracting. Consequently $x = r' \cos \theta'$, $y = r' \sin \theta' \cos \omega'$, $z = r' \sin \theta' \sin \omega'$.

Substituting these values, we have

$$v = \int \frac{r'^2 \sin \theta' dr' d\theta' d\omega'}{\sqrt{r^2 - 2rr' \{ \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\omega' - \omega) \} + r'^2}}.$$

The equation to partial differences becomes

$$\frac{d^2 v}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \cdot \frac{dv}{d\theta} + \frac{1}{\sin^2 \theta} \cdot \frac{d^2 v}{d\omega^2} + r \frac{d^2 (rv)}{dr^2} = 0. \text{ Or, making } \cos \theta = \mu;$$

$$(1 - \mu^2) \frac{d^2 v}{d\mu^2} - 2\mu \frac{dv}{d\mu} + \frac{1}{1 - \mu^2} \frac{d^2 v}{d\omega^2} + r \frac{d^2 (rv)}{dr^2} = 0 \dots \dots (A).$$

Suppose the point attracted without the mass, and we may

develope v in a descending series. Thus, let $v = \frac{u^0}{r} + \frac{u^1}{r^2} + \frac{u^2}{r^3} + \&c.$ If we substitute this value of v in (A), and compare the co-efficients of the same powers of r ; we have

$$(1 - \mu^2) \frac{d^2 u^1}{d\mu^2} - 2\mu \frac{du^1}{d\mu} + \frac{1}{1 - \mu^2} \frac{d^2 u^1}{d\omega^2} + i(i + 1)u^1 = 0 \dots (B).$$

making $T = \left\{ r^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varpi' - \varpi)) + r'^2 \right\}^{-\frac{1}{2}}$.

or $T = q^0 \frac{1}{r} + q^1 \frac{r'}{r^2} + q^2 \frac{r'^2}{r^3} + \&c.$: we may obviously put

T for v in (A), or q^i for u^i in (B); that is the quantities T and q^i are subject to the equations (A) and (B) respectively.

Now $q^0 = 1$, and $u^0 = \text{mass}$. Also

$$q' = \cos\theta' \cos\theta + \sin\theta' \cos\varpi' \sin\theta' \cos\varpi + \sin\theta' \sin\varpi' \sin\theta \sin\varpi;$$

But if the origin be the centre of gravity, we have by the property of that centre $\int r' \cos\theta' dM = \int x dM = 0$, $\int r' \sin\theta' \cos\varpi' dM = \int y dM = 0$, $\int r' \sin\theta' \sin\varpi' dM = \int z dM = 0$. Therefore $u' = 0$.

$$\text{Generally } u^i = \int r'^{i+2} dr' \sin\theta' d\theta' d\varpi' q^i =$$

$$\frac{1}{i+3} \int r'^{i+3} q^i d\mu' d\varpi'; \text{ the integral taken from } r'=0 \text{ to } r'=r'.$$

The remaining integrals must be taken from $\mu' = -1$ to $\mu' = +1$, and from $\varpi' = 0$ to $\varpi' = 360^\circ$.

If the point attracted be within the mass attracting, we must make $v = v^0 + rv^1 + r^2 v^2 + \&c.$; where v^i is also subject to

the equation (B). In this case $T = q^0 \frac{1}{r} + q^1 \frac{r'}{r^2} +$

$q^2 \frac{r'^2}{r^3} + \&c.$ $q^0, q^1, \&c.$ the same as before. Here

$$v^i = \int \frac{r'^i dr'}{r^{i+1}} \sin\theta' d\theta' d\varpi' q^i.$$

What precedes is to be found in the *Mechanice Celestis*; but as that work is scarce, and may not be in the hands of every reader who may wish to understand this paper, I have judged it best to supply all that is requisite to make the subject intelligible without any reference to that author.

In order to determine the form of r' we must have recourse to the principles of Hydronamics.

The co-ordinates of a particle of the fluid being as before, ex-

cept that y and z are now supposed fixed in space; $\frac{dv}{dx}, \frac{dv}{dy},$

$\frac{dv}{dz}$ will be the forces which act on it arising from the mutual

attraction of the particles, and tending to augment the co-ordinates. If p be the pressure, the well-known equations of the motion of the particle are

$$\frac{dv}{dx} - \frac{dp}{dx} - \frac{d^2x}{dt^2} = 0, \quad \frac{dv}{dy} - \frac{dp}{dy} - \frac{d^2y}{dt^2} = 0,$$

$$\frac{dv}{dz} - \frac{dp}{dz} - \frac{d^2z}{dt^2} = 0.$$

Now let the particle be supposed to move in a circle, the plane of which is perpendicular to the axis of rotation; $f = (y^2 + z^2)^{\frac{1}{2}} = \text{rad. of the circle}$; n the velocity of rotation to rad. unity. Then $y = f \cos nt$, $z = f \sin nt$, $x = \text{a constant}$;
 $\frac{d^2y}{dt^2} = -n^2 f \cos nt = -n^2 y$, $\frac{d^2z}{dt^2} = -n^2 f \sin nt = -n^2 z$;

and our equations become

$$\frac{dv}{dx} - \frac{dp}{dx} = 0, \quad \frac{dv}{dy} - \frac{dp}{dy} + n^2 y = 0, \quad \frac{dv}{dz} - \frac{dp}{dz} + n^2 z = 0.$$

These are the partial differentials relative to x , y , and z of one only integral equation, *viz.*

$$v - p + \frac{1}{2}n^2(y^2 + z^2) = A; \text{ or}$$

$$v + \frac{1}{2}n^2r^2 \sin^2 \theta = p + A.$$

This will appear by integrating them separately, and determining the arbitrary functions $\phi(y, z)$, $\phi_1(x, z)$, $\phi_2(x, y)$ which enter into their integrals; or by multiplying them by dx , dy , dz respectively, adding the results, and integrating. The preceding equation therefore confines the particle to the supposed motion, and contains all the conditions of equilibrium. At the surface $p = 0$; therefore

$$v + \frac{1}{2}n^2r^2 \sin^2 \theta = A.$$

This involves the equation to the surface. When $\theta = 0$, $v = A$: consequently A is the value of v at the pole. From this equation we determine the value of r for any point at the surface of the mass. The equation employed by La Place for this purpose renders the solution more easy; and is certainly a true one, and an ingenious contrivance: but it is superfluous, and the use of it in this case not scientific. Let $r = a(1 + \alpha y)$, putting a for the semi-axis, and α being a quantity of the order n^2 , or of the ellipticity.

Neglecting the second power of the ellipticity,

$$v = \frac{u^0}{r} + \frac{u^1}{a^2} + \frac{u^2}{a^3} + \&c. = \frac{M}{a}(1 - \alpha y) + \frac{u^2}{a^3} + \frac{u^3}{a^4} + \&c.$$

Putting this value of v in the equation of equilibrium, we find

$$\frac{M}{a} ay = \frac{u^2}{a^3} + \frac{u^2}{a^3} + \&c + \frac{M}{a} - \lambda + \frac{1}{2} n^2 a^2 \sin^2 \theta.$$

$$\text{But } \frac{1}{2} n^2 a^2 \sin^2 \theta = \frac{1}{2} n^2 a^2 + \frac{1}{2} n^2 a^2 \left(\frac{1}{2} - \cos^2 \theta \right).$$

The two terms on the right side of the last equation are now subjected to satisfy equation (B), the first in the case of $i = 0$ the second in the case of $i = 2$. To abridge let

$$\frac{M}{a} - \lambda + \frac{1}{2} n^2 a^2 = B. \quad \text{Then}$$

$$ay = \frac{1}{M} \left\{ \frac{u^2}{a^3} + \frac{u^2}{a^3} + \&c + \frac{1}{2} n^2 a^3 \left(\frac{1}{2} - \mu^2 \right) \right\} + \frac{aB}{M}.$$

Now r is the same function of θ and ϖ which r' is of θ' and ϖ' ; and u' the same function also of the former as u'' is of the latter. And their constant co-efficients in each case are the same: the change in r and u' from one point of the surface to another depending on the change of θ and ϖ , and not at all upon the co-efficients.

We must now step aside to give an important Theorem from La Place.

Let x^i , y'' be two functions of θ and ϖ which satisfy (B). Then integrating from $\mu = -1$ to $\mu = +1$, and from $\varpi = 0$ to $\varpi = 360^\circ$:

$$\int x^i y'' d\mu d\varpi = - \frac{1}{i(i+1)} \int y'' \left\{ \frac{d}{d\mu} \left(\frac{1}{1-\mu^2} \cdot \frac{dx^i}{d\mu} \right) \right\} d\mu d\varpi -$$

$$\frac{1}{i(i+1)} \int \frac{y'' \frac{d^2 x^i}{d\varpi^2}}{1-\mu^2} d\mu d\varpi \quad \text{by (B). Integrating relative to } \mu,$$

$$\int y'' \frac{d}{d\mu} \left\{ (1-\mu^2) \frac{dx^i}{d\mu} \right\} d\mu = (1-\mu^2) \frac{dx^i}{d\mu} y'' - (1-\mu^2) x^i \frac{dy''}{d\mu} +$$

$$\int x^i \frac{d}{d\mu} \left\{ (1-\mu^2) \frac{dy''}{d\mu} \right\} d\mu. \quad \text{The two first terms are nothing at}$$

the limits of the integral, which therefore reduces to its last term

Integrating relative to ϖ ,

$$\int y'' \frac{d^2 x^i}{d\varpi^2} d\varpi = y'' \frac{dx^i}{d\varpi} - x^i \frac{dy''}{d\varpi} + \int x^i \frac{d^2 y''}{d\varpi^2} d\varpi. \quad \text{The quan-}$$

ties y' , $\frac{dx^i}{d\varpi}$, x^i , $\frac{dy''}{d\varpi}$, are the same at both limits, which

reduces the integral to the last term. Therefore again by (v)

$$\begin{aligned} \int x^i y' d\mu d\omega &= -\frac{1}{i(i+1)} \int x^i d\mu d\omega \left\{ \frac{d}{d\mu} \left((1-\mu^2) \frac{dy'}{d\mu} \right) + \frac{\left(\frac{d^2 y''}{d\omega^2} \right)}{1-\mu^2} \right\} \\ &= \frac{i(i+1)}{i(i+1)} \int x^i y' d\mu d\omega, \text{ which is impossible as long as } i' \text{ is} \\ &\text{different from } i, \text{ unless } \int x^i y' d\mu d\omega = 0. \end{aligned}$$

Substituting for r' in $u^i = \frac{1}{i+3} \int r'^{i+3} q' d\mu' d\omega'$ we may by virtue of the theorem just demonstrated, leave out all terms in the value of r'^{i+3} which are not of the order i ,

$$\therefore u^i = a^{i+3} \int a y' q' d\mu' d\omega' = \frac{a^3}{M} \int u^i q' d\mu d\omega'.$$

But in the case of $i = 2$, we shall have

$$u^2 = \frac{a^3}{M} \int \left\{ u^{2'} + \frac{1}{2} n^2 a^2 \left(\frac{1}{3} - \mu'^2 \right) \right\} q^2 d\mu' d\omega'.$$

We now want the development of q^i . Let $\zeta \cos n(\omega' - \omega)$ be its general term. Substituting this for q^i or u^i in (B), and dividing by $\cos n(\omega' - \omega)$; we find the equation to ordinary differentials

$$(1 - \mu^2) \frac{d^2 \zeta}{d\mu^2} - 2\mu \frac{d\zeta}{d\mu} - \frac{n^2}{1 - \mu^2} \zeta + i(i+1) \zeta = 0.$$

Developing a few terms of the radical τ , we find ζ of the form $(1 - \mu^2)^{\frac{n}{2}} \cdot \{ A \mu^{i-n} + A' \mu^{i-n-2} + \&c. \}$. Putting this in the preceding equation, we have

$$\begin{aligned} \zeta &= A(1 - \mu^2)^{\frac{n}{2}} \cdot \left\{ \mu^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} \mu^{i-n-2} + \right. \\ &\quad \left. \frac{(i-n)(i-n-1)(i-n-2)(i-n-3)}{2 \cdot 4(2i-1)(2i-3)} \mu^{i-n-4} - \&c. \right\}. \end{aligned}$$

But ζ is the same function of μ' as of μ .

Therefore

$$A = \gamma(1 - \mu'^2)^{\frac{n}{2}}, \left\{ \mu'^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)} \mu'^{i-n-2} + \&c. \right\}.$$

Instead of ζ therefore we will write $\gamma \zeta'$, where γ is a constant to be determined.

To find γ , let $i - n$ be even; and suppose $\mu = 0, \mu' = 0$:

$$\text{which gives } \gamma \zeta \zeta' = \frac{\gamma \{1 \cdot 2 \cdot 3 \dots (i-n)\}^2}{\{2 \cdot 4 \dots (i-n)(2i-1)(2i-3) \dots (i+n+2)\}^2} = \frac{\gamma \{1 \cdot 3 \cdot 5 \dots (i-n-1) \cdot 1 \cdot 3 \cdot 5 \dots (i+n-1)\}^2}{\{1 \cdot 3 \cdot 5 \dots (2i-1)\}^2}.$$

If $i - n$ be odd, preserving only the first powers of μ and μ' :

$$\begin{aligned} \gamma \zeta \zeta' &= \frac{\gamma \mu \mu' \{1 \cdot 2 \cdot 3 \dots (i-n)\}^2}{2 \cdot 4 \dots (i-n-1) \cdot (2i-1) 2i-3 \dots (i+n+2)\}^2} \\ &= \frac{\gamma \mu \mu' \{1 \cdot 3 \cdot 5 \dots (i-n) \cdot 1 \cdot 3 \cdot 5 \dots (i+n)\}^2}{\{1 \cdot 3 \cdot 5 \dots (2i-1)\}^2}. \end{aligned}$$

Neglecting the powers of μ and μ' above the first

$$T = \{r^2 - 2rr' \cos(\varpi' - \varpi) + r'^2\}^{-\frac{1}{2}} + rr' \mu \mu' \{r^2 - 2rr' \cos(\varpi' - \varpi) + r'^2\}^{-\frac{3}{2}}$$

Developing the radicals, the coefficient of

$$\frac{r'^i}{r^{i+1}} \cos n(\varpi' - \varpi) \text{ in the first is}$$

$$\frac{2 \cdot 1 \cdot 3 \cdot 5 \dots (i+n-1) \cdot 1 \cdot 3 \cdot 5 \dots (i-n-1)}{2 \cdot 4 \cdot 6 \dots (i+n) \cdot 2 \cdot 4 \cdot 6 \dots (i-n)}.$$

That of $\frac{r'^i}{r^{i+1}} \mu \mu' \cos n(\varpi' - \varpi)$ in the second is

$$\frac{2 \cdot 1 \cdot 3 \cdot 5 \dots (i+n) \cdot 1 \cdot 3 \cdot 5 \dots (i-n)}{2 \cdot 4 \cdot 6 \dots (i+n-1) \cdot 2 \cdot 4 \cdot 6 \dots (i-n-1)}.$$

Comparing these values of $\gamma \zeta \zeta'$ with the former,

$$\gamma = 2 \left(\frac{1 \cdot 3 \cdot 5 \dots (2i-1)}{1 \cdot 2 \cdot 3 \dots i} \right)^2 \cdot \frac{i(i-1) \dots (i'-n+1)}{(i+1)(i+2) \dots (i+n)}, \text{ in both cases.}$$

$$\text{But when } n = 0, \text{ we must make } \gamma = \left(\frac{1 \cdot 3 \cdot 5 \dots (2i-1)}{1 \cdot 2 \cdot 3 \dots i} \right)^2.$$

We are now prepared to solve the equation $u^i =$

$$\frac{a^3}{m} \int u^i q^i d\mu' d\varpi'. \text{ The general term of } q^i \text{ is}$$

$$\gamma \zeta \zeta' \cos n(\varpi' - \varpi) = \gamma \zeta' \cos n\varpi' \zeta \cos n\varpi + \gamma \zeta' \sin n\varpi' \zeta \sin n\varpi.$$

$$\text{And as } u^i \text{ is the same function of } \theta \text{ and } \varpi \text{ with } q^i, u^i = B_n^i \zeta \cos n\varpi + C_n^i \zeta \sin n\varpi. \text{ Also } u^i = B_n^i \zeta' \cos n\varpi' + C_n^i \zeta' \sin n\varpi'.$$

These are the general terms of u^i and u'^i ; and these are the only terms of u^i and q^i that multiplied together give any thing

in the integral. For terms involving $\sin p\pi'$, $\cos p\pi'$ vanish between the limits. Substituting these we have by comparison

$$B_n^i - B_n^i \frac{a^3\pi}{M} \gamma \int \zeta'^2 d\mu' = 0, \quad C_n^i - C_n^i \frac{a^3\pi}{M} \gamma \int \zeta'^2 d\mu' = 0. \quad \text{In}$$

the case of $n = 0$ the coefficient of the last term must be multiplied by 2.

From the formulæ preceding we shall find $\gamma \int \zeta'^2 d\mu' = \frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{4}{11}, \&c.$ respectively, whatever be n . And as those formula observe regular laws, we may be sure that the general value of the integral is $\frac{4}{2i+1}$. We therefore have generally

$$B_n^i \left(1 - \frac{4a^3\pi}{M(2i+1)} \right) = 0. \quad \text{If we neglect the ellipticity of}$$

$$M = \frac{4}{3} a^3\pi. \quad \text{Consequently } B_n^i \left(1 - \frac{3}{2i+1} \right) = 0. \quad \text{As long}$$

as i is equal or greater than 2, the factor $1 - \frac{3}{2i+1}$ cannot be zero. Therefore $B_n^i = 0$.

We may make $M = \frac{4}{3} a^3\pi (1 + \delta)$, δ being an affirmative quantity; for it is impossible to suppose the equatorial diameter less than the polar. But this value of M would render the equation still more impossible unless $B_n^i = 0$. It may be remarked that La Place's method of determining the integral $\int \zeta'^2 d\mu'$ is not legitimate. We might as justly determine it from the equation $B_n^i \left(1 - \frac{3}{4} \gamma \int \zeta'^2 d\mu' \right) = 0$, for this is the very equation from which that author determines it. His result is right, but that is purely accidental; and is owing to the peculiar form he has assumed for r' , which makes his equation somewhat different from the above. The factor B_n^i being zero, it does not follow that the other factor is so.

From what has just been proved it appears that $u^3, u^4, u^5, \&c.$ are all nothing; as also are all the terms of u^2 except the first.

B_0^2 being the co-efficient of $\frac{1}{3} - \mu^2$, that term gives us $B_0^2 - \frac{4a^2\pi}{5M} B_0^2 = \frac{2\pi^2 a^2\pi}{5M}$, or neglecting the second power of the ellipticity, $\frac{2}{5} B_0^2 = \frac{3}{10} \pi^2 a^2$, and $B_0^2 = \frac{3}{4} \pi^2 a^2$.

Hence $v = \frac{u^0}{r} + \frac{u^1}{r^2} + \frac{u^2}{r^3} + \&c.$ becomes

$v = \frac{M}{r} + \left(\frac{1}{3} - \mu^2\right) \frac{3\pi^2 a^2}{4r^3} = \frac{M}{r} + \left(\frac{1}{3} - \mu^2\right) \frac{\frac{4}{5} a^2 \pi e}{r^3}$ if e be the ellipticity, as we shall presently see.

For $r = a(1 + ay) = a + \frac{u^1 + \frac{1}{2} n^2 u^2 \left(\frac{1}{3} - \mu^2\right) + \frac{5}{6} n^2 a^3}{Ma} =$
 $a(1 + \frac{15}{16} \frac{\pi^2}{\pi} \sin^2 \theta) = a(1 + e \sin^2 \theta)$; and $e = \frac{15}{16} \frac{\pi^2}{\pi}$.

It appears therefore that the figure of the fluid mass is that of the ellipsoid of revolution.

To find the value of v for a point interior of the mass; we may divide that mass into three portions—first the protuberant matter, the thickness of which is $ax y'$ —next the spherical shell, the thickness of which is $a - r$ —and lastly the sphere, of which the radius is r .

Relative to the first $v^1 = \int \frac{dr'}{r'^{n-1}} q^1 d\mu' d\omega' =$
 $\frac{1}{a^{n-1}} \int^3 dr' d\mu' d\omega' q^1 = \frac{1}{a^{n-2}} \int^3 ax y' q^1 d\mu' d\omega'$. But relative to this matter $u^1 = a^{1+2} \int^3 ax y' q^1 d\mu' d\omega'$. Consequently $v^1 = \frac{u^1}{a^{n+1}}$; or $v^0 = \frac{M}{a} - \frac{4}{3} a^2 \pi$, $v^1 = 0$, $v^2 = \left(\frac{1}{3} - \mu^2\right) \frac{4}{5} \pi e$, $v^3, v^4, \&c. = 0$.

Relative to the shell $v^1 = \frac{1}{2 - n} \left(\frac{1}{a^{n-2}} - \frac{1}{r^{n-2}} \right) \int^3 q^1 d\mu' d\omega'$; and $v^0 = 2a^2\pi - 2r^2\pi$, $v^1, v^2, \&c. = 0$.

For the sphere, our first formulæ gives $v = \frac{M}{r} = \frac{4}{3} r^2 \pi$.

Therefore relative to the whole mass

$$v = \frac{M}{a} + \frac{2}{3} a^2 \pi - \left\{ \frac{2}{3} \pi - \left(\frac{1}{3} - \mu^2 \right) \frac{4}{5} \pi c \right\} r^2 =$$

$$2a^2 \pi \left(1 + \frac{4}{3} c \right) - \left\{ \frac{2}{3} - \left(\frac{1}{3} - \mu^2 \right) \frac{4}{5} c \right\} \pi r^2. \text{ This is the very}$$

same as the value of v for a spheroid similar and concentric with the whole, passing through the point attracted.

If we wish to take account of the square of the ellipticity, we must make $r = a(1 + ay + a^2 z)$. Then by substitution in the equation of equilibrium

$$\frac{M}{a} (ay + a^2 z + a^2 y) = \frac{u^2}{r^3} + \frac{u^1}{r^1} + \&c. + \frac{1}{2} n^2 r^2 \sin^2 \theta - \Lambda + \frac{M}{a};$$

$$\frac{M}{a} a^2 z = \frac{M}{a} a^2 y^2 - \frac{3u^2}{a^3} ay + \frac{u^4}{a^3} + n^2 a^2 ay \sin^2 \theta - D; \text{ where}$$

D is the value of $\frac{u^4}{a^3}$ at the pole, and where such quantities have been left out as destroy one another by virtue of former equations.

$$\frac{M}{a} a^2 z = \frac{M}{a} c^2 \sin^2 \theta + a^2 n^2 c \sin^2 \theta - \frac{3u^2}{a^3} c \sin^2 \theta + \frac{u^4}{a^3} - D;$$

$$a^2 z = \frac{6}{5} c^2 \sin^2 \theta + \frac{3}{4a^2 \pi} \left(\frac{u^4}{5a^3} - D \right). \text{ Consequently}$$

$$a^2 z' = \frac{6}{5} c^2 \sin^2 \theta' + \frac{3}{4a^2 \pi} \left(\frac{u'^4}{5a^3} - D' \right).$$

We must form the functions under the integral sign into the orders $i = 0, 2, 4$, so as to make them satisfy (B). Then

$$u^4 = a^7 \int^{\omega} (a^2 z' + 3a^2 y'^2) q^4 d\mu' d\omega$$

$$= a^7 \int^{\omega} \left\{ \frac{6}{5} c^2 \sin^2 \theta' + 3c^2 \sin^2 \theta' + \frac{3}{4a^2 \pi} u'^4 \right\} q^4 d\mu' d\omega$$

$$= a^7 \left\{ \frac{3u'^4}{4a^2 \pi} + 3c^2 (\mu'^4 - \frac{6}{7} \mu'^2 + \frac{3}{85}) \right\} q^4 d\mu' d\omega$$

$$= \left(\frac{1}{3} B_0^4 + \frac{4}{3} \pi a^7 c^4 \right) \left(\cos^4 \theta - \frac{6}{7} \cos^2 \theta + \frac{3}{85} \right). \text{ Therefore}$$

by comparison with the like coefficient of u^4 ; we have $B_0^4 = 2\pi a^7 c^4$. And hence

$$u^4 = 2\pi a^7 c^4 \left(\cos^4 \theta - \frac{6}{7} \cos^2 \theta + \frac{3}{85} \right).$$

c c 2

In like manner we find $\Delta u^2 = a^2 \int^2 (a^2 z' + 2a^2 y'^2) q^2 d\mu' d\omega' =$
 $\frac{128}{35} a^2 c^2 \int^2 \left(\frac{1}{3} - \mu'^2 \right) q^2 d\mu' d\omega' = \frac{488}{175} \pi a^2 c^2 \left(\frac{1}{3} - \cos^2 \theta \right).$

Adding this to the value of u^2 before found, and taking into account in that value the ellipticity in $\frac{a^2}{M}$; we have for the corrected co-efficient $b_0^2 = \frac{3}{4} n^2 a^2 + \frac{104}{35} \pi a^2 c^2$; and therefore

$$u^2 = \left(\frac{3}{4} n^2 a^2 + \frac{104}{35} \pi a^2 c^2 \right) \left(\frac{1}{3} - \cos^2 \theta \right).$$

$r = a \left\{ 1 + \left(\frac{15}{16} \frac{n^2}{\pi} - \frac{2}{7} c^2 \right) \sin^2 \theta + \frac{3}{2} c^2 \sin^4 \theta \right\}$. The value of r for the ellipsoid of revolution is

$$r = a \left\{ 1 + \left(c - \frac{3}{2} c^2 \right) \sin^2 \theta + \frac{3}{2} c^2 \sin^4 \theta \right\}. \text{ The comparison}$$

of these values of r gives $c = \left(\frac{15}{16} \right) \frac{n^2}{\pi} + \frac{17}{14} \left(\frac{15}{16} \right)^2 \frac{n^4}{\pi^2}$.

Afterwards we find

$$v = \frac{M}{r} + \left(\frac{1}{3} - \mu^2 \right) \left(\frac{4}{5} \pi a^2 c + 2\pi a^2 c^2 \right) \frac{1}{r^3} +$$

$$\left(\frac{3}{85} - \frac{6}{7} \mu^2 + \mu^4 \right) 2\pi a^2 c^3 \frac{1}{r^3}. \text{ This agrees exactly}$$

to the ellipsoid of revolution.

We might in exactly the same way take account of higher powers of the ellipticity. And if we preserved $\frac{15}{16} \frac{n^2}{\pi}$ and its

powers in the values of r and v instead of introducing c , we should have the co-efficient of the second term of r expressed by a function of this quantity. And equalling this to the function of c by which the same co-efficient is expressed, we should determine all the values of c , or all the solutions which the problem admits of. And thus our solution is as complete as it can be by series.

We may also, when the point attracted is within the mass, take account of terms involving $c^2, c^3, \&c.$, from the values of $u^2, u^3, \&c.$

The figure of the mass has been found from the equation of equilibrium at the surface only. That equation therefore suffices alone for the conditions of equilibrium. Examining this

equation a little; since $-\frac{dv}{dr}$ is the force of gravity, and $n^2 r \sin^2 \theta$ the centrifugal force on a particle tending along the radius r : it follows that $v + \frac{1}{2} n^2 r^2 \sin^2 \theta$ is the weight of a whole column of fluid from the surface to the centre, taken with a contrary sign, in any place, the latitude of which is $\frac{\pi}{2} - \theta$. And Λ is the same thing with regard to a column on the axis. The equation $v + \frac{1}{2} n^2 r^2 \sin^2 \theta = \Lambda$ therefore implies that the weight of a column every where, taken from the surface to the centre, is the same. The equation $v - p + \frac{1}{2} n^2 r^2 \sin^2 \theta = \Lambda$ implies exactly the same thing. For $v + \frac{1}{2} n^2 r^2 \sin^2 \theta$ is mi-

nus the weight of a column up to the particle to which it appertains, and p is the weight of the superincumbent column. But as the integration by which this last equation was found may be supposed to be taken in any directions whatever, it implies that the pressure on a point any where taken in the mass is the same in all directions, which indeed necessarily follows from the equal weight of the columns.

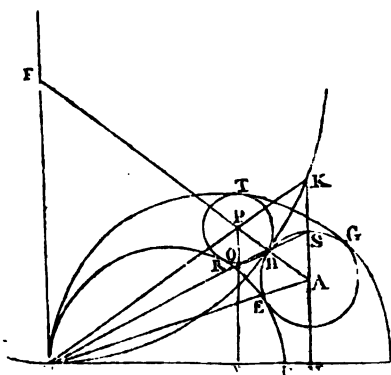
The differential of the former equation, or $dv + n^2(ydy + zdz) = 0$ implies that the resultant of the forces centripetal and centrifugal is perpendicular to the surface, as may be very easily made apparent. It is the resultant multiplied by the element of its direction, or by the differential equation of the surface. If we make p constant in the second equation, and differentiate; the resulting equation implies the same thing with regard to the surface of equal pressure. And this surface is similar to the outer surface, as will appear from the integral equation. For $p + \Lambda$ is the value of Λ for the mass included within this surface, and the value of v is that which belongs to this mass only; the shell above it having been proved to exercise no influence upon it otherwise than by pressure. This may also be made apparent by putting for v and r their values, for the result will be $p = \text{a constant}$.

ARTICLE XXIII.

*On the Arbelon.**By Mr. DAVIES, Bath.**(Concluded from page 214.)*

III. Let any two circles GHE , RHT touch the Arbelon BED in E , G , and R , T respectively, and each other in H . Let A , P be the two centres, and draw the line AP to meet the radical axis of the Arbelon in F .*

1. Then by the common properties of the radical axis and of the poles of similitude, the line AP is harmonically divided in the points F and H , and $FH = HB$. With centre F and radius FB describe a circle: it will touch the circles RHT , EHG in the point H .



2. Again, if the line BH be drawn; and cut the two circles in S and O respectively, and the radii SA , PO be drawn, these will be perpendicular to BC . For, since H is the pole of similitude of the circles whose centres are H and F the arcs cut off by AH are similar, and therefore they subtend equal angles BFH , HAS . It hence follows that AS is parallel to BF , or perpendicular to BC . In the same way it will appear that PO is also perpendicular to BC .

3. Also, since AP is harmonically divided in H and F , and lines are drawn from the points of section to a point B , one of which, BF , is parallel to AM , the intercepted portion AK of this

* Two other cases besides this are given by Pappus: but as the figures are extremely easy to construct, and would if drawn sufficiently large to be quite clear, require more room than could be conveniently commanded on an octavo page, it is thought better to omit them altogether. The point D is in both cases in the prolongation of BC , and these cases are separately formed by drawing the circle whose centre is F so as to be touched by the Arbelon internally and externally respectively.

line is bisected in s . Hence since AS is the radius, AK is equal to the diameter of the circle EGH .

4. Further, by the similar triangles furnished by the figure as already constructed, we have

$$AK (= AM + 2AS) : PN :: KB : PB :: AF : FP :: AH : HP.$$

IV. We shall now proceed to a demonstration of those celebrated properties of this figure, which excited so much interest among the ancient geometers.

1. Let the circle EGH touch BC . Then it will follow from what has been just shewn, that

$$PN : 3AH :: PH : AH,$$

and hence

$$PN = 3PH.$$

Describe another circle touching RHT in i , and let x be its centre: and draw Bx meeting PN in Y , and xz perpendicular to BC .* Then

$$XZ : NP + PY (= NP + 2PI) :: XI : PI,$$

that is

$$XZ : 5PI :: XI : PI,$$

or

$$XZ = 5 \cdot PI.$$

Proceeding in the same way we find the distance of the centre of the n th circle from BC equal to $2n - 1$ times the radius of that circle. Papp. iv. 18.

2. Let the first circle be described upon BC as a diameter. Here AM vanishes, or A coincides with M , and our proportion becomes simply

$$2AS : PN :: AH : HP, \text{ or}$$

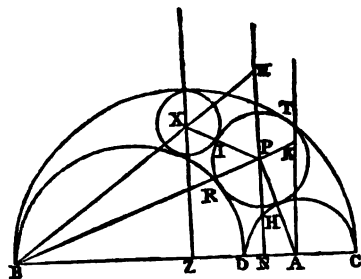
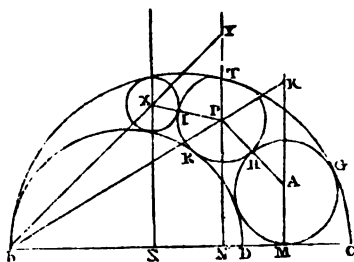
$$PN = 2PH.$$

That is, the distance of P from BC is equal to the diameter of the circle itself, or to twice its radius.

Again,

$$XZ : NP + PY :: XI : PI,$$

becomes



† By the second and third cases of the proposition, it will readily follow that the circles may be carried down the part RDH , and likewise down HCT .

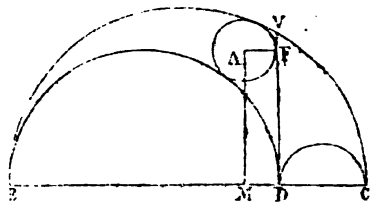
$$XZ : ZMF :: XI : PI, \text{ or}$$

$$XZ = 4PI.$$

That is, to twice the diameter of the circle whose centre is X .

In like manner, the distance of the n th centre from BC is n times the diameter, or $2n$ times the radius of the n th circle. Papp. iv. 16.

3. By referring to page 213, of the last number of the Repository, we readily get (changing to the notation of that page),



$$2AF \cdot BC = DV^2 = DF \cdot CV.$$

If, therefore, $BC = n CV$ ($n = \text{integer}$), the diameter of the first circle inscribed in the segments of the Arbelon made by DV , will be the same, or n th part of their perpendicular distances from BC . Papp. iv. 16.

It will follow from this, that when CB is an integral multiple of CV , the circles being described on the side of the circle whose centre is A lying towards CD , then the n th of these circles will have CD for a diameter.

It will also follow that if $\frac{n-1}{2}$ be integral, then the circle

whose numerical order is $\frac{n-1}{2}$ will touch the line BC .

Schol. The reader who is familiar with the works of the ancient geometers, will recognise in this latter part, the substance of the 14—18th propositions of the fourth book of Pappus. The subject is also treated very elegantly by the late lamented Professor Leslie, in the second book of his Geometrical Analysis. The method here pursued differs in some respects from both these authors; whilst in others there is an almost perfect identity between all three.

In a future paper I purpose to give a more general theorem which shall contain this as a particular case, and which at the same time opens the road to a totally new series of properties of the figure.

ARTICLE XXIV.

ON THE POWER OF MACHINES.

By MR. W. MARRAT, *Liverpool.*

1. The doctrine of the equilibrium and power of machines, is at once the most difficult and most useful branch of mechanics; and it is well known that persons who have not studied this subject with attention, have very imperfect ideas on the general effects which machinery produces.

No mass or parcel of matter can give motion to itself; it always requires some external agent to put matter in motion; it must be pushed or drawn by some external force; and to continue the motion which a body has already acquired, fresh impulses must be added at short but stated intervals.

The force which any agent expends in giving motion to a machine, is necessarily limited; thus, when a horse draws a body by means of a machine along a plane, he loses a part of his whole strength, or power of action, in endeavouring to give motion to the body; and if on account of its weight, together with the obstacles arising from friction, &c., he cannot overcome the resistance which it opposes, no motion will be produced, and this arises from the limited power of the horse. When the body moves, it is easily perceived that the absolute force of the horse is employed in three different portions. The first is that portion which is required to give himself motion; the second is expended in overcoming friction and other resistances; and the third is that portion which is employed to move the body itself.

2. This last is what is called the *moving* force, and it is measured by the product of the mass moved, or the body, into the velocity with which it is moved. It is the same for every other kind of agent, and we may consider every moving force as having for its elements, or for its factors, the weight of the body, and the velocity with which it moves. We have always a fixed and determinate moving force, which we are to employ in order to overcome a certain determinate resistance, or, which is the same, to move a given weight. Thus in machines which move weights by a uniform motion, the effect produced is measured by the product of the weight moved by its velocity; also

the effect of the moving force is measured, by the velocity of the power, into the power itself; that is, if p denote the power, v its velocity, w the weight, and V its velocity, we shall in all cases have $p v = w V$; and hence it follows that, the moving force always remains the same, whatever means we make use of to give it effect. If it be allowed that a man can act with a force or power of 50 lbs., and with a velocity of 3 feet per second; then p and v are constant quantities, and $50 \times 3 = w \times V$; if, therefore, the weight w which we are desirous to move, be great, its velocity must be slow, if the weight w be small, its velocity will be great. If the weight w be 150 lbs., its velocity

$$v = \frac{p v}{w} = \frac{50 \times 3}{150}, \text{ will be at the rate of one foot in a second}$$

of time; if the weight w be 3 lbs., its velocity will be 50 feet per second; whence it is plain that the velocity and weight may be varied in any manner we please, within certain given limits.

It is to no purpose, then, that, with a view to augment the force, we multiply levers, or wheels, or screws, or pulleys, for none of these instruments have any power of action in themselves; they only act with the force transmitted to them, and a considerable portion of the moving force is destroyed by them, by the friction which they oppose to its action. The use of machines, then, is only to modify the forces we can make use of in transporting weights from one place to another, or in raising large masses to given altitudes. Hence also it follows that if we wish to augment the load, we must diminish the velocity in the same ratio—if we would increase the velocity, we must diminish the weight to be moved.

3. In machines, where an equilibrium only is required, the situation of the forces has a great effect upon their intensity; but as the greater number of machines are destined to move heavy bodies, the moving forces cannot be varied, for the velocity *must* be diminished if the weight be increased, or the body *must* be diminished if the velocity be increased.

4. Let a weight of one pound be applied at the extremity of one arm of a lever which is ten feet long, it will be in equilibrium with a weight of 10 lbs. at the extremity of the other arm, which is one foot long; but if the lever be put in motion, and the less weight be called the power, and it be made to move with a velocity sufficient to carry it through a space of one foot in a second, the ten pounds weight will be moved through a space of only one-tenth of a foot in a second: for the velocities of the two weights are as the like arcs which they describe, and these are again as the arms of the lever. The use, then of machines,

is, to transmit, according to a certain law, a given moving force to a weight required to be moved. The machine offers us great facility, for by it we can either augment the weight, or increase the velocity, and the facility of means thus offered is of the greatest importance; for if we are desirous of moving a considerable weight, and have sufficient time for the purpose, or if we wish to raise a small weight with a considerable velocity, either of these purposes can easily be effected by the intervention of a machine; but to attempt to move a large weight with a great velocity, by a limited power, either with or without the aid of machinery, is evidently impossible. It is manifest then, that in all sorts of machinery, what is gained on one side is lost on the other, or what is gained in power is lost in velocity, or in time.

5. In the construction of machines, the less complicated they are, the better they will answer any destined purpose. Friction and every other kind of resistance should be avoided as much as possible. The most perfect machine is that in which the force is transmitted to the weight to be moved with the least possible waste of power; and this is all that, in reality, can be effected; we must endeavour then to diminish all waste of power, for every other advantage which men attribute to machines, is founded in ignorance, and in practice will be found to be merely an illusion.

6. In the choice of machines we ought to select such as are suitable to the effect required to be produced; thus, when a heavy body is to be rolled along the ground, we may employ a lever, as the most suitable instrument for that purpose; but when water is to be drawn from a well, or stones to the top of a building, we should make use of a wheel and axle, or, which is the same, of a crank and axle. It is sometimes difficult to determine what sort of machine would be most suitable to effect a given purpose, but the experienced engineer, who is well versed in the theory of mechanics, will seldom err in the choice he may make, while the practical man, if ignorant of the theory, will be just as liable to be wrong as right.

7. When we know the resistance that is to be overcome, we can easily determine the force that will be necessary to produce an equilibrium, and consequently the power that will be required to produce motion. Having calculated the power that will be necessary for the equilibrium of a machine, actually at rest, but in a state just bordering on motion, we have then only to augment the power. The motion thus generated will gradually increase, and the acceleration will continue so long as the power is greater than the resistance; but if the power diminish, or if

the power remaining the same, the resistance increases, the acceleration will diminish; and when the power and resistance have acquired values simply requisite to produce an equilibrium, the motion will become uniform:—the uniform motion will continue in consequence of the inertia of the matter moved, and the power will be sufficient to overcome the accumulating resistance continually opposed to it.

8. Machines moved by water, show, in a very sensible manner, how, when the force diminishes, the motion of the machine approximates towards a uniform motion. Let a current of water strike against the float-boards of a water wheel, at rest; at first the water strikes against the float-board with its whole force, but as soon as the wheel begins to move, the pressure of the water against the float-board gradually decreases, and if the wheel could acquire the same velocity as the current, all acceleration would cease. This circumstance, however, never actually takes place; because, if the wheel and the water were to move with the same velocity, the water would not act upon the float-boards with any force; they continue however, for some time to approximate towards an equality, and the example is properly chosen to shew how the motion of the wheel at first is gradually accelerated, and how its motion afterwards becomes nearly uniform. It may be proper to observe, that strictly speaking, no machine ever acquires a uniform motion in any finite time; like the asymptote to a curve, it continually approaches it, and arrives so near, that no difference can be distinguished by our senses, but the actual state of uniformity never takes place,

9. In the following articles I shall make use of the principle of D'ALEMBERT, as explained and illustrated by BOSSUT, and the more especially, because this method is not only more lucid, and more general, but more suitable to the purpose, than any other method with which I am acquainted.

10. Let P and w act at the extremities of a lever, the arms of which are denoted by r, r' , there will be an equilibrium when $Pr = wr'$, but if Pr be greater than wr' , P will preponderate, and the lever will turn on its fulcrum. Put the velocity of $P = v$,

that of $w = v$; then $v : v :: r' : r$, and $v = \frac{vr'}{r} =$ velocity

of P . Now if P and w were at liberty to fall freely, the velocity of each would be $= g$: the velocity lost by P , will, therefore, be expressed by $g - \frac{vr'}{r}$ and the velocity gained by w will be

$g + v$; their momenta must be equal, consequently

$$Pr \left(g - \frac{vr'}{r} \right) = wr' (g + v), \text{ and } v = r'g \left(\frac{Pr - wr'}{Pr^2 + wr'^2} \right).$$

$$\text{Because } t = \frac{2s}{v} \text{ we have } t = \frac{2s}{g} \left(\frac{Pr^2 + wr'^2}{Pr'r - wr'^2} \right)$$

where s is given.

11. Since w , P , and r' may remain constant, while r is variable, a value of r may be found that will give the time a minimum, or the angular velocity a maximum. Making r variable in the above expression for v , taking the differential, and reducing,

$$\text{we have } r = \frac{wr'}{P} \pm \frac{r}{P} \sqrt{(PW + w^2)}.$$

When the power and weight are equal, or $P = w$, $r = r' (1 + \sqrt{2}) =$

$$2.4142 r', \text{ or, } r = \frac{12}{5} r' \text{ nearly.}$$

12. If the power P be given, and we wish to find the weight w so that the work done may be the greatest; in this case the

momentum $w \times \frac{Pr'r' - r'^2 w}{Pr^2 + wr'^2}$ must be a maximum. Making

w variable; and differentiating this expression, by reduction, we

have $w = P \left(\sqrt{\left(\frac{r^4}{r'^4} + \frac{r^2}{r'^2} \right)} - \frac{r^2}{r'^2} \right)$; and when the arms r

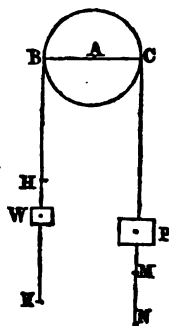
and r' are equal, $w = P (\sqrt{2} - 1) = .4142 P = \frac{5}{12} P$ nearly;

that is, the arms being equal, when the most work is done, the

weight is $\frac{5}{12}$ ths of the power.

If two unequal weights P and w be connected to the ends of a string, which passes over a fixed pulley; let it be required to determine the circumstances of the motion of the bodies.

Let us suppose that the two bodies, P , w , if they had been at liberty to fall freely, would have fallen through the equal spaces PN , KW ; in the first moment of time, but that in consequence of their mutual reaction, P has only descended through PM , while w has ascended through $WH = PM$. It is manifest that MN will express the velocity lost by P , and that KH will denote the velocity gained by w , in the same



time. Now the motion lost by P is equal to the motion gained by w , in consequence of which, we shall have $P \cdot MN = w \cdot KH$. But the small spaces PN , PM , WH , run through in consequence of the accelerative forces acting on the bodies P and w , may be regarded as the expressions of these forces; putting therefore, g for the force of gravity $= PN$, f for the actual force PM , or WH , the above equation will become $P(g - f) = w(g + f)$; whence $f = g \cdot \frac{P - w}{P + w} \dots (a)$

14. To find w when the work done is the greatest, we have, since the force is as the velocity, $\frac{w \cdot (P - w)}{P + w}$ for the momentum of w ; whence differentiating with w variable, we find $w = P(\sqrt{2} - 1) = \frac{5}{12} P$, as in art. 12.

15. If an agent be applied to a machine, as a man, a horse, &c., the whole force of the agent is not expended in giving motion to the machine; a part is consumed in giving motion to itself, the other part is expended in giving motion to the machine. Let g be the force of gravity, f the force required to move the machine, P the whole force with which the agent is capable of acting, w the resistance which the machine opposes to the agent; — the force required from the agent to move the machine, will be denoted by $P(g - f)$; but by the equation (a) $f = g \cdot \frac{P - w}{P + w}$; whence, by substitution $P(g - f) = \frac{2gPW}{P + w}$. The remaining force of the agent is expressed by $P \times f = g \times \frac{P^2 - PW}{P + w}$.

16. The tension of the part of the cord from which P is suspended is expressed by $P(g - f) = \frac{2gPW}{P + w}$, and that from which w is suspended is $w(g + f) = \frac{2gPW}{P + w}$; hence the tensions are equal, and the pressure upon the axle of the pulley is equal to their sum $= \frac{4gPW}{P + w}$; hence also, we may perceive, that when the bodies are in motion, the pressure on the axle is less than when the bodies are at rest: for when the bodies are at rest the pressure on the axle is $P + w$, a given quantity; but

when the bodies are in motion P must be greater than w , or else w greater than P . Now the rectangle of Pw is the greatest when $P = w$, consequently when P and w are unequal the rectangle is less, and so also is the pressure on the axle. Let

$P = 2\text{lbs}$, and $w = 1\text{lb}$, then $\frac{4 \times 2 \times 1}{2 + 1} = \frac{8}{3} = 2\frac{2}{3}$, is the pres-

sure on the axle when these bodies are in motion, but $2 + 1 = 3\text{lbs}$ is the pressure when the bodies are supposed to be at rest.

17. Let v be the last acquired velocity of a body after having fallen through a given space, and let v be the velocity of P , or w , after having moved through the same space; then, because the spaces are the same, the forces will be as the squares of the

velocities; that is, $v^2 : r^2 :: g : f :: g : \frac{g(P - w)}{P + w} ::$

$P + w : P - w$; consequently, $Pv^2 + wv^2 = Pv^2 - wv^2$. Hence it is plain that the principle of *living forces*, or the "Conservatio virium vivarum" as it was denominated by John Bernouilli, takes place in bodies subject to accelerating forces, in the same manner as in the shock of bodies perfectly elastic. It takes place also in many other cases, and is at bottom only a secondary consequence from the general principles of the communication of motion.

18. In (art. 19.) the inertia of the pulley has been neglected, in the following solution, the inertia is taken into consideration. In this case the momentum lost by the power P , in turning the pulley about its axis, is equal to the sum of the momenta gained by the weight w , and the mass of the pulley about its centre.

Let s denote the sum of the products of each particle of the pulley by the squares of their distances from the centre. The momentum lost by P will evidently be denoted by $Pr(g - f)$; the momentum gained by w will be denoted by $wr(g + f)$,

that of the pulley by $\frac{sf}{r}$, (because every point in the circumference of the pulley moves with the same velocity as P or w);

whence, $Pr(g - f) = wr(g + f) + \frac{sf}{r}$ and by reducing the

equation we find $f = g \cdot \frac{Pr^2 - wr^2}{Pr^2 + wr^2 + s}$. Calling this force F ,

we have the space $s = g F t^2$, and $v = 2gFt$.

19. The tension of the cord to which P is attached = $(g - f)$, becomes, by substituting for f its value found above

$= \frac{gP(2Wr^2 + s)}{Pr^2 + Wr^2 + s}$; also the tension of the cord to which w is

attached $= w(g + f)$, becomes $\frac{gW(2Pr^2 + s)}{Pr^2 + Wr^2 + s}$; therefore,

the pressure on the axle is the sum of these tensions, and if to this sum we add the weight of the pulley, we shall have the total pressure which the axle has to sustain.

20. When the friction is to be considered, let the axle of the pulley be denoted by a , and let the ratio of the friction to the pressure be as $n : 1$. The momentum lost by the power P , in moving the pulley about its axle, is equal to the sum of the momenta gained by w , of that gained by the pulley, and of that gained by the friction. Now the momentum lost by P is expressed by $Pr(g - f)$; that gained by w is expressed by $wr(g + f)$; that of the pulley, by $\frac{fs}{r}$; and since the tension of

each part of the cord to which P and w are fastened is denoted by $P(g - f)$ and $w(g + f)$, the sum of which expresses the pressure on the axle, the momentum of the friction, with respect to this axle, will be expressed by $na(P(g - f) + w(g + f))$;

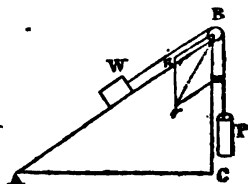
therefore $Pr(g - f) = wr(g + f) + \frac{fs}{r} + na(P(g - f) + w(g + f))$, and from this equation we find $f =$

$$\frac{g(Pr^2 - Wr^2 - narP - narW)}{Pr^2 + Wr^2 - narP + narW + s}. \text{ Putting this force } = F,$$

we have the space $s = gft^2$, and the velocity $v = 2gft$.

The solution of this case by Bossut is incorrect. He supposes that the tension of each string, when the bodies are in motion, is the same, and that the pressure on the axle is equal to twice the tension of the string from which P is suspended, which is not true; see Whewell's Dynamics, p. 269.

21. Let the power P draw the weight w along the inclined plane AB , by means of a cord passing over the pulley at B ; when the direction of the end is parallel to the plane AB , let it be required to determine the circumstances of the motions of P and w , when the inertia of the pulley is taken into the account.



Put the altitude of the plane $BC = h$, $AB = l$; then, $Pr(g-f)$ is the momentum lost by P ; and $wr\left(\frac{gh}{l} + f\right)$ is the momentum gained by w ; also, $\frac{wf}{r}$ is the momentum gained by the pulley; whence $Pr(g-f) = wr\left(\frac{gh}{l} + f\right) + \frac{wf}{r}$, and $f = \frac{g(lPr^2 - hwr^2)}{l(Pr^2 + wr^2 + s)}$. Calling this force F , we have $s = gFt^2$, and $v = 2gFt$.

22. The tension of $BP = P(g-f)$

$$= \frac{Pg(lWr^2 + hwr^2 + ls)}{l(Pr^2 + wr^2 + s)} \dots (b)$$

and the tension of the string $BW = w\left(\frac{hg}{l} + f\right)$

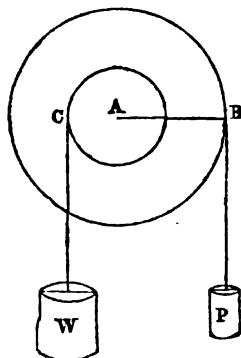
$$= \frac{wg(lPr^2 + hPr^2 + hs)}{l(Pr^2 + wr^2 + s)} \dots (c)$$

But the tension of P is in the direction BC , and that of wr , is in the direction AB ; to determine, therefore, the pressure which these two forces exercise on the axle of the pulley, let these forces be denoted by Bm and Bn ; the diagonal Bn of the parallelogram $nrmB$, will be the pressure on the axle of the pulley; put the angle $nBm = \theta$, then the pressure on the axle of the pulley will be equal to the sum of the above found tensions, or

$\{(b)+(c)\} \times \frac{\sin. \theta}{\sin. \frac{1}{2} \theta}$. The pressure of w against AB , is the same

as if w were at rest.

23. Let P and w be attached to the ends of two cords passing over a wheel and axle, as in the annexed diagram, and let P draw up w ; it is required to determine the circumstances of the motions of P and w , taking into the account the inertia both of the wheel and axle. Since the wheel and axle, although they move together, cannot be in the same plane, each has its own particular mass; let the sum of the products of the particles in each mass multiplied by the squares of their distances from the centre, be denoted by s



s , let also R and r , denote the radii of the wheel and axle, f and f' the forces which accelerate P and w respectively; then $f : f' :: R : r$, or $rf = Rr'$.

The momentum lost by P is $= PR(g - f)$, the momentum gained by w is $= wr(g + f)$; the momentum gained by the wheel $= \frac{sf}{R}$; the momentum gained by the axle $= \frac{sf'}{r}$;

hence, as in the preceding articles we have $PR(g - f) = wr(g + f) + \frac{sf}{R} + \frac{sf'}{r}$; consequently, since $rf = Rr'$ we shall have

$$f = \frac{g(PR^2 - WRr)}{PR^2 + Wr^2 + s + s}; \text{ and } f' = \frac{g(PRr - WR^2)}{PR^2 + Wr^2 + s + s}.$$

Hence we have the ratio of the accelerating forces f, f' , to the force of gravity, and the velocity, space and time may be found as in the preceding articles.

24. The tension of the cord $BP = P(g - f)$

$$= \frac{gP(Wr^2 + WRr + s + s)}{PR^2 + Wr^2 + s + s} \dots (d)$$

$$\text{tension of } CW = W(g + f) + \frac{sf}{R} + \frac{sf'}{r}$$

$$= \frac{gW(PR^2 + PR + s + s) + (s + s)g(PR - Wr)}{PR^2 + Wr^2 + s + s} \dots (e)$$

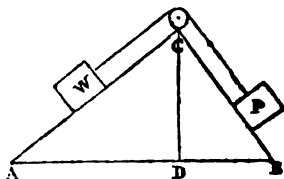
The pressure on the axle, arising from the weights P and w , is equal to the sum of the above tensions $= (d) + (e)$: The expressions found for the tension of w (e) by *Bossut*, is incorrect.

25. Let P moving down the inclined plane CB , draw w up the plane AC ; to determine the circumstances of the motion of the bodies, when the string passing over the pulley C , acts in directions parallel to CB, CA . Put $BC = m, AC = n$, and $CD = a$; we shall have

$$P\left(\frac{ga}{m} - f\right) = w\left(\frac{ga}{n} + f\right);$$

$$\text{whence, } f = \left(\frac{Pn - Wm}{Pn + Wn}\right).$$

By substituting this value of f , we shall easily obtain expressions for the velocity, space and time, by the common theorems.



26. Let it be required to ascertain the length of AC, when P will draw w along AC, in the least time possible.

In this case we have $s = n = \frac{ga}{m} \left(\frac{Pn - Wm}{Pn + Wn} \right) \times gt^2$, or $t^2 = \frac{n(Pn + Wn)}{Pn - Wm}$; but when any quantity is a maximum, its

square is a maximum also, taking, therefore, the differentials, making n variable, we have $Pn = 2Wm$, or $P : 2W :: m : n$.

In Art. 25, if the inertia of the pulley is to be considered, putting the radius of the pulley $= r$, and the rest as before;

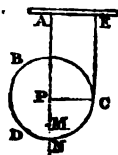
we shall have $P \left(\frac{ga}{m} - f \right) = W \left(\frac{ga}{n} + f \right) + \frac{fs}{r}$: whence, f

$= \frac{gar(nP - mW)}{mnr(P + W) + mns}$. This case has not been hitherto con-

sidered by any of the writers on mechanical philosophy; it would not be an easy matter to find the force (f) of acceleration on the two planes, when the inertia of the pulley is taken into the account, by the common method made use of by writers on this subject.

27. Let BCD be the section of a cylinder, round which a string ECDB passes, one end of which is fastened at E; when the part of the string EC is vertical, and the cylinder is left to descend by unrolling, it is required to determine the circumstances of the motion of the cylinder.

Because EC is vertical, the string reacts in the direction CE, parallel to the action of gravity, and turns the cylinder round its axis; hence the centre of gravity descends vertically, and the part of the string EC, will preserve its vertical position; draw the horizontal radius CP, then since CE = AP, the velocity with which the point c revolves will be the same as the velocity of the centre of gravity P. Let PN denote the space through which the centre of gravity P would descend if at liberty to fall freely, PM the space through which P descends when retarded by the reaction of the string. Call the natural force of gravity PN, g ; the accelerating force PM, $f = r$, the force of rotation of the point c; also put the radius PC = r , the weight of the cylinder = P , and let s denote the same as in the preceding articles. The motion lost by $P = P(g - f)$ which also represents the tension of the string; and since the tension of the string is the same in the directions EC and CE,



we may conceive that the force $P(g - f)$ reacts in the direction CE , and that it is caused in consequence of the rotatory motion of the cylinder about its axis. The momentum then, of $P(g - f)$ with respect to the point P , is equal to the momentum of rota-

tion of the cylinder; hence $Pr(g - f) = \frac{8f}{r}$, and f

$= \frac{gr^2 P}{r^2 P + s}$; which gives the relation of the accelerative force

to the force of gravity g , and shews that the motion of the cylinder is uniformly accelerated. By substituting the value of f , we have $v = 2gft$, and $s = gft^2$.

The tension of the string $= P(g - f)$, is also

$$= \frac{gPs}{r^2 P + s} \dots (f)$$

In the cylinder $s = \frac{1}{2} r^2 P$; therefore,

$$f = \frac{gr^2 P}{r^2 P + s} = \frac{gr^2 P}{r^2 P + \frac{1}{2} r^2 P} = \frac{2}{3} g.$$

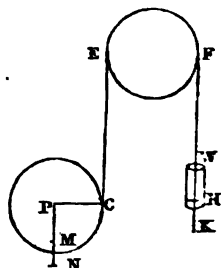
In the sphere $s = \frac{2}{3} r^2 P$; therefore,

$$f = \frac{gr^2 P}{r^2 P + s} = \frac{gr^2 P}{r^2 P + \frac{2}{3} r^2 P} = \frac{5}{7} g.$$

The tension of the string in the cylinder $= \frac{1}{3} P$; in the globe $= \frac{2}{7} P$.

28. If the string instead of being fastened at E , passes over a pulley, with a weight w hanging from its extremity w ; let it be required to find expressions for the accelerative forces of P and w .

Let $WK = PN$ denote the force of gravity $= g$, WH , PM , the spaces really passed over by the moving forces, $= f$ and f' ; also let F denote the force of rotation about the point C . The motion lost by w , which may be considered as acting in the direction CE , and at the point P , is $= HK$, that of P is $= MN$, whence we have evidently $w.HK = P.MN$, that is $w(g - f) = P(g - f') \dots (h)$. The momentum of the weight w with respect to



the centre P , is equal to the momentum gained by the cylinder in turning about its axis, passing through P ; that is, $w(g - f)$

$= \frac{8P}{r} \dots (i)$. Also, if the point P were fixed, we should have

$f = \mathfrak{P}$, but because while the point c falls through the space f , the weight w falls through the space f' , the force $\mathfrak{P}$ will evidently be equal to the sum of f and f' , that is $\mathfrak{P} = f + f' \dots (k)$. From the equations (h) (i) (k) , we easily obtain,

$$f = \frac{g (P W r^2 - (P - W) s)}{P W r^2 + (P + W) s} \dots (l) = \text{force accelerating } w$$

$$f' = \frac{g (P W r^2 + (P - W) s)}{P W r^2 + (P + W) s} \dots (m) = \text{force accelerating } P$$

$$\mathfrak{P} = \frac{2g P W r^2}{P W r^2 + (P + W) s} \dots (n) = \text{force of rotation about } c.$$

From these formulæ we may perceive that in the equation (l) if we have $P W r^2 = (P - W) s$, we shall have $w = \frac{P s}{P r^2 + s}$,

which being substituted for w in the numerator of the above formula, gives $f = 0$; consequently, in this case, w neither ascends nor descends, and the problem is the same as when $\mathfrak{E}$ was fastened, as in Art. 27.

If $(P - W) s$ be greater than $P W r^2$, or if w be less than $\frac{P s}{P r^2 + s}$, the weight w instead of descending will rise, because, in this case, the value of f becomes negative. If $w s$ be greater than $P W r^2 + P s$; or w be greater than $\frac{P \cdot s}{s - P r^2}$;

the cylinder will rise, instead of descending, because in this case the value of f' becomes negative.

29. Because the tension of the string is every where equal, it is equal to the motion lost by w , or equal to $w (g - f)$; and this gives the tension

$$= \frac{2 g P \cdot W s}{P W r^2 + (P + W) s}.$$

In the cylinder $s = \frac{1}{2} r^2 P$ and

$$\text{force accelerating } w, \text{ or } f = \frac{g (3W - P)}{3W + P},$$

$$\text{force accelerating } P, \text{ or } f' = \frac{g (W + P)}{3W + P},$$

$$\text{force accelerating } c, \text{ or } \mathfrak{P} = \frac{4 g W}{3W + P},$$

$$\text{tension of the string} = \frac{2 P W}{3W + P}.$$

Other deductions from the general theorems may easily be effected.

30. Suppose the cylinder to descend, by the unfolding of the string down an inclined plane, the accelerating force will be diminished in the proportion of the height of the plane to its length; hence the force which accelerates a cylinder on an inclined plane $= \frac{2}{3} \times \frac{h}{l}$, that which accelerates a globe $= \frac{5}{7} \times \frac{h}{l}$.

If the string be removed, and the friction between the body and plane be just sufficient to produce the same rotatory motion, the same effect will be produced, and the force with which a cylinder slides down an inclined plane will be to the force with which it rolls down, as $\frac{h}{l}$ to $\frac{2}{3} \times \frac{h}{l}$, or as 3 to 2; and in the

sphere, the forces of sliding and rolling will be as 7 to 5. There is then only a certain particular angle of elevation of the plane, suitable for the purpose of making the bodies roll, in the manner described; for with a less angle the bodies would not move, with a greater they would partly roll and partly slide, and the circumstances of their motion could not be easily ascertained. When the bodies and the plane are both perfectly smooth, the bodies would slide under all circumstances. In this manner the ratio of the friction of bodies to their pressure may be determined, not indeed, a priori, but always by the conditions of the circumstances under consideration.

31. Let it be required to determine the motion of two weights acted upon by a combination of wheels and axles.

In this case we shall have, $PR (g - f)$

$$= Wr (g + f) + \frac{sf}{r} + \frac{s'f}{r'} + \frac{s''f}{r''} \dots$$

which gives

$$f = \frac{g(PR - Wr)}{PR + Wr + \frac{s}{r} + \frac{s'}{r'} + \frac{s''}{r''} \dots}$$

and from this expression of the force being given, whatever else may be required is easily determined, as in the preceding cases.

The above general principle may be applied to the solution of many other problems in mechanics, but enough has been given to shew how it may be adapted to other similar cases.

ARTICLE XXV.

To the Editor of the Mathematical Repository.

MY DEAR SIR,

In replying to Mr. Davies I shall be as brief as possible, as the subject is one of very little importance, and, in an intellectual point of view, of too common a nature to interest your readers. When I availed myself of the opportunity of noticing the erroneous principle involved in Mr. D's solution, which accidentally occurred to me, I did so with that view to conciseness, which the useful appropriation of your pages demands, and, conscious that my remarks would be found sufficiently comprehensive, I certainly did not expect any thing in the shape of a reply from Mr. D. unless it might be a candid acknowledgement of the justness of my objections. My object was far from that of creating any unnecessary or unprofitable discussion. It is of very little concern to me whether or not I gain the conviction of Mr. D., although I should be sorry, from any want of explicitness, to allow my limited remarks to be in the least degree mystified by the very superficial observations set forth in his "Article" "On the deviation between two curve lines." Instead then of offering any allusion to his spun-out remarks on the subject of correlative points, it will be enough for my purpose merely to repeat what I conceive to be the precise meaning which is generally attached to the deviation of one curve from that of another, to which it roughly approximates.

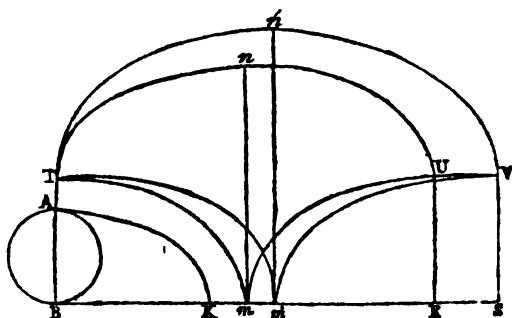
That the deviation between the two curves cannot be consistently measured by the distance between two of their points assumed in an arbitrary manner, or, which is the same thing in an analytical sense, by an arbitrary condition, or by a voluntary selection of an independent variable to be equated between the two curves.

That the usual acceptation, and that which always ought to be understood, of the term deviation as applied to two approximate curve lines, is definite; that one of the curves must be recognised as the standard, and that by the deviation of the other curve from it at any point we naturally understand its deviation or distance to the right or left, which is represented by the perpendicular drawn to the standard curve. That when this deviation attains its greatest or least value, this line, which represents it, will be perpendicular to the deviating as well as the standard curve, or a common normal to both curves.

Such are my ideas on this subject which are quite intelligible to myself, and such the "tissue of errors" so repugnant to the views of Mr. Davies; and I may now very safely indeed leave them to the calm decision of your readers.

It may be worth while to take this opportunity of noticing another circumstance connected with Mr. D's solution. This is in fact called for by the strange discovery which he states at the termination of his letter in the following words:—"It is sufficient for our present purpose to remark that the only point in the curves before us where the normals will be common to both curves is at τ . (fig. p. 28) and this point is precisely that where the deviation is least of all, or Zero!" and he adds, "This is unfortunate for Mr. Woolhouse's doctrine."

Now Sir, before Mr. Davies indulged his feelings with this last exclamation, it certainly behoved him to inform himself of the real cause of this seeming peculiarity. Had Mr. D. confined his attention to this point instead of giving way to the unnecessary remarks at the commencement of his letter, he would have learnt that his solution was altogether incorrect, and most probably would have been silent on the subject. The fact of the non-existence of a common normal line indicative of the greatest deviation, is a necessary consequence of the conditions on which he has proceeded in his solution, the inconsistency of which will at once appear on correcting and completing his figure by the addition of the other cycloidal cheek of the pendulum. Thus, as in the diagram annexed, the cycloid



$\tau m'$ will extend beyond the outline τm of the rim, and the points m, m' , will not coincide as Mr. D. has improperly assumed in his figure; because we must have

$$Bm = BK + Km = \pi r + a$$

$$Bm' = \pi \left(\frac{1}{2} BT \right) = \pi \left(r + \frac{1}{2} a \right)$$

and consequently $Bm' > Bm$, the difference being

$$mm' = \left(\frac{\pi}{2} - 1 \right) a$$

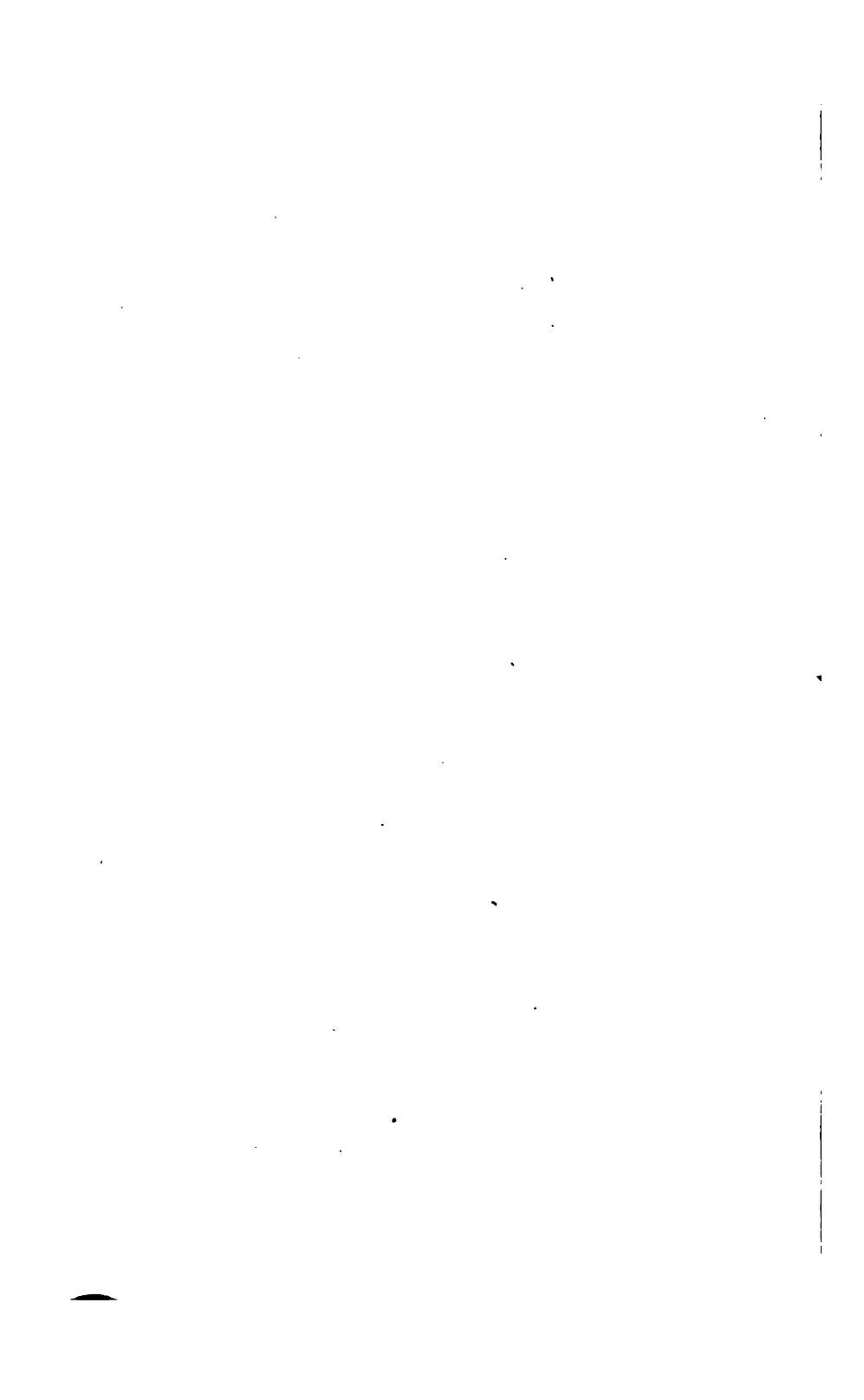
By adding, therefore, the remainder of the figure, the two pairs of cheeks tm , mu , and tm' , $m'v$, will be entirely dislocated, the extreme points u , v , being separated by the double of this distance mm' . It is also very obvious that the deviation of the two involute curves tau , tau' , described by the pendulum is *least* of all and *Zero* at t , where a *common normal* has place, and that it increases all the way from this point, and attains its *greatest* value at the points u , v , where a *common normal line* again takes place. Your readers cannot fail to perceive how very "fortunate" these circumstances are for my doctrine, and how extremely "unfortunate" for the whole of Mr. D's solution; and I am sure they will be possessed of that liberality which will fairly estimate the motive that at first induced me to attract their attention to this subject. The object of my notice was not the question, nor the solution, but only the arbitrary and unwarranted principle which Mr. Davies had employed; and I trust that I have now said quite enough for the accomplishment of my original intention, in guarding such persons, as might be too easy of assent, from the blind adoption of the same erroneous principle, for other deviations, on subjects of more interest and importance.

Your's obediently,

W. S. B. WOOLHOUSE.

Nautical Almanac Office,
Somerset House, Sept. 9, 1833.

END OF VOL. VI. PART II.



THE
MATHEMATICAL REPOSITORY.

VOL. VI. PART III.

ARTICLE I.

Pascal's Conics.

Communicated by Mr. THOMAS STEPHENS DAVIES, Bath.

Preliminary Note. Is it not a little astonishing that two works, which extended the methods of enquiry into the Conic Sections, so much beyond the boundaries marked out by Apollonius and Archimedes—works written less than two centuries ago, and amidst the glorious career of science waking with all the freshness of giant strength—and written, too, by men no less celebrated than Desargues and Pascal,—should be utterly lost! yet so it is. We can attribute this only to the neglect which resulted from men's minds being all at once turned into another direction, perhaps very naturally, by the splendid hopes, created by the discovery of the new calculus, and the absorbing devotion which it claimed from the master spirits of the following ages. Be this cause whatever it may, the only wrecks that remain of those works, are contained in a printed fragment of Pascal's (1640), which occupies four very open octavo pages, and a letter of about the same length from Leibnitz to Pascal's nephew and executor, (1676). That the works enumerated in the letter of Leibnitz were ever printed may be reasonably doubted, when we consider that the editor of his works, ABBE BOSSU, in 1779, thus expresses himself: "*J'ai écrit et fait écrire de tous côtés pour me procurer ces ouvrages de Pascal dont parle Leibnitz; mais jusqu'à présent mes recherches à ce sujet ont été presque inutiles.*" Besides, the date of the letter, 1676, just upon the return of Leibnitz from his famous English excursion, naturally leads us to suppose that his mind was full of that new doctrine which has proved the foundation of his celebrity: and that in such a mood, high as his veneration of Pascal might be, he would be

glad to escape from all other considerations than that of his Differential Calculus. The MSS, it is true, *might* yet be in existence; but this is *barely possible*: the interest which the French mathematicians take in this enquiry has doubtless led them to search every library both likely and unlikely that has fallen in their way.

Lahire, whose Conics were published in 1673, 1678, and 1685, was entirely unacquainted with both these writings, as he has not alluded to them in any way, nor do his enquiries or his language indicate the slightest influence upon his method, such as must have been created by them, had he ever seen them. It is not, however, my present intention to enter upon the history of the Geometry of Transversals, though in reference to some future time, I do contemplate a complete history of this branch of science in common with other branches of pure Mathematics, and such of their applications as have had any valuable influence upon the progress of Natural Philosophy. Suffice it to say at present, that the method of transversals is one of the most powerful methods of investigating the properties of the conic sections that has ever been devised, often superior to the analytical geometry in its most improved state, and scarcely ever inferior to it as an instrument of investigation; whilst for all practical purposes, especially of a descriptive kind, it is incalculably preferable. One of its most extensive properties is to be found in the printed Tract of Pascal, and is quoted from Desargues. Indeed, that one theorem followed out will furnish demonstrations of almost every property that is yet known, (and that perhaps can be known) concerning lines of the second order, which do not involve considerations concerning curvature. However the details of these propositions are lost: and we can only judge of them from the following fragments. They are found in the complete editions of Pascal's works of 1779 and 1819: and though easy enough to obtain, they are yet to be obtained only by procuring his whole works, one volume alone of which out of five are upon mathematical subjects, and even those subjects scarcely calculated to gratify the general taste of mathematicians in the present day. It is then, I conceive, doing service to geometers of a less antiquarian cast to occasionally reprint such tracts as these, which, however interesting in themselves, unless thus presented, would scarcely be sought for in the places to which circumstances had naturally consigned them.

Bath, Sept. 1829.

T. S. D.

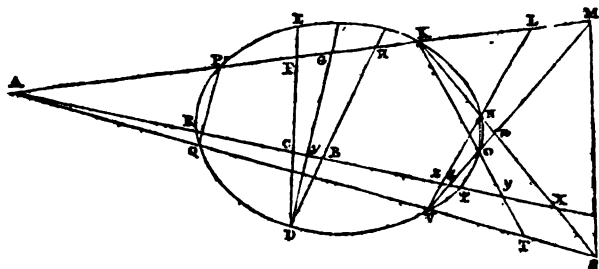
DEF. I. When several right lines converge to a point, or are all parallel to one another, they are said to be of the *same order*; and the number of these lines is called *the order of these lines*.

DEF. II. By the word *conic section*, I mean the circumference of the circle, the rectilinear angle, the ellipse, the hyperbola, and the parabola; accordingly as the cone is cut parallel to its base, or by its summit, or in the three other directions which produce the ellipse, the hyperbola or the parabola, and gives in its surface the circumference of a circle, or an angle, or the ellipse, or the hyperbola, or the parabola.

DEF. III. By the word *line* I mean the right line.

LEMMA I.

If in the plane MSA are drawn two lines MK , MV and from the point S be drawn two lines SK , SV ; and that K be the inter-



section of MK , SK ; V that of MV , SV ; A that of MA , SA ; μ that of MV , SK : and that through two of the four points A , K , μ , V which are not in the same line with the points M , S , as, for instance, K , V , pass the circumference of a circle cutting the lines MV , MP , SV , SK in the points O , P , Q , N : then I say that the lines MS , NO , PQ are of the same order.

LEMMA II.

If by the same line pass several planes, which are cut by another plane, the lines of section of these planes are of the same order with the line by which the said planes pass.

LEMMA III.

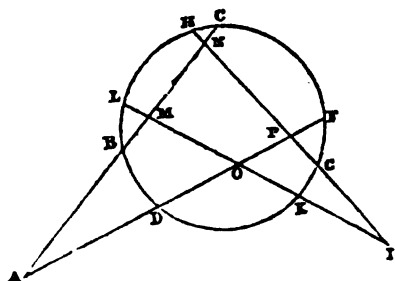
The two foregoing lemmas and some simple corollaries being admitted, we may demonstrate that the same things being supposed as in the first lemma, if by the points K , V pass any conic section which cuts the four lines MK , MV , SK , SV in the points P , O , N , Q ; the lines MS , NO , PQ are of the same order. This will be our third lemma.

Then from these three lemmas and their more obvious consequences we may derive the elements of a complete system of conics; such as the properties of diameters and their ordinates, of tangents, &c. the restitution of the cone upon all the data, the description of conic sections by points, &c.

In doing this we may state the properties upon which we touch in a manner more general than is commonly done. For instance, this: it in the plane MSQ in the section of a cone, PKV , are drawn the lines AK , AV , meeting the section in the points P , K , Q , V ; and that from two of these four points which are not in the same line with A , as K , V , and by the two points N , O , taken in the circumference of the section be drawn four lines KN , KO , VN , VO cutting the lines AV , AP in the points L , M , T , S ; I say that the ratio compounded of the ratios of the line PM to the line MA , and of the line AS to the line SQ , is the same with the ratio compounded of the ratios of the line PL to the line LA , and of the line AT to TQ .

We may shew also, that if there be three lines DE , DG , DH which cut the lines AP , AB in the points F , G , H , C , γ , B ; and that in the line DC there be determined the point E : the ratio compounded of the ratios of the rectangle EF , FG , to the rectangle EC , $C\gamma$, and of the line $A\gamma$ to the line AG , is the same with the ratio compounded of the ratios of the rectangle EF , EN to the rectangle EC , CB and of the line AB to the line AH : and that it is also the same with the ratio of the rectangle of the lines FE , ED to the rectangle of the lines CE , CD . And thence, if by the points E , D a conic section be made to pass which cuts the lines AH , AB , in the points P , K , R , ψ ; the ratio compounded of the ratios of the rectangle of EF , FC , to the rectangle of EC , $C\gamma$, and of γA to AG , will be the same as that compounded of the ratios of the rectangle KF , FP to that of CR , $R\psi$ and of the rectangle of the lines AR , $A\psi$ to that of the lines AK , AP .

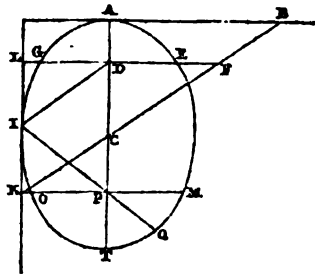
We may shew also that if four lines AC , AF , EH , EL intersect one another in the points N , P , M , O , and that a conic section cuts the said lines in the points C , B , F , D , H , G , L , K : the ratio compounded of the ratios of the rectangle MC , MB to the rectangle PF , PD , and of the rectangle of the lines AD , AF to that of AB , AC , is the same as the ratio com-



pounded of the ratios of the rectangle ML, MK to that of PH, PG , and of the rectangle EH, EG to that of EK, EL .

We may also demonstrate (fig. 1.) the following property, the author of which is M. Desargues, of Lyons, one of the greatest men of our time, and most profoundly versed in mathematics, and amongst other branches in conics, of which his writings afford ample testimony to those who are able to comprehend them. I must candidly acknowledge that I owe the little I have discovered upon this matter to his writings, and that I have attempted to imitate as far as I could his method of investigation upon this subject (which he has treated without employing the triangle passing by the axis) in discussing generally all the sections of the cone. The singular property in question is this: If in the plane MSQ there be a conic section PVQ in the circumference of which having taken the four points K, N, O, V , having drawn the four lines KN, KO, VN, VO , so that by any one of the four points there shall not pass more than two straight lines, and that another line be drawn to meet the section in R, ψ , and the lines KN, KO, VN, VO in the points x, y, z, δ ; I say that as the rectangle of $zR, z\psi$ is to that of $\delta R, \delta\psi$, so is the rectangle of $xR, R\psi$ to that of $yR, y\psi$.

We may further shew that if in the plane of the hyperbola or of the ellipse, or of the circle, $AGTE$, of which the centre is C we draw the line AB touching the section at A , and that having drawn the diameter AT we take the straight line AB the square of which is equal to one fourth of the rectangle of the figure*, and that we draw CB : then any line as DE being drawn parallel to AB cutting the section in E and the lines AC, CB in D, F : if the section AGE is an ellipse or a circle the sum of the squares of the lines DE, DF will be equal to the square of AB ; and in the hyperbola the difference of the squares of DE, DF will be equal to the square of AB .



We may deduce also some problems; for instance, *from a given point to draw a straight line touching a given conic section.*

* The product of the diameter by its parameter. *French Editor of 1779.*

*To find two conjugate diameters with a given inclination.
To find two diameters with a given inclination and in a given ratio.*

I have several other problems and theorems, and several deductions from those already given; but the diffidence which my little experience and capacity create, will not allow me to proceed further till this has undergone the scrutiny of learned men (who would greatly oblige me by taking the trouble); after which, should they judge that the enquiry deserves to be continued, I shall attempt to proceed as far as God shall enable me to do.

Letter of M. LEIBNITZ to M. PERIER, (Nephew of Pascal),

SIR,—I am extremely obliged by your communicating the manuscripts* of the late M. Pascal, concerning conics. For besides the mark of your good will which I esteem very highly, you have afforded me the means of profiting by the perusal of the meditations of one of the finest minds of the present age: though I wish I could have read them with a little more application; but the great number of distractions which do not allow me to dispose entirely of my own time, have prevented me. However, I think I have read them so as to be able to answer your enquiry, and to say that I think them sufficiently complete and finished for publication; and in order that you may judge whether I speak with foundation, I make a list of the pieces of which they are composed, and of the manner in which I think they should be arranged.

I. It is necessary to begin with the piece of which the title is *Generatio conic sectionum tangentium et secantium, seu projectio peripheriæ, tangentium, et secantium circuli, in quibuscunque oculi, plani ac tabellæ positionibus*: for this is the foundation of all the rest.

II. After having explained the generation of the sections of the cone, made optically by the projection of a circle upon a plane which cuts the cone of rays, he explains the singular properties of a certain figure composed of six right lines, which he calls *hexagramme mystique*. I have put before it

* I have written, and caused to be written in all directions to procure the works of which Leibnitz speaks, but hitherto my search has been altogether useless.—*French Editor.*

these words, *de hexagrammo mystico et conico*. One part of this piece is found repeated, and inserted word for word, in another, namely, the definitions, (with their corollaries); and the propositions, (but without their demonstrations), which are found repeated in the *Treatise de loco solido*, will supply the deficiency of some that are wanting in this, *de hexagrammo*.*

III. The third treatise should be, according to my view, that which bears this inscription: *De quatuor tangentibus, et rectis puncta lactuum jungentibus, unde rectarum harmonicè sectarum et diametrorum proprietates oriuntur*. For it is here that the use of the hexagram appears, and that the properties of centres and diameters of conic sections are explained. I think there is nothing deficient.

IV. The fourth treatise is: *de proportionibus segmentorum secantium et tangentium*. For the fundametal properties of conic sections which depend upon the knowledge of the centres and of the diameters, being explained in the preceding treatise, he judged it advisable to deduce some of the fine properties universally known, concerning the proportions of straight lines drawn in a conic section: and it is upon this which depends all we can state concerning ordinates. I have put before this treatise a sheet bearing for title, these words *de correspondentibus diametrorum*, of which the third page treats *de summâ et differentiâ laterum seu focis*.

V. The fifth treatise is: *de tactionibus conicis*, that is to say, (in order that the the title may have no ambiguity), *de punctis et rectis quas sectio conica attingit*; but I have not found all the figures.

VI. The sixth will be *de loco solido*: I have given this title to it because there is not one. It is upon this subject that M. M. Descartes and Fermat laboured, when they gave the composition of a solid locus each in his own way, a hint of

* The expression itself is not very clear, but it is easy to guess at its meaning: viz. that the propositions which were found in the *Treatise de loco solido* besides those which were repeated from the first *Treatise, de hexagrammo*, would supply the want of some which seemed to be deficient in it. The French is—"et les propositions (mais sans les demonstrations) que se trouvent répétées dans le *Traité de loco solido*, suppléeront au défaut des quelques-uns qui manquent dans celui-ci de *hexagrammo*."

Pappus having given rise to the discussion. But I think that M. Pascal has wished to give this treatise separately, or at least of communicating it to his friends, because he has repeated many of the things of the second treatise, word for word, and some even more at length. On this account he commences by: *definitiones excerptæ ex conicis*, that is from the second treatise already referred to, where he explains what he means by the words *hexagrammum mysticum*, *conicum*, &c. We may judge from this that the first, the second, the third, and perhaps the fifth ought to compose the conics; and this word is found also on the back of the first treatise. The large figures appertain to the sixth treatise.

I have put together some fragments. There is one paper of which the title is *Essais des Coniques*; and as there are two copies altogether alike. I hope you will allow me, Sir, to keep one of them. There is a fragment, *de restitutione conicæ*: viz. the diameters being given, to find the conic sections. This discourse appears to be complete, and has figures. There is also another fragment, at the commencement of which is found these words, *magnum problema*; and I think that this is what it comprises: *data puncta in sublimi, et solido conico ex eo descriptio, solidum ita secare ut exhibeat sectionem conicam datæ similem*: but this is not put distinctly.

There are some problems upon another sheet which are with them, but the first is wanting; I should think they might form an appendix, but the body of the work composed of the six first treatises appear to me very neat and perfect.

I conclude, then, that this work is in a state to be printed, and it will not be asked whether it deserves it or not. I think, also, that it will not be well to delay it long, because I see treatises appearing which have some relation to one part of this. On which account, I think it will be well to give it without delay, before it shall have lost the grace of novelty.

* * * * *

I am, Sir, your very obedient servant,

LEIBNITZ.

Paris, Aug. 30, 1676.

ARTICLE II.

A DEMONSTRATION OF LAWSON'S GEOMETRICAL THEOREMS:

By the late Rev. CHARLES WILDBORE.

From the Memoirs of the Literary and Philosophical Society of Manchester. Vol. II. Second Series.

The author at page 18, of his Pamphlet on the Analysis of the Antients, very justly observes, that in the resolution of problems there is often need of a previous preparation, a kind of mental contrivance and construction, in order to form a connexion between the *data* and *quæsitæ*. And I would not have it concealed that herein consists the great difficulty of this branch of science. Nor do I know any advice so proper to give the admirers of these rational amusements, as to endeavour to attain a facility of investigating or inventing one geometrical property from another. It is for their assistance herein, and not from any supposed excellence of the solutions above (though most of them are different from those of the original authors themselves,) that I have taken the trouble to run through, and investigate the 60 Theorems. I believe I may safely say that any person that will take the trouble to follow me herein, will find it worth his while, and may in a short time, from hence find out many times this number of theorems of like nature, and equally curious with these. And as this may possibly fall into the hands of some more learned readers, I would wish them to think, whether or no this may not possibly be a specimen of a method of investigation similar to that of the Ancients, which has been a *desideratum* ever since the Saracens burnt the library at Alexandria in Egypt.

DIAGRAM I.

Draw ED the perpendicular of the isosceles triangle BEC , and AH through the vertex, parallel to the base; from any point A in which, draw lines through the extremes of the base and perpendicular, viz. AC , AB , AD ; and through D , the extreme of the perpendicular, a line *ad libitum*, cutting AB in F , AC in G , AH in H ; and bisect DH in N . Then because the triangles DGC , AGH are similar, $DC : AH :: DG : GH$; and because BED , AFH are similar, $BD = DC : AH ::$

$FD : FH$. Therefore by equality of ratios $DG : GH :: FD : FH$. Which is the third Proposition.

Join EF , EG cutting BC in k and l , through G draw $gi \parallel$ to EH cutting ED in o and EF in i ; then by reason of the parallel lines, $AH : EH :: BD : Dk :: DC : Dl$, and because $BD = DC$, $\therefore Dk = Dl$; consequently ED bisects the $\angle FEG$, and $EF : EG :: FD : DG$. Which is the fifth Proposition.

Let fall fm perp. to ED produced; then because $FD : DG :: FH : GH$, therefore $dm : do :: em : eo$, and $FE : EG :: FD : DG :: FE : Ei$. Therefore if in any line emo , be taken two points d, e such, that $dm : do :: em : eo$, and mf, og be drawn perp. to em , and through the point d , be drawn any line to meet mf, og in f and g , and EF, EG be joined; then $FE : EG :: FD : DG$, and $FE : EG :: FE : Ei$. Which is the sixth Proposition.

Since gi is bisected by the perp. do , $\therefore di = dg$; and because $\angle DEH$ is right, $\therefore EH$ is equal to EH , and parallel to id ; because ig is parallel to EH , therefore the lines fg, EH are similarly divided in the points d and n , or $fa : FH :: FD : FG$. Which is the first Proposition.

On the centre n , describe the semicircle DEH , join FE , cutting the circle again in h , erect ag perp. to FH , cutting

LAWSON'S GEOMETRICAL THEROEMS.

PROP. I.

If a right line AB be bisected in E , and two points C and D taken therein such that $AC : CB :: AD : DB$; then I say the rectangle $DCB =$ the rectangle ACB .

The converse of this proposition is also true, which is this.

If a right line AB be bisected in E , and two points C and D taken therein such that $DCB = ACB$; then I say $AC : CB :: AD : DB$.

PROP. II. If in AB the diameter of a circle two points C and D be assumed such that $AC : CB :: AD : DB$, and from D an indefinite perpendicular to the same diameter as LD be erected, and through C any line be drawn to cut the same in E , and the circle in F and G ; I say $FC : CG :: FE : EG$.

The converse of this proposition is also true, which is this.

If any right line as LD be drawn perpendicular to the diameter AB of any circle and meets the same in D , and if from a point in the same diameter, as C , any line be drawn to meet the same perpendicular in E , and the circle in F and G , so that $FC : CG :: FE : EG$; I say that $AC : CB :: AD : DB$.

PROP. III. Let there be a triangle ABC , whose base AC is bisected in D , and through the vertex A a line AE be drawn parallel to BC and

FE in g , and the circle in T ; then since $FG : FD :: FH : FN$, therefore by division, $FG : FD :: GH : DN = HN$, and $FG : GH :: DG : GN$, or $DG \cdot GH = FG \cdot GN$, but $DG \cdot GH = TG^2$; consequently the points F, T, N are in a semicircle, and FT a tangent to the circle DEH . *Which is the fifteenth Proposition.*

Produce EG, HG till they cut the circle again in h', E' , and TG to T' ; join FT' , which must be equal to FT , and join EE' , cutting FH in f ; then because the angle $h'EH' = h'E'h'$, and $h'EH' = h'E'h'$, therefore $h'h'E' = EE'h'$, $h'E' = h'E$, $h'E' = h'E$; also because $h'GE = h'GE'$, $\therefore GH = GH', GE = GE'$; but $GT = GT'$, therefore EE' , as also hh' , is parallel to TT' ; and because FKE is a right line, $\therefore FH'E'$ is a right line and the triangles $TGE, T'GE'$ equal and similar, and therefore $gEG, g'E'G$ are so; consequently the angle $EGG = E'GG' = hgg$, and $gE : hg :: EG : Gh = GH' :: Ef : hg :: EE' : hh' :: FE : FH$. Hence if in DH the diameter of a circle, two points F and G be assumed, such that $FD : FH :: DG : GH$, and from G an indefinite perp. be erected, and through F any line be drawn to cut the same in g , and the circle in h and E , then $Fh : FE :: hg : gE$. *Which is the second Proposition.*

Also $EF : FH' :: EG : GH' = GH$. *Which is the fourth Proposition.*

any line drawn through D to meet AB, AC, AE in F, G, H ; then I say $GD : DF :: GH : HF$.

PROP. IV. If in AB the diameter of a circle two points C and D be taken such that $AC : CB :: AD : DB$, and through the point D any line be drawn to meet the circle in E and F , and CE, CF be joined; then I say $EC : CF :: ED : DF$.

PROP. V. If the base BC of a triangle be bisected in D , and through the vertex A a parallel thereto be drawn, and from D a perpendicular to BC be drawn to meet the parallel in E , and through D any line be drawn to meet AB, AC in F and G , and EF, EG be joined; then I say $EF : EG :: FD : DG$.

PROP. VI. If in the line AB be taken two points C and D such that $AC : CB :: AD : DB$, and AE, BF be drawn perpendicular to AB and through the point C be drawn any line to meet AE, BF in G and H , and DE, DH be joined; then I say that $DG : DH :: GC : CH$.

PROP. VII. If in the diameter of a circle AB be taken any point C , and CDE be drawn meeting the circle in D and E , and DF be perpendicular to AB meeting it in F , and the circle again in G , and EG be joined meeting AB in H ; I say that $AC : CB :: AH : HB$.

Also as the converse, that if in the diameter AB two points be taken as C and H such that $AC : CB :: AH : HB$, and from the points C and H two lines CE, HE be inflected to any point of the circumference E meeting the same again in D and G ; when DG is drawn, it will be perpendicular to AB .

PROP. VIII. If in the diameter of a circle AB two points O and N

Also if in the diameter of a circle, any point F be taken, and FhE be drawn meeting the circle in h and E , and hq be perpendicular to DH meeting it in q , and the circle again in h' and Ek' be joined meeting DH in G ; then $FD : FH :: DG : GH$. Which is the seventh Proposition.

Again, if in the diameter of a circle DH , two points F , G be taken such that $FD : FH :: DG : GH$, and from the points F and G , be inflected to any point of the circumference E , two lines FE , GE meeting the same again h and h' . Then $FH : FE :: GH' : EG$. Which is the eighth Proposition.

Perpendicular to FE , draw ee , cutting the diameter in c , and the circle in e ; then the angle $ECF = FEf = FGG$, and $h'ef = h'e'e = FEF$; $\therefore h'e$ is parallel to FH , and $fh = fh' : FE :: Gh' : EG :: ce : EC$. Which is the ninth Proposition.

DIAGRAM II.

Perpendicular to e the centre, or any other point of the diameter DH , erect nq , and make the $\angle \phi n = FEG$; then $\phi n = GEE$; produce ϕp till it meets ee in m ; then $\phi n = h'eh' = huf$; therefore fm is parallel to he , and $h'G : GE :: fh : FE :: em : em$. Which is Proposition tenth, part 1st.

be taken such that $AC : CB :: AH : HB$, and from the points C and H be inflected to any point of the circumference E two lines CE , HE meeting the same again in D and G ; I say that $EC : CD :: EH : HG$.

PROP. IX. If in AB the diameter of a circle be taken any point c , and CD be drawn meeting the circumference in D and E , and from the point D be drawn DF perpendicular to CD , which meets the diameter AB in F and the circumference in G , then I say that $DC : CE : DF : FE$.

PROP. X. If in AB the diameter of a circle two points C and D be taken such that $AC : CB :: AD : DB$, and through the centre E a perpendicular to AB be drawn, and from C a line CE be drawn to meet the same in F , and if through D any line DG be drawn to meet the circle in G and H , and from the point G be drawn GK on the same side of DG as F is of the diameter AB to make the angle DGK equal to the angle CFE , and let the line GK meet the circle in L and the line CF in M ; then I say that $GM : ML :: GD : DH$.

PROP. XI. If from any point C in the diameter of a circle produced a perpendicular be raised and from any point D in the same a line be drawn to cut the circle in E and F ; then I say the rectangle EDF is equal to the rectangle ACB together with the square of CD .

PROP. XII. If from any point C in the diameter of a circle produced a perpendicular be raised and thereon CD be taken whose square is equal the rectangle ACB , and CE be put equal CD , and from any point in DE as H a line be drawn to cut the circle in F and G ; then I say twice the rectangle FHG is equal to the sum of the squares of BD and HF .

Also if through the point h , any line hL be drawn to the circle, and at G the $\angle ggm$ be made equal to LhE , the points m, g, h are in a circle, therefore the angle $hmg =$ the supplement of hgg , and consequently of $hE'E$; therefore $hLE = hmg$, mg parallel to LE , and $hm : mL :: hg : gE :: gh' : EG :: fh : fL$. Which is Proposition tenth, part 2d.

At F erect a perpendicular to FH , produce hh till it meets it in δ , and join dh ; then the $\angle dhH$ being right, F, δ, h, D are in a circle; therefore $hh \cdot H\delta = FH \cdot DH = FH^2 - FD \times FH = H\delta^2 - H\delta \cdot \delta h$; $\therefore H\delta \cdot \delta h = H\delta^2 - FH^2 + FD \cdot FH = F\delta^2 + FD \cdot FH$ Which is the eleventh Proposition.

If $fd' = fa = fq = ft$, then $\delta a = fq + f\delta$, $\delta d' = fq - f\delta$, $\delta a^2 + \delta d'^2 = 2fq^2 + 2f\delta^2 = 2fd \cdot FH + 2f\delta^2 = 2H\delta \cdot \delta h$. Hence if from any point F in the diam. of a circle produced, a perpendicular be raised, and thereon $f'd$ be taken whose square is equal to the rectangle DFH , and fa be put equal to $f'd$ and from any point in da as δ a line be drawn to cut the circle in any two points as h and H , then twice the rectangle $h\delta H$ is equal to the sum of the squares of δd and δa . Which is the twelfth Proposition.

If δ be drawn through q cutting the circle in r and s , then the rectangle $= r\delta s = h\delta H =$ (by the last) δq^2 . Which is the thirteenth Proposition.

PROP. XIII. If in AB the diameter of a circle two points c and D be so taken that, c being without, and D either within or without the circle, the square of CD be equal to the rectangle ACB , and from c a perpendicular to AB erected, and any line drawn through D to cut the same in E and the circle in F ; then I say the square of CD will be equal to the rectangle EGF .

The converse is also true which is this.

If dc be perpendicular to AB the diameter of a circle and meets it without the circle in c , and if from c a line be drawn to cut the circle in E and F , and the diameter either within or without in D , and the square of CD be equal to the rectangle EGF ; then I say the square of CD will be equal to the rectangle ACB .

PROP. XIV. Things remaining as in the last proposition, if the perpendiculars eg and fh be demitted; then I say that the rectangle gch is equal to the square of CD .

PROP. XV. If from c any point in the diameter of a circle AB produced a tangent be drawn, and from the point of contact D a perpendicular to the diameter DE be demitted; then I say that $AC : CB :: AE : EB$.

Or conversely thus:

If in AB the diameter of a circle be taken two points c and E such that $AC : CB :: AE : EB$, and from E a perpendicular ED raised, and CD drawn; then I say CD touches the circle in D .

Or thus:

If in AB the diameter of a circle produced a point c be taken, and

Let fall the perpendicular sw , rw , then $rw : rq :: sr : dq ::$ (by the last) $dq : ds :: rq : rw$; therefore the rectangle $rw = rq^2$. Which is the fourteenth Proposition.

Through the points h and t describe a circle cutting hr , tr produced in x and y , then the angle $DTF = DHT = xy$; consequently xy is parallel to DT . If therefore xh be any chord in a circle, and yt another, cutting the former in r , xy being joined, from t draw TD parallel to xy to meet xh in D , then the rectangle $HTD = FT^2$. Which is the sixteenth Proposition.

Join TH , $T'H$ and from T' set off $T'B$ so that $TT'^2 =$ the rectangle $HT'B$, and let tx be any line cutting $T'A$ in A , and the circle again in E' and from E' let a parallel to TT' be drawn to meet $T'H$ in γ , then (by the last) $AE'^2 = A\gamma \cdot AH$ and the $\Delta s A\gamma E'$, AHE' and consequently $\Delta T'T$ similar, as also $E'Hy$. $T'HE'$, therefore as $AE'^2 = A\gamma \cdot AH : AT'^2 :: E'H^2 : TT'^2$; but $\gamma H : E'H :: E'H : T'H$, $\therefore E'H^2 = \gamma H \cdot TH : TT'^2 = T'B \cdot TH :: \gamma H : T'B$; consequently $HA \cdot A\gamma : AT'^2 :: \gamma H : T'B$. Which is the seventeenth Proposition.

Let hh cut TT' in i , then $GH : TH :: TH : DH$ and $GH : iH :: hH : DH$; therefore $TH^2 = iH \cdot hH$. Which is the eighteenth Proposition.

Or describing the semicircle that passes through F , T , s ,

therefrom a tangent as CD be drawn, and in the diameter a point e be taken such that $AC : CB :: AE : EB$; then I say ED being drawn will be perpendicular to the diameter AB .

PROP. XVI. Let AB be any chord in a circle and CD another cutting the former in E , CB being joined, from D draw DF parallel to CB to meet AB in F ; I say that the rectangle AEF is equal to the square of DE .

PROP. XVII. If ABC be a triangle inscribed in a circle whose sides CA and CB are equal, and the rectangle CBD equal to the square of AB , and let AE be any line cutting CB in F and the circle again in E , and from E let a parallel to AB be drawn to meet CB in G ; then I say that the rectangle $CFG : BF^2 :: CG : BD$.

PROP. XVIII. Let ABC be a triangle inscribed in a circle, whose sides AB and AC are equal, and from A any line be drawn meeting the circle again in D and BC in E ; I say that the rectangle DAE is equal to the square of AB .

PROP. XIX. Things remaining as in the last proposition, if lines touching the circle in A and C be drawn to meet in F , and FD be drawn cutting BC in G ; I say that the rectangle BCE is equal to the square of CE .

PROP. XX. Let ABC be a triangle inscribed in a circle whose sides AB and AC are equal, and let AD be parallel to BC , and taking any point therein D let the rectangle under AD and P be equal to the square of AP or AC , and from the points A and D let the lines AE , DE be inflected to

cutting FE in t , we have $FN : FT :: FT : FG$, and $FG : FG :: FT : FN$; $\therefore FT^2 = FG \cdot FN$. Which is Proposition 18.

Join d t cutting TT' in z , dT touching the circle in T being first drawn, then $rd = dT$ the angle $dFT = FTT'$, the triangles FdT , FTT' equiangular, and $rd : FT :: FT : TT'$; therefore $rd \cdot TT' = FG \cdot FN = FG \cdot Ft$ and $rd : Ft :: FG : TT' :: zg : gt$, $zg \cdot TT' = FG \cdot gt$ (because r , t , T' , F are in a circle) $Tg \cdot GT' = Tg \cdot (TT' - Tg)$, consequently $zg \cdot TT' + Tg^2 = Tg \cdot TT'$ and $Tg^2 = TT' \cdot Tz$. Which is the nineteenth Proposition.

If the line FO be taken of any length, so that the rectangle under FO and a given line P may be equal to FT^2 and ot cut TT' in z' , then $FT^2 = FO \cdot P =$ (by the 18th) $FG \cdot Ft$, hence $FO : Ft :: FG : P :: z'g : gt$; therefore $P \cdot z'g = FG \cdot gt = Tg \cdot GT'$. Which is the twentieth Proposition.

Let EL perpendicular to NE meet GT produced in λ , then because the $\angle s$ LEN , LGK are right, λ , E , N , G are in a circle whose diameter is λN , therefore the angle $LEN = LGK = GEE' = \angle GEE' = \frac{1}{2}hNE$, therefore λN bisects the angle hNE , and because $nh = nE$, therefore $ht = tE$ is perpendicular to λL , therefore the same circle passes through h , and $h\lambda = EL$ is perpendicular to hn , and $ag \cdot g\lambda = hg \cdot gE = Tg \cdot GT'$, add ag^2 and $ag \cdot g\lambda = Tg \cdot GT' + ag^2 = Tg^2$. Which is Proposition 21st. both cases.

any point E in the circle, meeting BC in F and G ; I say the rectangle under FO and $P =$ the rectangle EFC .

PROP. XXI. If in AB the diameter of a circle be taken two points C and D such that $AC : CB :: AD : DB$, and D be within the circle, and DE be perpendicular to AB meeting the circle in E and F , and if through C any line be drawn meeting the circle in G and H , and the line DE in K , and GL touch the circle in G , and meet DE in L ; then I say the rectangle LDK is equal to the square of DE .

PROP. XXII. If in AB the diameter of a circle be taken two points C and D such that $AC : CB :: AD : DB$, and D be without the circle, and DE be perpendicular to AB , and through C be drawn any line meeting the circle in G and H , and the line DE in K , and GL touch the circle in G , and meet DE in L ; then I say the rectangle LDK is equal to the rectangle ADB .

PROP. XXIII. If AB be the diameter of a circle and CD perpendicular thereto meeting it in C , and from the points A and B be inflected AE , BE to any point E in the circumference, meeting CD in F and G ; I say the rectangle GCF is equal to the rectangle ACB .

PROP. XXIV. In AB the diameter of a circle let two points C and D be taken such that $AC : CB :: AD : DB$, and the point D be within the circle, and DE be perpendicular to AB , meeting the circumference in E and F , and let through C any line be drawn meeting the same in G and H , and from the points G and H let GN , HN be inflected to any point in

DIAGRAM III.

Through h' draw ek cutting the perpendicular through r in k , and produce el till it cuts it in l , then the triangles $eh'e$, $fh'k$, $gh'g$ are similar as also ghg , and the points c , h , λ , e are in a circle, the triangle λge , and consequently lfe is similar to hgg , and consequently to $fh'k$, therefore $ek : fh' :: fe : fl$ and $ek . fl = fh' . fe = fd . fh$. Which is the twenty-second Proposition.

Produce hd to γ , then by similar triangles $fd : fy :: fd : fh$. Which is the twenty-third Proposition.

From h and e to any point N in the circle, let lines be inflected cutting cl in k and L , then because the angle $enh = ee'h = lch$ the points h , L , N , G are in a circle, consequently $lk . kg = kh . kn = tk . kt = tg^2 - gk^2$; therefore $tg^2 = gk^2 + lk . gk = LG . Gk$. Which is the 24th Proposition.

Since (by the first) $fn : fh :: fd : fg :: fn : hn :: fd : dg :: fn - fd : hn - dg$ or $fn : hn :: hn : gn$. Which is the twenty-fifth Proposition.

Produce kt , et' till they cut the perpendicular in Δ and Γ ; then the $\angle t\Gamma\Delta = \Gamma t'\Gamma$ and the angle $\Delta t\Gamma$ made with the tangent is equal to the angle $t\Gamma'e$ in the segment = $\Delta\Gamma e$, therefore the triangles $\Delta t\Gamma$, $t\Gamma't'$ are similar, therefore $t\Gamma : t\Gamma' = t\Gamma : t\Gamma$, and $t\Gamma . t\Gamma' = fd . fh$. Which is the twenty-sixth Proposition.

the same N , and let them meet de in M and L ; I say the rectangle LDM is equal to the square of de .

PROP. XXV. Let ab be the diameter of a circle and cd perpendicular to the same meeting the circumference in c and d , and let e be the centre, and from c and d let cf , df be inflected to any point F in the circumference meeting the diameter ab in G and H ; I say the rectangle GEH is equal to the square of the radius ae .

PROP. XXVI. In ab the diameter of a circle let two points c and d be taken such that $ac : cb :: ad : db$, and let d be without the circle, and de perpendicular to bd , through the point c let any line be drawn meeting the circumference in F and G , and from the points r and e let fh and gh be inflected to any point H in the circumference meeting de in k and L ; I say the rectangle KDL is equal to the rectangle ADB .

PROP. XXVII. In ab the diameter of a circle let be taken the point c , and cd be perpendicular to ab , meeting the circumference in d and N , in cd let be taken two points E and F on the same side of c with d such that the rectangle ECF may be equal to the square of cd , and from the points E and F let eg , fg be inflected to any point G in the circumference, meeting the same in H and k , and let hk when drawn meet the diameter ab in L ; then I say that $al : lb :: ac : cb$.

PROP. XXVIII. In ab the diameter of a circle produced let be

The 28th is the converse of this, viz. if $FG \cdot FD = FB \cdot FH$ then $DC : CH :: FD : FH$. And the 27th of the 24th.

If $FA = FT$ and AE, ah be drawn cutting the circle in p and o , then since $fh : fa :: fa : fe$, the triangles fha, fae are equiangular, therefore the $\angle fha = fae$, also $ao : ap :: ah : ah$: therefore the triangles aop, ahe are equiangular, $\angle aop = ahe$; $\therefore ope = fha = fae$; $\therefore op$ is parallel to af . Which is Proposition 29th.

The 30th is very evident from the printed figure, for since $AE \cdot AF = AB^2 = AC \cdot AD$, the points C, D, E, F are in a circle, therefore the external angle $ACE = EFD$ and $= DHG$; $\therefore EFD$ being $= DHG$, GH is parallel to AB . Take $fl' = fl$, then $fl' \cdot FK = fl^2 = FD \cdot FH = Fq \cdot FN$, two lines FN', KN' being inflected to any point N' in the circle cutting it in q and T' , draw $l'q$ cutting the circle again in T , then the points l', K, q, N are in a circle because $fl' \cdot FK = Fq \cdot FN'$, therefore the angle $fl'q = FN'K = qTT'$ consequently TT' is parallel to FK . Hence if ET touch a circle in T , and any line $l'f$ be drawn, and therein be taken two points l' and K on the same side of F such that the rectangle $l'FK$ may be equal to the square of FT , and from F any line be drawn to meet the circle in q and N' , and $l'q, KN'$ be drawn meeting the circle again in T and T' , TT' being drawn will be parallel to $l'f$. Which is Proposition 31st.

(The 32d is evident from the printed figure, because $FA \cdot AG = DA \cdot AC$, F, DG, C are in a circle; therefore $\angle FCA =$

taken the point C , and CD be perpendicular to AB , and therein be taken two points E and F on different sides of C such that the rectangle ECF may be equal to the rectangle ACB , and from the points E and F let EG, FG be inflected to any point G in the circle, meeting the same in H and K , and let HK when drawn meet the diameter AB in L : then I say that $AB : LB :: AC : CB$.

PROP. XXIX. Let AB touch a circle in B , and any line AE be drawn equal to AB , and likewise from A let any line be drawn to cut the circle in C and D , and let EC, ED be drawn meeting the circle again in F and G ; then FG being drawn will be parallel to AE .

PROP. XXX. Let AB touch a circle in B , and therein be taken two points E and F on the same side of A such that the rectangle EAF may be equal to the square of AB , and from A let any line be drawn meeting the circle in C and D , and EC, FD be drawn meeting the circle again in G and H , then GH being drawn will be parallel to AB .

PROP. XXXI. Let AB touch a circle in B , and any line AE be drawn and therein be taken two points E and F on the same side of A such that the rectangle EAF may be equal to the square of AB , and from A any line be drawn to meet the circle in C and D , and EC, FD be drawn meeting the circle again in G and H ; GH being drawn will be parallel to AE .

PROP. XXXII. Through any point A within a circle let a line be

$\text{DCA} = \text{DKH}$, and KH parallel to BG .) If R be taken so that $\text{RK} \cdot \text{KL}' =$ the square of a tangent to the circle from K , then $\text{R}, \text{L}', \text{T}', \text{N}'$ are in a circle; through K draw Kq cutting the circle again in s , then $\text{R}, \text{L}', \text{q}, \text{s}$ are in a circle, and $\angle' \text{lqk} = \angle' \text{RS}$, but $\angle' \text{qk} = \text{sqT} = \text{TT's}$, therefore $\angle' \text{RS} = \text{TT's}$ and $\text{R}, \text{s}, \text{T'}$ are in a right line. Hence if from the extremes of FK two lines be drawn to touch a circle, and in FK let a point L' be taken on the same side of F with K such that the rectangle $\text{KL}'\text{F}$ may be equal to the square of FT the tangent from F , and also in FK another point R on the same side of K with L' , such that the rectangle $\text{L}'\text{KR}$ may be equal to the square of a tangent from R to the circle, and through F any line be drawn meeting the circle in q and N' , and Kq, KN be drawn meeting the circle again in s and T' , then the points $\text{s}, \text{T'}, \text{R}$, are in a right line. *Which is the thirty fifth Proposition.*

DIAGRAM IV.

In the preceding Diagrams it is shewn that $\delta\text{H} \cdot \delta\text{h}$ which is the square of a tangent from δ to the circle is $= \text{Fd}' + \text{Fd} \cdot \text{FH}$ (Prop. 11.) $= \text{Fd}' + \text{Fd} \cdot \text{F}\gamma$ (Prop. 23.) $= \text{Fd} \times \delta\gamma$; also $\gamma\delta \cdot \gamma\text{F} = \gamma\text{h} \cdot \gamma\text{D}$ (because in the preceding Diagrams, $\text{r}, \delta, \text{h}, \text{D}$ are in a circle) $=$ the square of a tangent from γ to the circle; therefore $\delta\gamma \times (\text{Fd} + \text{F}\gamma) = \delta\gamma^2 =$ the sum of their squares. Hence we have a ready way of finding two such points δ, γ in the perpendicular that $\delta\gamma^2$ may equal the sum

drawn meeting it in B and E , and therein two points F and G be taken such on different sides of A that the rectangle FAG may be equal to the rectangle BAE , and through A any line be drawn meeting the circle in C and D , and FC, GD being drawn to meet the circle again in H and K ; then HK being drawn will be parallel to AB .

PROP. XXXIII. Let AB be a line without a circle and from A and B two lines be drawn to touch the circle in C and D , and let the square of AB be equal to the sum of the squares of AC and BD and from A any line be drawn to meet the circle in E and F , and BE, BF be drawn meeting the circle again in G and H ; the points $\text{A}, \text{G}, \text{H}$, will be in a right line.

PROP. XXXIV. Let AB meet a circle in C and D , and A be without and B within the same, and let the rectangle CAD be equal to the square of AB together with the rectangle CBD , and through A any line be drawn meeting the circle in E and F , and BE, BF be drawn meeting the circle again in G and H ; then the points $\text{A}, \text{G}, \text{H}$, are in a right line.

PROP. XXXV. From the extremes of AB let two lines AC, BD be drawn to touch a circle in C and D , and in AB let a point E be taken on the same side of A with B such that the rectangle BAE may be equal to the square of AC , and also in AB another point F on the same side of B with E such that the rectangle EBF may be equal to the square

of the squares of the tangents from these points to the circle. Let δ , γ in Diagram 4, be two such points, and from γ , let a line be drawn meeting the circle in any two points E' and H' , then $\delta H'$, $\delta E'$ being drawn meeting the circle again in h and d' , because $\delta H \cdot \delta h = \delta F' \cdot \delta \gamma = \delta D \cdot \delta E$; therefore F, h, H', γ are in a circle, as also F, d', E', γ ; therefore the $\angle F\gamma H = \delta D'F = \delta hF$, therefore δ, h, d', F are in a circle, and $\angle \delta D'h = \delta Fh = \delta H'\gamma$, also $Fd'\gamma = F\hat{c}h$; consequently the three angles $\delta D'h, \delta D'F, \gamma D'F$ at the point D' being respectively equal to $\delta Fh, \delta H'\gamma, F\hat{c}h$, those of the triangle $F\hat{c}h$, their sum must be equal to two right angles, and consequently γ, d', h are in a right line. *Which is the thirty-third Proposition.*

Also because $\delta D'h = \delta Fh = \gamma D'E' = \gamma FE'$; therefore Fh, FE' make $\angle$ s with $\delta\gamma$ and FE' is perpendicular to the diameter DH , and parallel to TT' ; draw FD' cutting TT' in G' , and the circle in N , then $FD' \cdot FN = FT'^2 = FG'^2 + G'T'^2 = FG'^2 + GG'^2 + TG'^2 \cdot G'T' = FG'^2 + D'G' \cdot G'N$. Therefore TT' is the locus of all points G' dividing lines intercepted between F and the periphery so that $FD' \cdot FN = G'F^2 + D'G' \cdot G'N$. From E and h through G' draw lines meeting the circle again in s and r , and describe as in the 2d Diagram, the circle $FTNT'$ cutting FN in l . Then because $FG' \cdot G'l = TG' \cdot G'T'$, and this equal $sg' \cdot G'E = hg' \cdot G'R$, therefore the points F, l, s, E are in a circle, and the angle $FLE = FSE$. Also $D'G' \cdot G'N + FG'^2 = (TG' \cdot G'T') G'l \cdot FG' + FG'^2 = FG' \cdot Fl = FD' \cdot FN = Fh \cdot FE$, therefore h, G', l, E are in a circle, consequently the angle $ehR = esR$, and the sum of the angles ehR, FLE is equal to two right ones, therefore the sum of their equals esR, ESF must be equal to two right, and F, s, R in a right line. *Which is the thirty-fourth Proposition.*

of AD , and through A any line be drawn meeting the circle in G and H , and BC, BH be drawn meeting the circle again in K and L ; then the points L, K, F , are in a right line.

PROP. XXXVI. If from A the vertex of a triangle ABC be drawn AD to any point D in the base, and DE be drawn parallel to AC , and DF to AB ; I say the sum of the rectangles BAB, CAF will be equal to the square of AD together with the rectangle BDC .

PROP. XXXVII. Let A and B be two points in the diameter of a circle whose centre is C , and let the rectangle ACB be equal to the square of the semidiameter; bisect AB in D , and raise the perpendicular DM ; from the point A draw AF to any point F in the circumference, and FE perpendicular to DM ; then I say that the square of AF is equal to twice the rectangle contained by AC and FE .

PROP. XXXVIII. If any regular figure be circumscribed about a circle, and from any point within the figure there be drawn perpendiculars to all the sides of the figure; the sum of the perpendiculars

DIAGRAM V.

To any point g' in TT' from the centre n draw $g'n$, perpendicular to which through g' draw ba cutting the two tangents in a and b ; then the angles at T and g' being right, the points g, a, T, n are in a circle, and the angle $nag' = ntg' = nT'g'$; again the angles at g' and T' being right, the points g', T', b, n are in a circle, and the $\angle nbg' = nT'g'$, consequently $nbg' = nag'$, and ng' bisects both ab and the chord of the circle. *Which is the forty-second Proposition.*

Let fall TC perpendicular to $T'n$ then by similar triangles $TG = \frac{1}{2}TT' : T'n :: T'C : TT'$, therefore $T'n \cdot T'C = \frac{1}{2}TT'^2$. *Which is the forty-third Proposition.*

If any diameter AB be drawn to this circle, and $TA, T'B$ be drawn intersecting in O , through which drawing of intersecting AB in F and producing BT, AT' till they intersect in H ; then because BA is a diameter, the angles at T and T' are right, and the points T, O, T', H in a circle; $\therefore$ the $\angle THT' = TOB =$ the complement of TBO ; but because FT is a tangent, the $\angle TBO$ in the segment is $= FTG$, consequently $THT' = TFG = T'FG$, therefore FT' being $= FT$, and the angle TFT' double THT' F must be the centre of the circle $TO'T'H$, consequently the diameter HO passes through F . Also since the $\angle OHT' = OTT' = OBA =$ the complement of $F'A'H$, the angle HFA is right. *Which is the forty-ninth Proposition.*

Moreover $FE'^2 : FT'^2 = FE' \cdot FH :: FE' : FH' ::$ (by Prop. 13) $E'g' : g'h'$, therefore $FE'^2 : FT'^2 :: E'g' : g'h'$. *Which is the fifty-sixth Proposition.*

will be equal to the multiple of the semidiameter of the circle by the number of the sides of the figure.

PROP. XXXIX. Let there be any number of right lines intersecting in a point, and making all the angles about the point equal, and let any circle pass through the same point; I say the circumference thereof will be divided by the intersecting lines into as many equal parts as there are lines.

PROP. XL. If there be two triangles ABC, DEF , which have one angle A in one equal to one angle D in the other, and another angle B in the first equal to the sum of the angles D and E in the second; then shall the sides AC, BC, DE, DF be proportional.

PROP. XLI. The square of the line bisecting the vertical angle of any triangle is a mean proportional between the difference of the squares of each side including that angle, and the square of the adjacent segment of the base made thereby.

PROP. XLII. If from the same point two tangents be drawn to a

DIAGRAM VI.

Having described a circle about the triangle BAC , and produced AD till it cuts in G , draw DF , ED parallel to AB , AC respectively, and then another circle through G , F , C cutting AG in L , and join LF , GC . Then the $\angle AFL = \angle ACG$, $\angle ALF = \angle ACG =$ the supplement of $\angle ABC$, therefore $\angle ABC = \angle DLF$, and $\angle ADF = \angle BAG$ by construction; consequently the triangles DLF , ABG are similar, and $AC : BA :: DF = AE : DL$ and $AG : AC :: AF : AL$; but $DL + AL = AD$, therefore $BA \cdot AE + CA \cdot AF = AG \cdot (DL + AL) = AG \cdot AD = GD \cdot AD + AD^2 = BD \cdot DC + AD^2$. Which is Proposition 36.

DIAGRAM VII.

The construction being as in the proposition, through B draw BF , join FC , make $EH = EF$, and join AH ; then by hypothesis $AC : FC :: FC : BC$, therefore the $\angle FBC = \angle AFC$, $\angle CFB = \angle FAC = \angle HFA$, but the $\angle AHF = \angle BFH = \angle FBC = \angle AFC$, therefore $\angle HAF = \angle FCA$, and the triangles HFA , FCA similar, consequently $MF = 2EF : AF :: AF : AC$. Which is the thirty-seventh Proposition.

Moreover, the two triangles AFC , AFB have the angle A common, and the angle $AFC = \angle FBC = A + \angle AFB$, and taking $FI = BF$, the angle $AIF = \angle FBC = \angle AFC$, therefore $AC : FC :: AF : FI = FB :: (drawing GK parallel to FB) AG : GK$. Which is Proposition 40.

As to the two intermediate Propositions, viz the 38th and 39th; since the double area of any regular polygon is equal the continual product of the radius of the inscribed circle, the number of sides and the length of one side; and if it be

circle, and a line be drawn joining the points of contact, and another line to be intercepted between the tangents cut the foregoing which joins the point of contact, so as to be bisected in the point of intersection; then I say that the part of that line which is a chord of the circle will also be bisected by the same point.

And, conversely, if the chord cutting the line joining the points of contact be bisected by the point of intersection; then the continuation of the same to meet the tangents will also be bisected by the same point.

PROP. XLIII. If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side; then I say that the rectangle contained under that side and it's segment intercepted by the perpendicular and the base is equal to half the square of the base.

PROP. XLIV. If in a line AB two points C and D be taken; then I say that $AB + AD \times BC + DC^2 = 2AC \times BC + BCD$.

divided into triangles equinumerous with the sides, the sum of their double areas, must be equal to the product of the sum of their perpendiculars and the side of the polygon. Therefore the radius $\times$ number of sides = the sum of the perpendiculars. *Which is Proposition 38.*

And since equal arches of the same circle subtend equal angles as well at the circumference as centre, therefore the 39th Proposition is manifest.

A C D B.

Because $AD + DB = AB$, therefore $AD + BC = CD + AB$, and $AB + AD + BC = 2AB + CD$, consequently $(AB + AD) \cdot CB + CB^2 = 2ABC + BCD$. Also $AB + BD + AD = 2AD + BC$, therefore $(AB + AD) \cdot CD + CD^2 = 2ADC + BCD$. *Which is Proposition 44.*

Again if $AB : AD :: AD : DB$, multiplying by AB , $AB^2 : AB \cdot AD :: AB \cdot AD : AB \cdot DB$. *Which is Proposition 51.*

Also $AB^2 : AD^2 :: AB : DB$, and by composition $AB^2 + AD^2 : AD^2 = AB : DB :: AB + DB : DB$, and multiplying the consequents $AB \cdot DB$ and DB by AD , and dividing them by DB , we have $AB^2 + AD^2 : AB \cdot AD :: AB + DB : AD$. *Which is Proposition 52.* For it is evident that this holds whether AB be equal to $AD + DB$ or not.

DIAGRAM VIII.

The rectangle under the difference of two lines or quantities, and the difference of two other lines or quantities is easily shewn to be equal the sum of the rectangles under the two greater and two less, minus the sum of the rectangles under each of the greater and each of the less. There-

And moreover that

A C D B

$$AB + AD \times CD + CD^2 = 2ADC + BCD.$$

PROP. XLV. If from the vertical angle of any triangle two lines be drawn to make equal angles with the sides containing it, and to cut the base; then I say that the square of one side is to the square of the other side as the rectangle under the segments of the base contiguous to the first side is to the rectangle under the segments contiguous to the other side.

PROP. XLVI. If in AB the diameter of a semicircle any point c be taken, and from thence any line as cd drawn to meet the circumference in d , and a perpendicular de be demitted: then I say that the square of the line ac is equal to the square of the line cd together with the rectangle under the sum of the distances of c from a and c from b and the line ae , when c is taken in the diameter ac produced; but equal to the square of cd together with the rectangle under the differ-

fore $(AC^2 - AF^2) \times (CB^2 - FB^2) = AC^2 \cdot CB^2 + AF^2 \cdot FB^2 - AC^2 \cdot FB^2 - CB^2 \cdot AF^2$. But CF bisects the angle ACB , consequently $CF^2 = AC \cdot CB - AF \cdot FB$, $AC \cdot FB = AF \cdot CB$, and $AC^2 \cdot FB^2 + CB^2 \cdot AF^2 = 2AC^2 \cdot FB^2 = 2AC \cdot FB \cdot AF \cdot CB$; therefore the quantity above equal $AC^2 \cdot CB^2 + AF^2 \cdot FB^2 - 2AC \cdot CB \cdot AF \cdot FB$ = the square of $AC \cdot CB - AF \cdot FB$, and consequently equal $CF^2 \times CF^2$: therefore $AC^2 - AF^2 : CF^2 :: CF^2 : CB^2 - FB^2$. Which is Proposition 41.

If upon CF produced be taken E so that $BE = BF$, then the angle $BEF = BFE = AFC$, and the angles at C being equal, the two triangles ACF , BCE are similar, consequently $EB = FB : CE :: AF : CF$. Which is Proposition 58.

Having taken $CFL = CBF$, it will be $CB : CF :: CF : CL$, and the angle CLF being $= CFB$, ALF must be $= AFC$, and the triangles ALF , AFC similar; therefore $AC : AF :: AF : AL$. Which is Proposition 59.

If the circumscribing circle be drawn about the triangle ACB , and HL' be drawn parallel to AB to cut in H and L' , and CH , CL be joined cutting the base in M and N , and making equal angles ACH , BCL' with the sides; then the rectangle $AMB = CMH$, and $CNL' = ANB$, but $CM : CN :: MH : NL'$; therefore $CM^2 : CN^2 :: CM \cdot MH : CN \cdot NL' :: AMB : ANB$. Which is Proposition 45.

ence of the distances of C from A and C from B and the same line AE , when C is taken in the diameter itself.

PROP. XLVII. If from one angle A of a rectangle $ABCD$ a line be drawn to cut the two opposite sides BC , DC , the former in F , and the latter produced in E ; then I say that the rectangle EAF is equal to the sum of the rectangles EDC , CBF .

PROP. XLVIII. If a rectangle be inscribed in a right-angled triangle, so that one of its angles coincide with the angle of the triangle; then I say that the rectangle under the segment of the hypothenuse is equal to the sum of the rectangles under the segments of the sides about the right angle made by this inscription.

PROP. XLIX. If from the same point C two tangents be drawn to a semicircle whose diameter is AB , and if the extremes of the diameter and the points of contact be joined, either cross-ways by two lines intersecting in F , or other-ways by two lines intersecting in H ; then I say that CF or HC produced to meet the diameter AB will be perpendicular to the same.

PROP. L. If in a semicircle whose diameter is AB the chord of 60° equal to the radius be inscribed and from the centre E a perpendicular drawn thereto and produced to meet the circumference in F ; then I say that AF , EF , BF are continual proportionals.

PROP. LI. If a line be cut in extreme and mean proportion; then I say that the square of the whole, the rectangle under the whole and the greater segment, and the rectangle under the whole and the lesser segments are continual proportionals.

DIAGRAM IX.

Since $EC^2 = EC^2 - DE^2$; $\therefore AC^2 = AE^2 + 2AE \cdot EC + EC^2$
 $= AE^2 + AE \cdot EC + AE \cdot BC + AE \cdot EB + EC^2 = AE \cdot CA +$
 $AE \cdot BC + DE^2 + EC^2 = DC^2 + AE \cdot (AC + BC)$. Also AC'^2
 $= AE^2 + 2AE \times EC' + EC'^2 + EC'^2 = AE \cdot AC' + AE \cdot EB -$
 $AE \cdot C'B + EC'^2 = AE \times (AC' - C'B) + DE^2 + EC'^2 = DC'^2 +$
 $AE \times (AC' - C'B)$. Which is Proposition the 46th.

DIAGRAM X.

Having through the given rectangle ABCD drawn AE meeting DC produced in E, and cutting BC in F, let fall BL perpendicular thereto. Then by similar triangles EA : ED :: AB = DC : AL, and EA : AD = BC :: BF : LF; consequently the rectangle EDC + rectangle CBF = EA $\times$ (AL + LF) = EA $\times$ AF. Which is Proposition 47.

Draw FK parallel to DE, then FB : CE :: KF = CD = AB : AL, and FE : CF :: BF : LF; therefore FE = (AL + LF) = FE $\times$ AF = ECD + CFB (DKA). Which is Proposition 48.

Also AF : AB + BF :: AB + BF : AL + BL + BL + LF = AF + 2BL, Which is Proposition 53.

And AF - BF : BF - LF :: BF : LF, because AF : BF :: BF : LF. Proposition the 55th.

PROP. LII. If three lines are continual proportionals: the sum of the squares of the mean and the greater extreme is to the rectangle contained under the same, as the sum of the extremes is to the mean.

PROP. LIII. In every right-angled triangle, as the hypotenuse is to the sum of the sides about the right angle, so is the said sum, to the sum of the hypotenuse and twice the perpendicular from the right angle.

PROP. LIV. If a right line AD be any-ways cut in E, and from thence a perpendicular BE erected equal to a mean proportional between the whole AD and the part AB, and a circle be drawn through the points A, D, E, and from A a perpendicular be erected to meet the circumference in F; then I say that AF, AB, BE, AD, are four continual proportionals.

PROP. LV. In every right-angled triangle, as the difference between the hypotenuse and one side is to the difference between the same side and its adjacent segment, so is the same side to the same segment.

PROP. LVI. If HC be a tangent to a circle meeting the diameter DB produced in H, and from the point of contact C a perpendicular CK to that diameter be drawn, and likewise a line from H cutting the circle in F and G, and the perpendicular CK in I, and F be the nearest point to H; then I say that the square of HF is to the square of the tangent HC as FI to IG.

DIAGRAM XI.

If $CB = AE = EB = CE$, and $CF = FB$, then the angle $F\hat{C}B = F\hat{B}C$, and $\angle FAE = \angle AFE$, and the triangles CFB , AEF are similar, consequently $AF : AE = CB :: CB : BF$. Which is Proposition 50.

If CG be drawn perpendicular to EB and EL parallel to CG cutting AC in L ; then because the angle $FEB = CAB$, AC is parallel to EF , consequently $LE = CO = EO = BO$. Which is Proposition the 57th,

DIAGRAM XII.

Erect DC perpendicular to the centre D of the semicircle ACB ; join AC , CB , which produce till it meets a parallel FE to CD in F , that cuts the semicircle in H and AC in G ; then because $AE = GE$, and $FE = EB$, and $AE \cdot EB = HE^2 = FE \cdot GE$, therefore $GE : HE :: HE : FE$. Which is Proposition the 60th.

Upon HE produced as a diameter describe a semicircle through H and A meeting it in K , then $EK : AE = GE :: AE = GE : HE :: HE : FE$; take $ME = EK$, and parallel thereto $NG = HE$, and $OE = NG$; then $NG : ME :: FE : FG$, and $MO : FG :: NG = HE : FE :: GE = NO : HE = NG$, or $MO : NO :: FG : NG$, therefore the triangles FGN , MON are similar, and the angle $NMO = NFE$, consequently the points E , M , N , F are in a circle. And conversely when these points are in a circle, and $FE : NG :: NG : GE$, take $EO = NG$, and the triangles NOM , NGF are similar; therefore $FG : OM :: NG : NO = GE ::$ by hypothesis $FE : NG$, and $NG = OE : OM$

PROP. LVII. If one side AC of an equilateral triangle ABC be produced to E so that EC may be equal to AC , and from A a perpendicular to AC raised, and from E a line drawn through the vertex B to meet the perpendicular in D ; then I say that BD is equal to the radius of the circle which circumscribes the triangle.

PROP. LVIII. If BD bisect the vertical angle B of a triangle ABC and meet the base in D , and if with either of the other angular points A or C as centre and adjacent segment of the base as radius a circle be described to cut BD again in E ; then I say that BE is to BD as that segment used as a radius is to the other.

PROP. LIX. If BD bisect the vertical angle B of the triangle ABC , and if on BA or BC from B be put a third proportional to the other side and the bisecting line; then I say the rectangle under that side on which it is put and its remainder when the third proportional is taken from it is equal to the square of the adjacent segment of the base made by the bisecting line. i. e. $BCE = CD^2$, or $BAE = AB^2$.

$FE : FG$, and by division $NG : MB :: FE : GE$, or $FE : NG :: GE : ME$, consequently $FE : NG :: NG : GE :: GE : ME$, or ME, GE, NG, FE are four continual proportionals. Which is the 54th Proposition.

Its converse is here also proved. And likewise the following, viz. FE being divided by the 60th Proposition, so that $GE : EH :: EH : FE$, if EL be taken $= AE = GE$, a circle drawn through the points A, H, L will cut FE produced in K so that EK, GE, HE, FE , are four continual proportionals.

ARTICLE III.

On the Theory of Elliptic Transcendents, By JAMES IVORY, A.M. F.R.S. *Instit. Reg. Sc. Paris, et Soc. Reg. Sc. Gottin. Corresp.* and H.K.G.

From the Philosophical Transactions for 1831.

The branch of the integral calculus which treats of elliptic transcendents originated in the researches of FAGNANI, an Italian geometer of eminence. He discovered that two arcs of the periphery of a given ellipse may be determined in many ways, so that their difference shall be equal to an assignable straight line; and he proved that any arc of the lemniscata, like that of a circle, may be multiplied any number of times, or may be subdivided into any number of equal parts, by finite algebraic equations. These are particular results; and it was the discoveries of EULER that enabled geometers to advance to the investigation of the general properties of the elliptic functions. An integral in finite terms deduced by that geometer from an equation between the differentials of two similar transcendent quantities not separately integrable, led immediately to an algebraic equation between the amplitudes of three elliptic functions, of which one is the sum, or the difference of the other two. This sort of integrals, therefore, could now be added or subtracted in a manner analogous to circular arcs, or loga-

PROP. LX. If an isosceles triangle be inscribed in a semicircle and one of the equal sides be produced, and if from any point z in a diameter a perpendicular thereto be drawn to cut the side, the circle, and the side produced in the points e, H and F respectively; then I say that ze, zH and zF are continual proportionals.

rithms; the amplitude of the sum, or of the difference, being expressed algebraically by means of the amplitudes of the quantities added or subtracted. What Fagnani had accomplished with respect to the arcs of the *lemniscata*, which are expressed by a particular elliptic integral, Euler extended to all transcendents of the same class. To multiply a function of this kind, or to subdivide it into equal parts, was reduced to solving an algebraic equation. In general, all the properties of elliptic transcendents, in which the modulus remains unchanged, are deducible from the discoveries of Euler. Landen enlarged our knowledge of this kind of functions, and made a useful addition to analysis, by showing that the arcs of the hyperbola may be reduced, by a proper transformation to those of the ellipse. Every part of analysis is indebted to Lagrange, who enriched this particular branch with a general method for changing an elliptic function into another having a different modulus, a process which greatly facilitates the numerical calculation of this class of integrals. An elliptic function lies between an arc of the circle on one hand, and a logarithm on the other, approaching indefinitely to the first when the modulus is diminished to zero, and to the second when the modulus is augmented to unit, its other limit. By repeatedly applying the transformation of Lagrange, we may compute either a scale of decreasing moduli reducing the integral to a circular arc, or a scale of increasing moduli bringing it continually nearer to a logarithm. The approximation is very elegant and simple, and attains the end proposed with great rapidity.

The discoveries that have been mentioned occurred in the general cultivation of analysis; but Legendre has bestowed much of his attention and study upon this particular branch of the integral calculus. He distributed the elliptic functions in distinct classes, and reduced them to a regular theory. In a *Mémoire sur les Transcendentes Elliptiques*, published in 1793, and in his *Exercices de Calcul Intégral*, which appeared in 1817, he has developed many of their properties entirely new; investigated the easiest methods of approximating to their values; computed numerical tables to facilitate their application; and exemplified their use in some interesting problems of geometry and mechanics. In a publication so late as 1825, the author, returning to the same subject, has rendered his theory still more perfect; and made many additions to it which further researches had suggested. In particular we find a new method of making an elliptic function approach as near as we please to a cir-

cular arc, or to a logarithm, by a scale of reduction very different from that of which LAGRANGE is the author, the only one before known. This step in advance would unavoidably have conducted to a more extensive theory of this kind of integrals, which nearly about the same time, was being discovered by the researches of other geometers.

M. ABEL of Christiana, and M. JACOBI of Königsberg, entirely changed the aspect of this branch of analysis by the extent and importance of their discoveries. The first of these geometers, whom, to the great loss of science, a premature death cut off in the beginning of a career of the highest expectations, happily conceived the idea of expressing the amplitude of an elliptic function in terms of the function itself. By this procedure the sines and cosines of the amplitudes become periodical quantities like the sines and cosines of circular arcs; and analogy immediately points out many new and useful properties which it would be difficult to deduce by any other mode of investigation. This new way of considering the subject struck out by M. ABEL, not only disclosed to him some interesting and original views, but it conducted him to the general and recondite theorems, which without his knowledge, had been previously discovered by the geometer of Königsberg. M. JACOBI, following in his researches a different method from M. ABEL, proved that an elliptic function may be transformed innumerable ways into another similar function to which it bears constantly the same proportion. In the solution of this problem the modulus and the amplitude sought are deduced from the like given quantities, by equations which depend upon the division into an odd number of equal parts of the definite integral, having its amplitude equal to 90° ; and, as any odd number may be chosen at pleasure, the number of transformations is unlimited. In consequence of this discovery, an elliptic function can have its modulus augmented or diminished according to an infinite number of different scales. The new process for effecting the same reduction discovered by LEGENDRE in 1825, is only the most simple case of the extensive theorem of M. JACOBI; and, although the older transformation of LAGRANGE is no part of the same theorem, it bears to it a close resemblance in every respect. Such is the principal addition made to this branch of analysis by M. JACOBI; but the new methods of investigation introduced by him and M. ABEL, open a wide field of collateral research, which probably will long continue to furnish matter for exercising the ingenuity of mathematicians.

But it seldom happens that an inventor arrives by the shortest road at the results which he has created, or explains them in the simplest manner. The demonstrations of M. JACOBI require long and complicated calculations; and it can hardly be said that the train of deductions leads naturally to truths which are proved, or presents all the conclusions which the theory embraces in a connected point of view. The theorem does not comprehend the transformation of LAGRANGE, which must be separately demonstrated. This is an imperfection of no great moment; but it is always satisfactory to contemplate a theory in its full extent, and to deduce all the connected truths from the same principles. On a careful examination it will be found that the sines or cosines of the amplitudes used in the transformations are analogous to the sines or cosines of two circular arcs; one of which is a multiple of the other; insomuch that the former quantities are changed into the latter when the modulus is supposed to vanish in the algebraic expressions. We may therefore transfer to the elliptic transcendent the same methods of investigation that succeed in the circle. When this procedure is followed, there is no need to distinguish between an odd and an even number; the demonstrations are shortened; and the difficulties are mostly removed by the close analogy between the two cases. It is in this point of view that the subject is treated in this paper, in which it is proposed to demonstrate the principal theorems without going into the detail of the applications.

1. Elliptic functions of the first kind are of this form,* viz.

$$\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}},$$

$$\int_0^\psi \frac{d\psi}{\sqrt{1-h^2 \sin^2 \psi}};$$

the arcs φ and ψ being the amplitudes, and the quantities k and h , which are always less than unit, the moduli of the functions. For the sake of abridging, I shall denote the foregoing integrals by $\kappa(\varphi)$ and $\mathfrak{h}(\psi)$, the prefixes κ and $\mathfrak{h}$ having reference to the moduli k and h ; and, for the definite

integrals between the amplitudes 0 and $\frac{\pi}{2}$, I shall use indis-

* In what follows, the terms 'elliptic functions' and 'elliptic transcendents' are to be understood as applying to the first kind only, which alone are treated of.

criminally either $\kappa \left(\frac{\pi}{2} \right)$ and $h \left(\frac{\pi}{2} \right)$, or, more simply, κ and h .

The general equation to be investigated is the following,

$$\int \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \beta \int \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}, \quad (1)$$

β being a constant quantity equal to the first ratio of the nascent arcs ψ and ϕ . If we admit that this is a possible equation, and suppose that when ψ is successively equal to the arcs of the series,

$$0, \frac{\pi}{2}, 2\frac{\pi}{2}, 3\frac{\pi}{2}, \&c.,$$

ϕ attains the respective values,

$$0, \lambda_1, \lambda_2, \lambda_3, \&c.;$$

we shall have,

$$h = \beta \kappa(\lambda_1), 2h = \beta \kappa(\lambda_2), 3h = \beta \kappa(\lambda_3), \&c.;$$

and consequently,

$$\kappa(\lambda_2) = 2\kappa(\lambda_1), \kappa(\lambda_3) = 3\kappa(\lambda_1), \&c.$$

Thus the arcs $\lambda_2, \lambda_3, \&c.$ are the amplitudes of the multiples of the function $(\kappa\lambda_1)$, which itself remains indeterminate.

We may therefore suppose $p \times \kappa(\lambda_1) = \kappa \left(\frac{\pi}{2} \right)$, p representing any integer number; and, in consequence, we shall have

$$\kappa(\lambda_1) = \frac{1}{p} \kappa, \kappa(\lambda_2) = \frac{2}{p} \kappa, \dots \kappa(\lambda_m) = \frac{m}{p} \kappa.$$

Any proposed number being assumed for p , we may determine the amplitudes $\lambda_1, \lambda_2, \lambda_3, \&c.$ by the theory for the multiplication and subdivision of elliptic functions: but as the equations to be solved are complicated and impracticable, the arcs $\lambda_1, \lambda_2, \&c.$ may be treated as known quantities without any attempt to compute them.

An elliptic function becomes equal to the arc of its amplitude, when the modulus vanishes: and in this case the arcs $\lambda_1, \lambda_2, \lambda_3, \&c.$ are obtained by the subdivision of the quadrant of the circle, and are respectively equal to

$$\frac{1}{p} \cdot \frac{\pi}{2}, \frac{2}{p} \cdot \frac{\pi}{2}, \frac{3}{p} \cdot \frac{\pi}{2}, \&c.$$

Having made these observations, we shall for the present dismiss all consideration of the equation to be demonstrated,

and turn our attention to investigate two variable arcs ψ and ϕ , such that the first shall have the successive values,

$$0, \frac{\pi}{2}, 2 \times \frac{\pi}{2}, 3 \times \frac{\pi}{2}, \&c.$$

when the second becomes respectively equal to the several known amplitudes,

$$0, \lambda_1, \lambda_2, \lambda_3, \&c.$$

2. As we shall have occasion to refer to the formulas for the addition and subtraction of elliptic functions, it will be convenient to premise them.

Let a and b represent any two amplitudes, and put

$$\kappa(a) + \kappa(b) = \kappa(s)$$

$$\kappa(a) - \kappa(b) = \kappa(\sigma);$$

then, according to the formulas of EULER*,

$$\sin s = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} + \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\sin \sigma = \frac{\sin a \cos b \sqrt{1 - k^2 \sin^2 b} - \cos a \sin b \sqrt{1 - k^2 \sin^2 a}}{1 - k^2 \sin^2 a \sin^2 b}.$$

From these we immediately deduce,

$$\sin s \sin \sigma = \frac{\sin^2 a - \sin^2 b}{1 - k^2 \sin^2 a \sin^2 b} \dagger.$$

It may be observed that if $a = \lambda_m$, $b = \lambda_n$; then $s = \lambda_{m+n}$, $\sigma = \lambda_{m-n}$: for it is obvious that

$$\kappa(\lambda_m) + \kappa(\lambda_n) = (m+n) \kappa(\lambda_1) = \kappa(\lambda_{m+n})$$

$$\kappa(\lambda_m) - \kappa(\lambda_n) = (m-n) \kappa(\lambda_1) = \kappa(\lambda_{m-n}).$$

3. In order to avail ourselves of the analogy between the elliptic functions and the arcs of a circle, we must take that view of the matter first suggested by M. ABEL. Let

$$u = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \kappa(\phi);$$

then, as u is a variable quantity depending upon the amplitude ϕ , reciprocally this latter quantity will depend upon the first; which dependance we shall express in this manner,

$$\phi = \text{amplitude of } u = \Delta u,$$

$$\sin \phi = \sin \Delta u.$$

* *Traité des Fonctions Elliptiques*, tom. i. p. 22.

† This equation is called by M. ABEL "la propriété fondamentale."

For the sake of abridging, let $\omega = \frac{1}{p} \kappa$; so that $\lambda_1, \lambda_2, \lambda_3, \&c.$ will be the respective amplitudes of $\omega, 2\omega, 3\omega, \&c.$ and $\lambda_p =$ amplitude of $p\omega =$ amp. of $\kappa = 90^\circ$; and $\lambda_{2p} =$ amp. of $2p\omega =$ amp. of $2\kappa = 180^\circ$. From the nature of the integral, it follows that when u receives an addition equal to $2p\omega$ or 2κ the amplitude of u will be increased by 180° .

To the indeterminate quantity u let there be added the several even multiples of ω less than $2p\omega$; and let us consider the sines of the amplitudes of the functions so formed, viz.

$\sin \Delta u, \sin \Delta(u+2\omega), \sin \Delta(u+4\omega), \dots, \sin \Delta(u+2p\omega-2\omega)$: in this series, if we substitute in place of u , the successive quantities $u+2\omega, u+4\omega, u+6\omega, \&c.$, the same sines will constantly recur in periodical order, abstracting from the change of sign when an amplitude becomes greater than 180° , or than a multiple of 180° . Thus, if we put $u+2\omega$ in place of u , the second term of the foregoing series will stand first, and the last term will be $\sin \Delta(u+2p\omega) = -\sin \Delta(u)$. In like manner, if $u+4\omega$ be substituted for u , the third term of the series will stand first, and the two last terms will be $-\sin \Delta u, \sin \Delta(u+2\omega)$; and so on.

Let us now put

$y = \sin \Delta u \times \sin \Delta(3\omega) \times \sin \Delta(5\omega) \dots \times \sin \Delta(2p\omega-\omega)$
or, which is the same thing,

$y = \sin \lambda_1 \times \sin \lambda_3 \times \sin \lambda_5 \dots \times \sin \lambda_{2p-1}$;
and further, let us assume,

$$y = \frac{\sin \Delta u \times \sin \Delta(u+2\omega) \times \sin \Delta(u+4\omega) \dots \times \sin \Delta(u+2p\omega-2\omega)}{y} \dots\dots (B).$$

In this expression, if we substitute for u , the several odd multiples of ω in succession, viz.

$$\omega, 3\omega, 5\omega, 7\omega, \&c.$$

it follows, from what has been said, that the products in the numerator will always be the same, and equal to the denominator, but that their signs will change alternately as the successive quantities are substituted. Thus, when any odd multiple $(2n+1)\omega$ is substituted for u in the expression (B), the value of y is always $+1$ or -1 , according as $(2n+1)\omega$ holds an odd or an even rank in the series of the odd multiples of ω .

On the other hand when u is zero, or equal to $2p\omega$ any even multiple of ω , we shall have $y = 0$, one of the factors of the numerator necessarily vanishing; for in a sequence

of the even multiples of ω , of which the number is p , there must be one equal to $2p\omega$, or to a multiple of $2p\omega$; and therefore when $u = 2p\omega$, one of the factors must be the sine of an amplitude equal to 180° or to a multiple of 180° .

Further, let $\omega - z$ be substituted for u in the expression (B), z being less than ω ; then

$$y = \frac{\sin A(\omega - z) \sin A(3\omega - z) \dots \sin A(2p\omega - \omega - z)}{y}$$

Now, in the numerator, the partial products, of the first and last factors, of the second and last but one, and so on, are as follows:

$\sin A(\omega - z) \sin A(2p\omega - \omega - z) = \sin A(\omega - z) \sin A(\omega + z)$,
 $\sin A(3\omega - z) \sin A(2p\omega - 3\omega - z) = \sin A(3\omega - z) \sin A(3\omega + z)$, &c.,
 to which we must add the single factor $\sin A(p\omega - z)$, when p is an odd number. All the partial products, it will be observed, have the same value whether z be positive or negative; and they are all greatest, when $z = 0$, as will readily appear from what is proved in § 2. Wherefore y has the same value and the same sign, when u is at equal distances from the limits 0 and 2ω ; and it attains its greatest magnitude, equal to 1, when $u = \omega$. And, if we substitute $(2n + 1)\omega - z$ for u , this substitution will not change the foregoing factors, but only their order, and the sign of their product, which sign, while u is contained between the limits $2n\omega$ and $2n\omega + 2\omega$, will be + or —, according as $(2n + 1)\omega$ holds an odd or an even rank in the series of the odd multiples of ω .

We may now conclude, from what has been proved, that y , in the expression (B), represents the sine of an arc ψ , which increases from zero with the elliptic function u , and coincides with the successive terms of the series,

$$0, \frac{\pi}{2}, 2\frac{\pi}{2}, 3\frac{\pi}{2}, \&c. \text{ ad infinitum,}$$

at the same time that u attains the values,

$$0, \omega, 2\omega, 3\omega, \&c. \text{ ad infinitum,}$$

or, when the amplitude of u becomes equal to the several known arcs,

$$0, \lambda_1, \lambda_2, \lambda_3, \&c. \text{ ad infinitum:}$$

and further, that there is but one value of y , or of $\sin \psi$,

between the two consecutive terms $m \times \frac{\pi}{2}$ and $(m + 1) \times \frac{\pi}{2}$,

for any given value of u between the limits $m\omega$ and $(m+1)\omega$, or for any given amplitude between the arcs λ_m and λ_{m+1} :

4. In what has been proved, p may be either an odd or an even number; but we must now distinguish between the two cases, in like manner as it is necessary to do when we investigate the sine of a multiple of a circular arc. Representing the amplitude of z by ϕ , we have $z = \kappa(\phi)$, and $\sin \phi = \sin \Delta u$. When p is odd, there will be an even number of factors after the first in the numerator of the expression of y or $\sin \psi$; and any one of these, as $\sin \Delta(u + 2n\omega)$, will have another, namely, $\sin \Delta(u + 2p\omega - 2n\omega) = \sin \Delta(2n\omega - u)$, answering to it; and the product of this pair of factors, viz. $\sin \Delta(u + 2n\omega) \times \sin \Delta(2n\omega - u)$, will be found by the formula (Δ) of § 2, observing that $\sin a = \sin \Delta(2n\omega) = \sin \lambda_n$, $\sin b = \sin \Delta u = \sin \phi$, $\sin s = \sin \Delta(2n\omega + u)$, $\sin \sigma = \sin \Delta(2n\omega - u)$: thus we have,

$$\sin \Delta(u + 2n\omega) \sin \Delta(u + 2p\omega - 2n\omega) = \frac{\sin^2 \lambda_n - \sin^2 \phi}{1 - k^2 \sin^2 \lambda_n \sin^2 \phi}.$$

Wherefore, by taking in all the factors and writing z for $\sin \phi$, we shall obtain

$$\sin \psi = \frac{z}{\gamma} \cdot \frac{\sin^2 \lambda_2 - z^2}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{\sin^2 \lambda_4 - z^2}{1 - k^2 z^2 \sin^2 \lambda_4} \dots \frac{\sin^2 \lambda_{p-1} - z^2}{1 - k^2 z^2 \sin^2 \lambda_{p-1}}.$$

The expression of γ , viz.

$$\gamma = \sin \lambda_1 \cdot \sin \lambda_3 \cdot \sin \lambda_5 \dots \sin \lambda_{p-1},$$

may be written in this form,

$$\gamma = \sin^2 \lambda_1 \cdot \sin^2 \lambda_3 \cdot \sin^2 \lambda_5 \dots \sin^2 \lambda_{p-1},$$

omitting the factor $\sin \lambda_p = 1$: wherefore, if we assume

$$\beta = \frac{\sin^2 \lambda_2 \cdot \sin^2 \lambda_4 \cdot \sin^2 \lambda_6 \dots \sin^2 \lambda_{p-1}}{\sin^2 \lambda_1 \cdot \sin^2 \lambda_3 \cdot \sin^2 \lambda_5 \dots \sin^2 \lambda_{p-2}},$$

we shall have,

p being an odd number,

$$\sin \psi = \beta z \cdot \frac{1 - z^2}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{1 - z^2}{1 - k^2 z^2 \sin^2 \lambda_4} \dots \frac{1 - z^2}{1 - k^2 z^2 \sin^2 \lambda_{p-1}}. \quad (8)$$

When p is an even number, if we leave out the first factor in the numerator of the expression of y or $\sin \psi$, there will remain an odd number of factors, that which occupies the middle place, being $\sin \Delta(u + p\omega)$: and any factor, as $\sin \Delta(u + 2n\omega)$, between the first and the middle one, will

have another, viz. $\sin A(x + 2p\omega - 2n\omega)$, corresponding to it after the middle one; and the product of this pair of factors will be obtained as before, viz.

$$\sin A(x + 2n\omega) \sin A(x + 2p\omega - 2n\omega) = \frac{\sin^2 \lambda_m - \sin^2 \phi}{1 - k^2 \sin^2 \lambda_m \sin^2 \phi}.$$

With regard to the middle factor, we shall have, in the formulas of § 2, $\sin a = \sin A(p\omega) = \sin 90^\circ$, $\sin b = \sin A\pi = \sin \phi$, $\sin s = \sin A(x + p\omega)$; and

$$\sin A(x + p\omega) = \frac{\cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Wherefore, by proceeding as before, we shall have,

p being an even number,

$$\sin \psi = \frac{\beta x \sqrt{1-x^2}}{\sqrt{1-k^2 x^2}} \cdot \frac{1 - \frac{x^2}{\sin^2 \lambda_2}}{1 - k^2 x^2 \sin^2 \lambda_2} \cdot \frac{1 - \frac{x^2}{\sin^2 \lambda_4}}{1 - k^2 x^2 \sin^2 \lambda_4} \cdots \frac{1 - \frac{x^2}{\sin^2 \lambda_{p-2}}}{1 - k^2 x^2 \sin^2 \lambda_{p-2}} \quad (3)$$

$$\beta = \frac{\sin^2 \lambda_2 \cdot \sin^2 \lambda_4 \cdot \sin^2 \lambda_6 \cdots \sin^2 \lambda_{p-2}}{\sin^2 \lambda_1 \cdot \sin^2 \lambda_3 \cdot \sin^2 \lambda_5 \cdots \sin^2 \lambda_{p-1}}.$$

In both the formulas (2) and (3), it is obvious that β is the quotient of the product of the sines of all the even amplitudes, λ_2, λ_4 , &c. between the limits 0 and 180° , divided by the product of the sines of all the odd amplitudes, λ_1, λ_3 , &c. contained between the same limits. The general expression of β , common to the two cases, is therefore as follows,

$$\beta = \frac{\sin \lambda_2 \cdot \sin \lambda_4 \cdot \sin \lambda_6 \cdots \sin \lambda_{p-2}}{\sin \lambda_1 \cdot \sin \lambda_3 \cdot \sin \lambda_5 \cdots \sin \lambda_{p-1}}. \quad (4)$$

In the formula (3) let r and x stand for the products of the binomials in the numerator and denominator; then,

$$\sin \psi = \frac{\beta x r}{R};$$

and,

$$\cos^2 \psi = \frac{R^2 - \beta^2 x^2 r^2}{R^2}.$$

The numerator of this expression is a rational function of x^2 , and it will vanish whenever $\cos^2 \psi = 0$, or $\sin^2 \psi = 1$. that is, when x^2 is equal to $\sin^2 \lambda_{2n+1}$, $2n + 1$ being any odd number less than $2p$. Suppose that $2n + 1$ is any odd number less than p , the numerator of the value of $\cos^2 \psi$ will be di-

visible by $(1 - \frac{x^2}{\sin^2 \lambda_{2n+1}})$, and also by $(1 - \frac{x^2}{\sin^2 \lambda_{2p-2n-1}})$:

and as these binomials are equal, it will be divisible by their product $\left(1 - \frac{z^2}{\sin^2 \lambda_{2n+1}}\right)^2$; and p being an odd number, to the double divisors there must be added the single one $\left(1 - \frac{z^2}{\sin^2 \lambda_p}\right) = 1 - z^2$. The numerator is therefore divisible by the product,

$$(1 - z^2) \cdot \left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2 \lambda_3}\right)^2 \dots \left(1 - \frac{z^2}{\sin^2 \lambda_{p-2}}\right)^2;$$

and, as the two expressions have the same absolute term and same dimensions, they must be identical. Wherefore we have,

p being an odd number,

$$\cos \psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_3}}{1 - k^2 z^2 \sin^2 \lambda_4} \dots \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-2}}}{1 - k^2 z^2 \sin^2 \lambda_{p-1}}. \quad (5)$$

In like manner, if p and n represent the rational binomial products in the numerator and denominator of the formula (3), we shall have

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \times \frac{p}{R};$$

and

$$\cos^2 \psi = \frac{(1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) p^2}{(1 - k^2 z^2) R^2}.$$

Proceeding as before, it will appear that the numerator of this expression is divisible by the double divisor $\left(1 - \frac{z^2}{\sin^2 \lambda_{2n+1}}\right)^2$, $2n + 1$ being any odd number less than p ; and in this case when p is an even number, all the divisors are double. Wherefore the product

$$\left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2 \lambda_3}\right)^2 \cdot \left(1 - \frac{z^2}{\sin^2 \lambda_5}\right)^2 \dots \left(1 - \frac{z^2}{\sin^2 \lambda_{p-1}}\right)^2$$

will divide the numerator of the value of $\cos^2 \psi$; and it will be identical to it, because both the expressions have the same dimensions. Thus we obtain,

p being an even number,

$$\cos \psi = \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{\sqrt{1 - k^2 z^2}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_3}}{1 - k^2 z^2 \sin^2 \lambda_2} \cdot \dots \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-1}}}{1 - k^2 z^2 \sin^2 \lambda_{p-2}}. \quad (6)$$

From the equations (2) and (3) we deduce

$$\tan \psi = \frac{\beta z}{\sqrt{1-z^2}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_2}}{1 - \frac{z^2}{\sin^2 \lambda_1}} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_4}}{1 - \frac{z^2}{\sin^2 \lambda_3}} \dots \frac{1 - \frac{z^2}{\sin^2 \lambda_{p-1}}}{1 - \frac{z^2}{\sin^2 \lambda_{p-2}}}$$

but it will readily appear that

$$\frac{1 - \frac{\sin^2 \phi}{\sin^2 \lambda_{2n}}}{1 - \frac{\sin^2 \phi}{\sin^2 \lambda_{2n+1}}} = \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{2n}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{2n+1}}}$$

wherefore we obtain,

p being an odd number,

$$\tan \psi = \tan \phi \times \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_2}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_1}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_4}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_3}} \dots \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-1}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-2}}} \quad (7)$$

And in a similar manner we deduce from the equations (3) and (6),

p being an even number,

$$\tan \psi = \frac{\beta \tan \phi}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_1}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_2}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_3}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_4}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_5}} \dots \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-2}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-1}}} \quad (8)$$

The formulas (2), (5), (7), in which p is an odd number, are those used in the theorems of M. JACOBI; the other three, (3), (6), and (8), have been added here. All the formulas will be true in the circle, if we make $k=0$, and derive the arcs λ_1, λ_2 , &c. from the subdivision of the quadrant, in like manner as they have been obtained from the

subdivision of the definite integral $k \left(\frac{\pi}{2} \right)$. The coefficient β .

is the expression of the first ratio of the nascent arcs ψ and ϕ ; and it is equal to p in the circle.

All the formulas are, however, imperfect in one respect: they all suppose that the amplitudes λ_1, λ_2 , &c., derived from the subdivision of the definite integral $k \left(\frac{\pi}{2} \right)$, are known.

By means of these amplitudes, the general solution of the problem has been deduced from a particular case: but the formulas cannot be considered as complete till all the co-

efficients have been expressed in functions of the modulus k ; and, with respect to this point, the researches of analysts have not yet been entirely successful.

5. Having now investigated the relation between the arcs ψ and ϕ , we have next to demonstrate that the equation (1) is true when these amplitudes are substituted in it, and a proper value is assigned to the indeterminate modulus k ; but this requires some preparation, in order to avoid complicated operations.

First, p being an odd number we have

$$\sin \psi = \frac{\beta \cdot x^p}{R}, \quad \cos \psi = \frac{\sqrt{1-x^2} \cdot Q}{R};$$

x, P, Q , representing the rational binomial products in the denominators and numerators of the equations (2) and (5); we therefore obtain,

$$R^2 = \beta^2 x^2 P^2 + (1-x^2) Q^2.$$

This equation has been found on the supposition that x is less than 1; but as it contains no radical quantities, it will

be true for all values of x . We may therefore substitute $\frac{1}{kz}$

for x ; and, in the resulting equation, the symbol x will still represent a quantity unrestricted in its value. Now, the

substitution of $\frac{1}{kz}$ for x being made, we shall obtain,

$$R^2 = \beta^2 k^2 z^2 P^2 = (1-k^2 z^2) R'^2,$$

in which expression R and P denote the same functions of z as before, and the values of the new symbols k and R' are as follows,

$$k = k^p \sin^p \lambda_1 \cdot \sin^p \lambda_2 \cdot \sin^p \lambda_3 \dots \sin^p \lambda_{p-1},$$

$$R' = (1-k^2 z^2 \sin^2 \lambda_1) (1-k^2 z^2 \sin^2 \lambda_2) \dots (1-k^2 z^2 \sin^2 \lambda_{p-1}).$$

We thus have

$$\left. \begin{aligned} \sqrt{R^2 - \beta^2 x^2 P^2} &= \sqrt{1-x^2} \cdot Q \\ \sqrt{R^2 - \beta^2 k^2 z^2 P^2} &= \sqrt{1-k^2 z^2} \cdot R' \end{aligned} \right\} \quad (c)$$

Secondly, when p is an even number, R, P, Q will stand for the rational binomial products in the denominators and numerators of the equations (3) and (6): thus

$$\sin \psi = \frac{\beta z \sqrt{1-x^2}}{\sqrt{1-k^2 z^2}} \cdot \frac{P}{R}, \quad \cos \psi = \frac{1}{\sqrt{1-k^2 z^2}} \cdot \frac{Q}{R};$$

consequently

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And if in this equation we substitute $\frac{1}{kz}$ in place of z we shall obtain this result

$$(1 - k^2 z^2) R^2 = \beta^2 k^2 z^2 (1 - z^2) P^2 + R'^2$$

u and P representing the same functions of z as before, and the new symbols k and R' standing for these values

$$k = k' \cdot \sin^4 \lambda_1 \cdot \sin^4 \lambda_2 \cdot \sin^4 \lambda_3 \dots \sin^4 \lambda_{p-1}$$

$$R' = (1 - k^2 z^2 \sin^2 \lambda_1) (1 - k^2 z^2 \sin^2 \lambda_2) \dots (1 - k^2 z^2 \sin^2 \lambda_{p-1}).$$

From what has been proved we now have

$$\left. \begin{aligned} \sqrt{(1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2} &= Q, \\ \sqrt{(1 - k^2 z^2) R^2 - \beta^2 k^2 z^2 (1 - z^2) P^2} &= R' \end{aligned} \right\} \quad (D)$$

To these formulas we must add the following principle of analysis, on which the demonstration we have in view mainly turns. Let v and u denote rational functions of z : we shall have this identical equation,

$$\frac{d \cdot \left(\frac{v}{u} \right)}{dz} u^2 = \frac{1}{a} \left\{ (v + au) \frac{d \cdot (v - au)}{dz} - (v - au) \frac{d \cdot (v + au)}{dz} \right\};$$

from which it follows, that every double binomial factor either of $v + au$, or of $v - au$, is a simple binomial factor

of $\frac{d \cdot \left(\frac{v}{u} \right)}{dz} u^2$; and further, if $v + au$ and $v - au$ have no

common divisor, that every double binomial factor of $(v + au) \times (v - au) = v^2 - a^2 u^2$, is a simple binomial fac-

tor of $\frac{d \cdot \left(\frac{v}{u} \right)}{dz} \cdot u^2$.

6. The differential of the equation (1) may now be readily demonstrated, supposing that β has the value investigated in § 4, and k , the value assigned to it in § 5. And first when p is an odd number, we obtain from the equation (2);

$$\sin \psi = \frac{\beta \cdot zP}{R}, \quad z \pm \sin \phi:$$

and with these values the equation (1) will become

$$\frac{\frac{R^2}{dz} d \cdot \left(\frac{zP}{R} \right)}{\sqrt{(R^2 - \beta^2 z^2 P^2) (R^2 - \beta^2 k^2 z^2 P^2)}} = \frac{1}{\sqrt{1 - z^2} \cdot 1 - k^2 z^2}.$$

and, on account of the formulas (c),

$$\frac{\frac{R^2}{dz} d \cdot \left(\frac{zP}{R}\right)}{Q \cdot R'} = 1.$$

Now it is evident that $R + \beta \cdot zP$ and $R - \beta \cdot zP$, have no common divisor: for as R contains only the even powers of z , and zP only the odd powers, if $1 + cz$ be a factor of $R + \beta \cdot zP$, $1 - cz$ will necessarily be a factor of $R - \beta \cdot zP$. Wherefore, according to what has been proved above, every double binomial factor of $R^2 - \beta^2 z^2 P^2$, that is, every factor of Q , will be a factor of the function in the numerator of the left side of the last equation. In the very same manner it is proved that every double binomial factor of $R^2 - \beta^2 k^2 z^2 P^2$, that is, every factor of R' , will be a factor of the same function. Wherefore the numerator of the left side of the last equation is divisible by the product $Q \times R'$ in the denominator; and, as both the expressions have the same dimensions and the same absolute term, they are identical; which verifies the equation. Wherefore the equation (1) is demonstrated when p is an odd number.

Secondly, when p is an even number, we have by equation (3),

$$\sin \psi = \frac{\beta z \sqrt{1 - z^2}}{\sqrt{1 - k^2 z^2}} \cdot \frac{P}{R}, \quad \sin \phi = z:$$

and the differential of equation (1) will become by substitution,

$$\begin{aligned} & \frac{(1 - k^2 z^2) R^2}{dz} \cdot d \cdot \left(\frac{z \sqrt{1 - z^2} \cdot P}{\sqrt{1 - k^2 z^2} \cdot R} \right) \\ & \sqrt{((1 - k^2 z^2) R^2 - \beta^2 z^2 (1 - z^2) P^2) ((1 - k^2 z^2) R^2 - \beta^2 k^2 z^2 (1 - z^2) P^2)} \\ & = \frac{1}{\sqrt{1 - z^2} \cdot 1 - k^2 z^2}: \text{and, on account of the formulas (d),} \\ & \frac{(1 - k^2 z^2) R^2}{dz} d \cdot \left(\frac{z \sqrt{1 - z^2} \cdot P}{\sqrt{1 - k^2 z^2} \cdot R} \right) = \frac{1}{\sqrt{1 - z^2} \cdot 1 - k^2 z^2}. \end{aligned}$$

It will be proved by the like reasoning as before, that the numerator of the left side of this equation is divisible by the product in the denominator. Now if we perform the differentiation indicated, we shall find,

$$S = (1 - z^2 + k^2 z^4) PR + z(1 - z^2)(1 - k^2 z^2) R^2 \frac{d \left(\frac{P}{R} \right)}{dz}.$$

$$\frac{(1-k^2z^2)z^2}{dz} d \cdot \left(\frac{z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} \cdot \frac{P}{R} \right) = \frac{s}{\sqrt{1-z^2} \cdot \sqrt{1-k^2z^2}},$$

and it is evident that all the rational factors of the left side of this last formula, and consequently all the factors of $Q \times R'$, will be factors of s . By substitution the differential equation (1) will now become

$$\frac{s}{Q \times R'} = 1,$$

which is manifestly verified: for, as $Q \times R'$ divides s , and the two expressions have the same dimensions and the same absolute term, they are identical. The equation (1) is therefore demonstrated when p is an even number.

7. The transformation expressed by the equation,

$$\int_0^\psi \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \beta \int_0^\phi \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}},$$

has now been demonstrated for any number whether odd or even, the constant β being determined by the formula (4), and the modulus k by the special formulas in § 5, or, generally without distinguishing whether p is odd or even, by this formula,

$$k = k^p \cdot (\sin \lambda_1 \sin \lambda_2 \sin \lambda_3 \dots \sin \lambda_{p-1})^2, \quad (9)$$

the sines multiplied together being those of all the odd amplitudes less than 180° . The relation between the variable amplitudes ψ and ϕ is expressed by the several equations in § 4.

In order to render the solution of the problem more complete, it may be proper to add a useful method of computing the amplitude ψ .

In § 5 we have obtained this equation,

$$R^2 = \beta^2 z^2 P^2 + (1-z^2) Q^2.$$

And, if we represent by N and M the products of the binomials in the numerator and denominator of the equation (7), we shall have

$$\tan \psi = \frac{\beta \tan \phi N}{M}.$$

Let $x = \tan \phi$, then $z^2 = \sin^2 \phi = \frac{x^2}{1+x^2}$; and, observing that

$$1 - \frac{z^2}{\sin^2 \lambda_n} = \frac{1 - \frac{x^2}{\tan^2 \lambda_n}}{(1+x^2)},$$

it will readily appear that

$$\beta z P = \frac{\beta x^2 N}{(1+x^2)^{\frac{p}{2}}}, \quad \sqrt{1-z^2} \cdot Q = \frac{M}{(1+x^2)^{\frac{r}{2}}}.$$

These values being substituted in the foregoing equation, we get

$$M^2 + \beta^2 x^2 N^2 = (1+x^2)^p \cdot R^2.$$

And if R be transformed into a function of x^2 , we shall obtain

$$M^2 + \beta^2 x^2 N^2 = (1+x^2)(1+c_2^2 x^2)(1+c_4^2 x^2) \dots (1+c_{2p-2}^2 x^2),$$

the new symbol c_{2n} being determined by this formula,

$$\overline{c_{2n}}^2 = 1 - k^2 \sin^2 \lambda_{2n}.$$

The last equation may be resolved into these two,

$$M + \beta x N \sqrt{-1} = (1+x\sqrt{-1})(1+c_2 x\sqrt{-1})^2 \dots (1+c_{p-2} x\sqrt{-1})^2,$$

$$M - \beta x N \sqrt{-1} = (1-x\sqrt{-1})(1-c_2 x\sqrt{-1})^2 \dots (1-c_{p-2} x\sqrt{-1})^2,$$

the second of which being divided by the first, there will result,

$$\frac{1 - \tan \psi \sqrt{-1}}{1 + \tan \psi \sqrt{-1}} = \frac{1 - x \sqrt{-1}}{1 + x \sqrt{-1}} \cdot \left(\frac{1 - c_2 x \sqrt{-1}}{1 + c_2 x \sqrt{-1}} \right)^2 \dots \left(\frac{1 - c_{p-2} x \sqrt{-1}}{1 + c_{p-2} x \sqrt{-1}} \right)^2.$$

Now u being an arc of a circle, we have this well known formula,

$$u = \frac{1}{2\sqrt{-1}} \times \log. \left(\frac{1 - \tan u \sqrt{-1}}{1 + \tan u \sqrt{-1}} \right);$$

wherefore, if we take the logarithms of the factors of the foregoing expression, and substitute the equivalent circular arcs, we shall obtain,

p being an odd number,

$$\psi = \phi + 2\phi_2 + 2\phi_4 + 2\phi_6 \dots 2\phi_{p-1}, \quad (10)$$

the arc ϕ_{2n} being determined by the equation,

$$\tan \phi_{2n} = c_{2n} \times \tan \phi.$$

When p is an even number, we have this equation in § 5,

$$(1 - k^2 z^2) R^2 = \beta^2 z^2 (1 - z^2) P^2 + Q^2.$$

And, using N and M to denote the products of the binomials in the numerator and denominator of the equation (8), we have

$$\tan \psi = \frac{\beta \tan \phi N}{M}.$$

By the substitution of $\frac{x}{\sqrt{1+x^2}}$ for z as before, it will be found that,

$$\beta z \sqrt{1-z^2} \cdot P = \frac{\beta x N}{(1+x^2)^{\frac{p}{2}}}, \quad Q = \frac{M}{(1+x^2)^{\frac{p}{2}}}.$$

Wherefore we have,

$$M^2 + \beta^2 x^2 N^2 = (1+x^2)^p (1-k^2 z^2) R^2;$$

and by converting $(1-k^2 z^2)$, R^2 into a function of x^2 , we get

$$M^2 + \beta^2 x^2 N^2 = (1+x^2)(1+k'^2 x^2)(1+c_2^2 x^2)^2 \dots (1+c_{p-2}^2 x^2)^2;$$

$$k'^2 = 1-k^2, \quad c_{2n}^2 = 1-k^2 \sin^2 \lambda_{2n}.$$

By treating this equation as before, we get

$$\frac{1-\tan \psi \sqrt{-1}}{1+\tan \psi \sqrt{-1}} = \frac{1-x \sqrt{-1}}{1+x \sqrt{-1}} \cdot \frac{1-k'x \sqrt{-1}}{1+k'x \sqrt{-1}} \left(\frac{1-c_2 x \sqrt{-1}}{1+c_2 x \sqrt{-1}} \right)^2 \dots;$$

and from this we deduce,

p being an even number,

$$\psi = \phi + \phi' + \phi_2 + \phi_4 \dots + \phi_{p-2} \quad (11)$$

$$\tan \phi' = k' \tan \phi, \quad \tan \phi_{2n} = c_{2n} \tan \phi.$$

8. In what goes before, our attention has been confined to two related functions, which, for the sake of abridging, we have denoted by the prefixes $\mathbf{H}$ and $\mathbf{K}$; but as we shall have occasion, in what follows, to compare several functions differing from one another in their moduli and amplitudes, it will be proper to adopt the usual and more general notation, by means of the characteristic $\mathbf{F}$ prefixed to the modulus and amplitude. According to this notation, the equation (1) will be thus written,

$$\mathbf{F}(h, \psi) = \beta \mathbf{F}(k, \phi); \quad \mathbf{F}(k, \phi) = \frac{1}{\beta} \mathbf{F}(h, \psi).$$

The modulus k being given, we can compute the amplitudes, λ_1, λ_2 , &c., at least by approximation; and the amplitude ϕ being supposed known, the foregoing formulas will determine the modulus h , the multiplier β , and the amplitude ψ ; so that the function $\mathbf{F}(k, \phi)$ will be reduced to the similar function $\mathbf{F}(h, \psi)$, of which the modulus h is less than the given modulus k . And in like manner as the three quantities h, β, ψ were determined from the two k, ϕ , we can deduce, from the two h, ψ , three new quantities, k, β, ϕ , which will satisfy the equations,

$$\mathbf{F}(h, \psi) = \beta, \mathbf{F}(h, \psi); \quad \mathbf{F}(k, \phi) = \frac{1}{\beta \beta}, \times \mathbf{F}(h, \psi);$$

are nevertheless of great practical utility in this theory; and they express succinctly the conditions necessary, in order that the transformations in the equations (1) and (12) take place. A little attention will shew that the formulas (14) and (13) are entirely similar, the quantities β', h', k' occupying the same places in the first, that β, k, h do in the other. From this we learn that the equations (1) and (12) will still be true if we change β, k, h for β', h', k' , respectively. Thus we have,

$$F(h', \psi) = \beta' F(h, \phi),$$

the letters ψ and ϕ , it need hardly be noticed, although used on a former occasion, here express simply the variable amplitudes of the related functions. If therefore we divide

$H' = F\left(h', \frac{\pi}{2}\right)$ into p equal parts, and put $\mu_1, \mu_2, \mu_3, \&c.$,

for the respective amplitudes of $\frac{1}{p} H', \frac{2}{p} H', \frac{3}{p} H', \&c.$; we shall have by the formulas (4) and (9),

$$\left. \begin{aligned} \beta' &= \frac{\sin \mu_2 \sin \mu_4 \dots \sin \mu_{2p-2}}{\sin \mu_1 \sin \mu_3 \dots \sin \mu_{2p-1}} \\ k' &= h'^2 \cdot (\sin \mu_1 \sin \mu_3 \dots \sin \mu_{2p-1})^2 \end{aligned} \right\} \quad (16)$$

The multipliers β and β' being similar functions, the first of the amplitudes $\lambda_1, \lambda_2, \lambda_3, \&c.$, and the other of the amplitudes $\mu_1, \mu_2, \mu_3, \&c.$, the equation $\beta\beta' = p$, expresses a curious property of those functions.

And, in like manner, if we change β, k, h , respectively for β', h', k' in the equation (12); or, which is the same thing, if we derive an equation from (15) in the same manner that (12) was obtained from (1), we shall get

$$\left. \begin{aligned} F(k, \tau) &= \beta' F(h, \sigma), \\ \sin \tau &= \beta' \sin \sigma \times \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \mu_2}}{1 + \frac{\sin^2 \sigma}{\tan^2 \mu_1}} \cdot \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \mu_4}}{1 + \frac{\sin^2 \sigma}{\tan^2 \mu_3}} \dots \frac{1 + \frac{\sin^2 \sigma}{\tan^2 \mu_{2p-2}}}{1 + \frac{\sin^2 \sigma}{\tan^2 \mu_{2p-1}}} \end{aligned} \right\} \quad (17)$$

Although, in the investigation of this §, we have supposed that p is an odd number, yet it is obvious that they will succeed equally when p is an even number, the formula (8) being used instead of (7).

The analysis by which the equation (12), of which those that follow are consequences, has been deduced from the

equation (1), is precisely that by which the expression of a circular arc is made to pass into a logarithm; so that the whole of this analytical theory rests on one principle, namely, the analogy which an elliptic function bears to a circular arc and to a logarithm, which are its extreme limits.

10. Of the transformations in the last §, the principal one is contained in the formulas (17), which constitute what is called the second theorem of M. JACOBI. One of its chief uses is to supply the defect of the first theorem by furnishing a direct process for reducing an elliptic function to a logarithm, through a scale of increasing moduli. In the function $F(h, \sigma)$, the modulus h being given, we know $h' (= \sqrt{1-h^2})$ named the complement of h for the sake of abridging; we shall therefore obtain the amplitudes μ_1, μ_2 ,

&c., by the subdivision of the function $H' = F(h', \frac{\pi}{2})$; we

next compute the quantities β' and k' by the formulas (16); and, the amplitude σ being given, we deduce from the formulas (17), the amplitude τ , which will satisfy the equation,

$$F(h, \sigma) = \frac{1}{\beta'} \cdot F(k, \tau),$$

the modulus k of the new function being greater than h , because the complement k' is less than the complement h' . Taking now k' the complement of k , we deduce from it, by means of the formulas (16) and (17), the three quantities β', k', τ , in like manner as β', k', τ were deduced from h' ; and we shall have these equations,

$$F(k, \tau) = \beta' F(k', \tau); \quad F(h, \sigma) = \frac{1}{\beta' \beta'} \times F(k, \tau);$$

the modulus k , being greater than h , because the complement k' is less than h' . The like operations being continued, we shall at length arrive at a modulus k_n , as near the limit 1 as may be required.

Another use of the second theorem, when combined with the first, is to find any multiple of an elliptic function. or any aliquot part of it. By the first theorem, we have

$$F(h, \psi) = \beta F(k, \phi);$$

and by the second, making $\sigma = \psi$ in the equations (17),

$$F(k, \tau) = \beta' F(k, \psi);$$

and, by combining the two equations, observing that $\beta\beta' = \rho$, we get

$$F(k, \tau) = \rho F(k, \phi).$$

If p be an odd number, the amplitudes are obtained by the formulas (2) and (17), viz.

$$\sin \psi = \beta \sin \phi \times \frac{1 - \frac{\sin^2 \phi}{\sin^2 \lambda_2}}{1 - k^2 \sin^2 \phi \sin^2 \lambda_2} \cdots \frac{1 - \frac{\sin^2 \phi}{\sin^2 \lambda_{p-1}}}{1 - k^2 \sin^2 \phi \sin^2 \lambda_{p-1}},$$

$$\frac{1 + \frac{\sin^2 \psi}{\tan^2 \mu_2}}{1 + \frac{\sin^2 \psi}{\tan^2 \mu_1}} \cdots \frac{1 + \frac{\sin^2 \psi}{\tan^2 \mu_{p-1}}}{1 + \frac{\sin^2 \psi}{\tan^2 \mu_{p-2}}}.$$

When a multiple is required, we pass directly, by means of the two equations, from the given amplitude ϕ to τ which is sought. In the case of an aliquot part, the amplitude τ being given, the solution of the second equation, of which p is the dimension, will determine $\sin \psi$; and the amplitude ϕ which is sought, will then be found by solving the first equation, which is also of p dimensions. From the nature of the second equation, it has only one real root, and $p-1$ impossible roots, for every real value of $\sin \tau$; and therefore it follows from the first equation, that the ampli-

tude ϕ of the function $\frac{1}{p} F(k, \tau)$ admits in all of p^2 values, of which only p values are real quantities, and the rest impossible.

If p be an even number, the expression of $\sin \psi$ will contain radical quantities, but instead of it we may take the value of $\tan \psi$ in the formula (8); and the two equations for the amplitudes will be,

$$\tan \psi = \frac{\beta \tan \phi}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_1}} \cdot \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_2}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_2}} \cdots \frac{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-1}}}{1 - \frac{\tan^2 \phi}{\tan^2 \lambda_{p-1}}},$$

$$\sin \tau = \frac{\beta' \sin \psi}{1 + \frac{\sin^2 \psi}{\tan^2 \mu_1}} \cdot \frac{1 + \frac{\sin^2 \psi}{\tan^2 \mu_2}}{1 + \frac{\sin^2 \psi}{\tan^2 \mu_2}} \cdots \frac{1 + \frac{\sin^2 \psi}{\tan^2 \mu_{p-1}}}{1 + \frac{\sin^2 \psi}{\tan^2 \mu_{p-1}}};$$

from which the same general properties may be deduced, as when p is an odd number.

11. We have now demonstrated, as was proposed, the principal and leading points of this theory, for which we are indebted to M. JACOBI. For the subordinate details,

and for many curious and important collateral researches that have been suggested by the new views laid open to this branch of analysis, we must refer to M. JACOBI's own work, to the papers of M. ABEL, and to the writings of LEGENDRE. We shall conclude this paper by applying the formulas that have been investigated to two particular instances, taking for p the most simple values, namely 2 and 3.

Example 1. Supposing $p = 2$.

By the formulas (3) and (6) we have these equations between the amplitudes ψ and ϕ , z being $= \sin \phi$,

$$\sin \psi = \frac{\beta z \sqrt{1-z^2}}{\sqrt{1-k^2 z^2}}; \quad \cos \psi = \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{\sqrt{1-k^2 z^2}};$$

wherefore

$$1 - k^2 z^2 = \beta^2 z^2 (1 - z^2) + \left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2;$$

and

$$\beta^2 = \frac{1}{\sin^4 \lambda_1}; \quad \frac{2}{\sin^2 \lambda_1} - \beta^2 = k^2.$$

We now get

$$k'^2 = 1 - k^2; \quad \beta = 1 + k'; \quad \sin^2 \lambda_1 = \frac{1}{1 + k'}.$$

Also, by the formula (9),

$$h = k^2 \sin^4 \lambda_1 = \frac{k^2}{(1 + k')^2} = \frac{1 - k'}{1 + k'};$$

from which we deduce

$$(1 + h)(1 + k') = 2.$$

The equation (1), viz.

$$F(h, \psi) = \beta F(k, \phi),$$

may now be put in one or other of these two forms,

$$F(k, \phi) = \frac{1 + h}{2} F(h, \psi),$$

$$F(h, \psi) = (1 + k') F(k, \phi):$$

by the first we pass from the greater modulus k to the less h ; and by the second, from the less modulus h to the greater k .

In the first of the two cases we must derive the amplitude ψ from ϕ : and, for this purpose we immediately obtain from the formula (11),

$$\tan(\psi - \phi) = k' \tan \phi.$$

Wherefore, if

$$k, h, k_1, h_2, \&c.$$

represent a series of decreasing moduli, of which the complements are,

$$k', h', h'_1, h'_2, \&c.$$

the successive quantities being derived from one another by these formulas,

$$h = \frac{1-k'}{1+k'}, \quad h_1 = \frac{1-k'}{1+h'}, \quad h_2 = \frac{1-h'_1}{1+h'_1}, \&c.$$

and, if we likewise deduce a series of amplitudes in this manner,

$$\tan(\psi - \phi) = k' \tan \phi$$

$$\tan(\psi_1 - \psi) = h' \tan \chi$$

$$\tan(\psi_2 - \psi_1) = h'_1 \tan \psi_1, \&c.$$

we shall have these successive transformations, by which the value of the given function $F(k, \phi)$ is made to approach indefinitely to the arc of a circle,

$$F(k, \phi) = \frac{1+h}{2} F(h, \psi)$$

$$F(k, \phi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} F(h_1, \psi_1)$$

$$F(k, \phi) = \frac{1+h}{2} \cdot \frac{1+h_1}{2} \cdot \frac{1+h_2}{2} \cdot F(h_2, \psi_2), \&c.$$

In the second case, when we would pass from the less modulus h to the greater k , the amplitude ϕ must be deduced from ψ . For this purpose we have

$$\frac{\sin \psi}{\cos \psi} = \frac{\beta z \sqrt{1-z^2}}{1 - \frac{z^2}{\sin^2 \lambda}} = \frac{\beta \sin \phi \cos \phi}{1 - \beta \sin^2 \phi}.$$

$$\text{but } \beta = \frac{2}{1+h}; \quad \cos \phi \sin \phi = \frac{\sin 2\phi}{2}; \quad \text{and } 1 - 2 \sin^2 \phi = \cos 2\phi;$$

wherefore,

$$\frac{\sin \psi}{\cos \psi} = \frac{\sin 2\phi}{h + \cos 2\phi}; \quad \text{and, } (\psi - \phi) = h \sin \phi.$$

Wherefore if the quantities,

$$h, k, k_1, k_2, \&c.$$

represent a series of increasing moduli derived from one another by these equations,

$$k' = \frac{1-k}{1+k}, \quad k_1' = \frac{1-k}{1+k}, \quad k_2'' = \frac{1-k_1}{1+k_1}, \text{ \&c.};$$

and further, if the amplitudes ψ , ϕ , ϕ_1 , ϕ_2 , &c. be deduced from the formulas,

$$\sin(2\phi - \psi) = k \sin \psi,$$

$$\sin(2\phi_1 - \phi) = k \sin \phi,$$

$$\sin(2\phi_2 - \phi_1) = k_1 \sin \phi_1: \text{ \&c.}$$

we shall have these transformations in which the successive moduli tend to the limit 1,

$$F(k, \psi) = (1 + k') F(k, \phi),$$

$$F(k, \psi) = (1 + k') (1 + k_1') F(k_1, \phi_1),$$

$$F(k, \psi) = (1 + k') (1 + k_1') (1 + k_2') F(k_2, \phi_2), \text{ \&c.}$$

Example 2. Supposing $p = 3$.

By the formulas, (2) and (5) we have these equations between the amplitudes ψ and ϕ ,

$$\sin \psi = \frac{\beta z \left(1 - \frac{z^2}{\sin^2 \lambda_2}\right)}{1 - k^2 z^2 \sin^2 \lambda_2}; \quad \cos \psi = \sqrt{1 - z^2} \cdot \frac{1 - \frac{z^2}{\sin^2 \lambda_1}}{1 - k^2 z^2 \sin^2 \lambda_2};$$

wherefore,

$$(1 - k^2 z^2 \sin^2 \lambda_2)^2 = \beta^2 z^2 \left(1 - \frac{z^2}{\sin^2 \lambda_2}\right)^2 + (1 - z^2) \left(1 - \frac{z^2}{\sin^2 \lambda_1}\right)^2;$$

from which we get

$$2k^2 \sin^2 \lambda_2 = \frac{2}{\sin^2 \lambda_1} + 1 - \beta^2,$$

$$k^4 \sin^4 \lambda_2 = \frac{1}{\sin^4 \lambda_1} + \frac{2}{\sin^2 \lambda_1} - \frac{2\beta^2}{\sin^2 \lambda_2},$$

$$\beta = \frac{\sin^2 \lambda_2}{\sin^2 \lambda_1}.$$

Now observing that $\frac{\beta^2}{\sin^2 \lambda_2} = \frac{\beta}{\sin^2 \lambda_1}$, we obtain by equating the values of $k^4 \sin^4 \lambda_2$,

$$\left(\frac{1}{\sin^2 \lambda_1} + \frac{1 - \beta^2}{2}\right)^2 = \frac{1}{\sin^4 \lambda_1} + \frac{2(1 - \beta)}{\sin^2 \lambda_1};$$

from which we deduce

$$\beta = \frac{2}{\sin \lambda_1} - 1.$$

In order to simplify the formulas I shall put $\frac{1}{\sin \lambda_1} = 1 + s$

then $\beta = 1 + 2\epsilon$, and $\sin^2 \lambda_2 = \beta \sin^2 \lambda_1 = \frac{1 + 2\epsilon}{(1 + \epsilon)^2}$: and, having substituted these values in the first of the foregoing equations, we shall get

$$\epsilon^4 + 2\epsilon^3 - 2k'^2\epsilon - k'^2 = 0.$$

This equation may be resolved by the usual method into the two following quadratic factors,

$$\rho^2 = 4k'^2 k'^2,$$

$$\epsilon^2 + (1 + \sqrt{1 - \rho})\epsilon - \frac{1}{2}\sqrt{\rho^2 + 4k'^2} - \frac{1}{2}\rho = 0,$$

$$\epsilon^2 + (1 - \sqrt{1 - \rho})\epsilon + \frac{1}{2}\sqrt{\rho^2 + 4k'^2} - \frac{1}{2}\rho = 0.$$

The second of these equations has two impossible roots: the first has two real roots, one being negative and foreign to the question, and the other positive, which solves the problem. Thus ϵ has only one value, which may be constructed geometrically, but the algebraic expression of it need not be written down.

We have next to derive the amplitude ψ from ϕ . We readily obtain from the foregoing biquadratic equation,

$$\lambda^2 = \frac{(1 + \epsilon)^2(1 - \epsilon)}{1 + 2\epsilon};$$

and hence

$$1 - k^2 \sin^2 \lambda_2 = c_2^2 = \epsilon^2.$$

And the formula (10) will determine ψ when ϕ is given.

Finally, therefore, we have these determinations,

$$\beta = 1 + 2\epsilon, k = \frac{k^3}{(1 + \epsilon)^2}, \tan \left(\frac{\psi - \phi}{2} \right) = \epsilon \tan \phi;$$

$$F(k, \phi) = \frac{1}{1 + 2\epsilon} \times F(h, \psi),$$

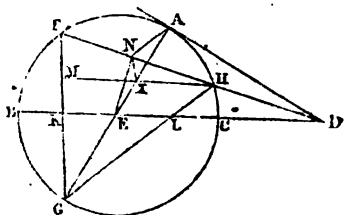
the modulus h being less than k^3 . By repeating the like operations, a succession of moduli rapidly decreasing may be formed, by means of which the given elliptic function will be reduced to a circular arc as near as may be required.

ARTICLE IV.

*Prop. 4, Book 4, of the Mathematical collections of Pappus Alexandrinus, in a more general form: to which are added some Propositions of a similar nature. By MATTHEW STEWART, Professor of Mathematics in the University of Edinburgh.**

Let ABC be a circle, of which the centre is E , diameter BC , and AD a tangent meeting BC in the point D . Draw DF , join A, E , and produce it to G , join FG and GLH . Then $KE = EL$.

Analysis. Make $KE = EL$, and draw HXM parallel to KL : then $MX = XH$. Draw also from the point E , EN perpendicular to FH ; then $FN = NH$: but $MX = XH \therefore NX$ is parallel to FM ; and $\angle HNX = \angle NFM = \angle HAX$; and A, N, X, H , are points in a circle. $\therefore \angle ANH =$



$\angle AXH = \angle AEL$; and A, E, N, D , are likewise points in a circle, for each of the angles EAD, END is a right angle.

Synthesis. Since each of the angles EAD, END is a right angle A, D, E, N , will be points in a circle. $\therefore \angle AND = \angle AED$. But $\angle AED = \angle AXH$, because ED, XH are parallel. $\therefore A, N, X, H$, are points in a circle and $\angle HAX = \angle HNX$: but $\angle HAX = \angle HFM$, therefore FM is parallel to NX , and $FN = NH$; wherefore

$$MX = XH: \text{ and as } XG : GE :: XM : EK \text{ and}$$

$$XG : GE :: XH : LE$$

$$\therefore XM : EK :: HX : LE$$

$$\text{but } XM = XH \therefore KE = EL.$$

On considering this prop. it occurred to me that its very beautiful analysis and synthesis would be equally true, although BC were not a diameter, for if it be bisected in the point E , the demonstration may be framed, with the assistance of the following Lemma, almost in the words of Pappus.

* From "Essays and Observations, Physical and Literary, read before the Philosophical Society in Edinburgh, and published by them."—*Edinburgh 1754—71. vol. 1. p. 141—83.*

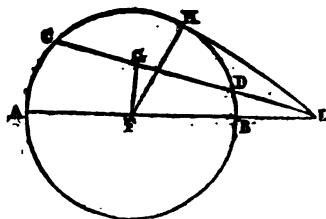
LEMMA.

Let AB , CD be two straight lines inscribed in a circle meeting in a point E , without it; and let them be bisected in the points F , G : let EH be drawn from the point E , a tangent to the circle at H . Then, E , F , G , H will be points in a circle.

Moreover if H be a point in the circumference, and E , F , G , H , points in a circle, EH when joined will be a tangent to the circle at H .

Case 1. When one of the straight lines AB , CD , (suppose AB ,) is the diameter. Join FG , FH .

Since the straight line CD is bisected in G , and F is the centre of the circle because diameter AB is bisected in F ; the angle FGE will be a right angle but $\angle FHE$ is a right angle because EH is a tangent to the circle at H , wherefore $\angle FGE = \angle FHE$. $\therefore E$, F , G , H , are points in a circle.

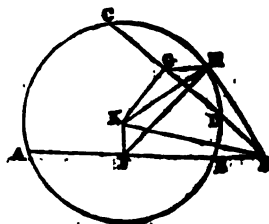


And, if H be a point in the circumference, and E , F , G , H points in a circle, EH when joined, will be a tangent to the circle in H .

Because the diameter AB is bisected in F , F will be the centre of the circle; and because the straight line CD is bisected in G , $\angle FGE$ will be a right angle; but the $\angle FHE = \angle FGE$, since E , F , G , H , are points in a circle; therefore $\angle FHE$ is a right angle, and consequently EH is a tangent to the circle at H .

Case 2. If neither of the straight lines AB , CD be a diameter let K be the centre of the circle, and join KF , KG , KH , FH , GH , and KE .

Because the straight lines AB , CD are bisected in the points F , G ; $\angle KFE$ and $\angle KGE$ are right angles and because EH is a tangent at H the $\angle EHK$ will be a right angle. Now, since KGE , KHE are right angles E , K , G , H are points in a circle; wherefore the angle $EGH = \angle EKH$, and since the angles EFK , EHK are



right angles, E , F , K , H are points in a circle; wherefore $\angle EFH = \angle EKH = \angle EGH$ and E , F , G , H are points in a circle.

And, if H be a point in the circumference, and E, F, G, K points in a circle, EH ; when joined will be a tangent to the circle in H .

Because AB, CD are bisected in F, G , and K is the centre of the circle, EKF, EGK will be right angles; and E, F, G, K points in a circle: $\therefore E, F, K, G, H$, are points in a circle: consequently $EHK = EGK$. But EGK is a right angle $\therefore EHK$ is a right angle, and EH is a tangent to the circle at H .

The Proposition of Pappus may be differently enunciated in the following form.

GENERAL PROPOSITION.

Let ABC be a circle, and BC a straight line meeting it in B, C : let AD be a tangent to the circle at A , meeting BC in D : let BC be bisected in E , let AE be joined and produced to meet the circle in G : through the point D let any straight line be drawn meeting the circle in F, H , and let GF, GH be joined, meeting the straight line BC in K, L . Then EK will be equal to EL .

See First Fig.

Analysis. Make $EK = EL$. Draw HM parallel to KL meeting the straight lines GF, GA in M, X ; bisect FH in N ; join NE, NX, NA , and AH .

Since $EK = EL, MX = XH$; but $FN = NH, \therefore NX, FM$ are parallel. Hence $\angle HNX = \angle NFM = \angle HAX$; and A, N, X, H , are points in a circle: $\therefore \angle ANH = \angle AXH = \angle AEL$, wherefore A, E, N, D are points in a circle, and (by preceding Lemma) AD is a tangent at A .

Synthesis. Since DA is a tangent at A , (by preceding Lemma) A, E, N, D are points in a circle; wherefore $\angle AND = \angle AED = \angle AXH, \therefore A, N, X, H$ are points in a circle; $\therefore \angle HNX = \angle HAX = \angle NFM$, and NX is parallel to FM : since also $FN = NH$, then $MX = XH$, and $EK = EL. Q. E. D.$

Some other propositions of a similar nature have occurred to me, a few of which are here appended.

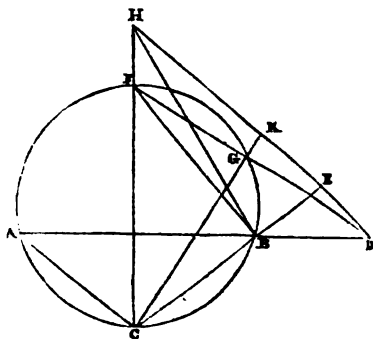
PROP. I.

Let ABC be a circle, and AB a straight line meeting it in A, B , and let C be a point in the circumference. Join CA, CB and

let the straight line DE be parallel to AC meeting the straight lines AB , CB in D , E : let any straight line be drawn through the point D meeting the circle in F , G : let CF , CG be joined meeting DE in H , K : then the rectangle $HEK = \text{rect. } CEB$.

Analysis. Make rect. $HEK = CEB$, and join BH , BF . Since rect. $HEK = CEB$, K , H , C , B are points in a circle, $\therefore \angle BHD = \angle BCG = \angle BFD$; and consequently B , H , F , D are points in a circle: wherefore $\angle ADH = \angle CFB = \angle CAB$, $\therefore DE$ is parallel to AC .

Synthesis. Because the straight line DE is parallel to AC , $\angle ADH = \angle CAB = \angle CFB$: $\therefore B$, H , F , D are points in a circle, $\therefore \angle BHD = \angle BFD = \angle BCG$: therefore K , H , C , B are points in a circle; and rectangle $HEK = \text{rect. } CFB$. Q. E. D.



LEMMA.

Let A , B , C , D ; E , F , G , H , be straight lines, and let $A : B :: C : D$, and $E : F :: G : H$; then rect. $(A, E) : \text{rect. } (B, F) :: \text{rect. } (C, G) : \text{rect. } (D, H)$:

For, rect. $(A, E) : \text{rect. } (B, E) :: A : B$

$:: C : D$

$:: \text{rect. } (C, G) : \text{rect. } (D, G)$

and rect. $(B, E) : \text{rect. } (B, F) :: E : F$

$:: G : H$

$:: (D, G) : (D, H)$

$\therefore$ ex æquo, rect. $(A, E) : (B, F) :: (C, G) : (D, H)$

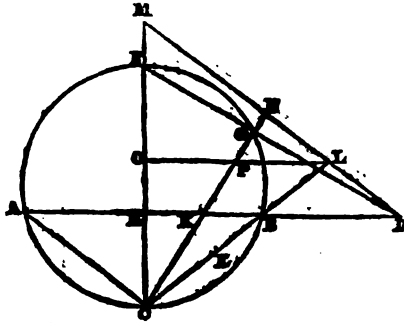
Q. E. D.

PROP. II.

Let ABC be a circle, and AB a straight line meeting it in A , B ; let C be a point in the circumference, let CA , CB be joined; and D a point in the straight line AB ; let $BE : BC :: BD : DA$; through the point D let any straight line be drawn meeting the circle in F , G ; let CF , CG be joined meeting AB in H , K ; then

rect. $HAK : \text{rect. } HBK :: AC^2 : \text{rect. } CBE$.

Fig. 3.



Analysis. Let this be granted. Draw DL parallel to AC , meeting the straight lines CB, CF, CG in L, M, N : through the point L draw a line parallel to AB , meeting the straight lines CF, CG in O, P ,

Because $\text{rect. HAK} : \text{rect. HBK} :: AC^2 : \text{rect. CBE}$,
alternando, $\text{rect. HAK} : AC^2 :: \text{rect. HBK} : \text{rect. CBE}$

But since $OL : LM :: HA : AC$

and $PL : LN :: KA : AC$.

(By Lemma) $\text{rect. OLP} : \text{rect. MLN} :: \text{rect. HAK} : AC^2$
:: $\text{rect. HBK} : \text{rect. CBE}$

and invertendo, $\text{rect. MLN} : \text{rect. OLP} :: \text{rect. CBE} : \text{rect. HBK}$

but $\text{rect. OLP} : CL^2 :: \text{rect. HBK} : BC^2$,

wherefore $\text{rect. MLN} : CL^2 :: \text{rect. CBE} : BC^2$,

:: $BE : BC$

and since $BE : BC :: BD : DA$

:: $BL : LC$

:: $\text{rect. CLB} : CL^2$

we have $\text{rect. MLN} : OL^2 :: \text{rect. CLB} : CL^2$

$\therefore \text{rect. MLN} = \text{rect. CLB}$.

Synthesis. Since $\text{rect. MLN} = \text{rect. CLB}$,

$\text{rect. MLN} : CL^2 :: \text{rect. CLB} : CL^2$

:: $BL : LC$.

and since $BL : LC :: BD : DA$

:: $BE : BC$

:: $\text{rect. CBE} : BC^2$

$\therefore \text{rect. MLN} : CL^2 :: \text{rect. CBE} : BC^2$

but $CL^2 : \text{rect. OLP} :: BC^2 : \text{rect. HBK}$

$\therefore \text{rect. MLN} : \text{rect. OLP} :: \text{rect. CBE} : \text{rect. HBK}$

and invertendo, $\text{rect. OLP} : \text{rect. MLN} :: \text{rect. HBK} : \text{rect. CBE}$.

But since $HA : AC :: OL : LM$

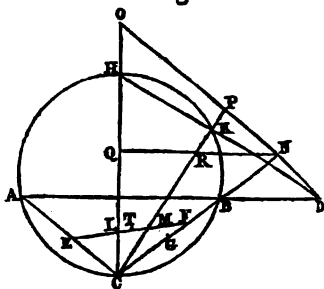
and $KA : AC :: PL : LN$

(by Lemma) rect. HAK : AC^2 :: rect. OLP : rect. MLN
 :: rect. HBK : rect. CBE;
 and alternando, rect. HAK : rect. HBK :: AC^2 : rect. CBE.
 Q. E. D.

PROP. III.

Let ABC be a circle, and AB a straight line meeting it in A, B: let C be a point in the circumference, join CA, CB, let D be a point in AB; EF any straight line meeting CA, CB in E, F; let FG : CF :: DB : DA; through D let any straight line be drawn meeting the circle in H, K; and CH, CK be joined, meeting EF in L, M: then rect. LEM : rect. LFM :: EC^2 : rect. CFG.

Fig. 6.



Synthesis. Let this be granted. Draw DN parallel to AC, meeting CB, CH, CK in N, O, P; through the point N draw a line parallel to EF, meeting CO, CP in Q, R.

Since rect. LEM : rect. LFM :: EC^2 : rect. CFG

rect. LEM : EC^2 :: rect. LFM : rect. CFG

But since QN : NO :: LE : EC

and RN : NP :: ME : EC

then (by Lemma, Proposition 2.)

rect. QRN : rect. ONP :: rect. LEM : EC^2

:: rect. LFM : rect. CFG

and invertendo, rect. ONP : QNR :: rect. CFG : rect. LFM

but rect. QNR : CN^2 :: rect. LFM : CF^2

$\therefore$ rect. ONP : CN^2 :: rect. CFG : CF^2

and since FG : CF :: FG : CF.

and since FG : CF :: DB : DA

:: BN : NC

:: rect. CNB : CN^2

$\therefore$ rect. ONP = rect. CNB.

Synthesis. Since rect. ONP = rect. CNB,

rect. ONP : CN^2 :: rect. CNB : CN^2

:: BN : NC

and since BN : NC :: BD : DA

:: GF : FC

:: rect. CFG : CF^2

$\therefore$ rect. ONP : CN^2 :: rect. CFG : CF^2

But CN^2 : rect. QNR :: CF^2 : rect. LFM

wherefore rect. ONP : rect. QNR :: rect. CFG : rect. LFM

and invertendo QNR : ONP :: rect. LFM : rect. CFG

But since $LE : EC :: QN : NO$
 and $ME : EC :: RN : NP$

(by Lemma, Prop. 2)

rect. LEM : $EC^2 ::$ rect. QNR : rect. ONP
 $::$ rect. LEM : rect. CFG ;

and alternando rect. LEM : rect. LFM : : EC^2 : rect. CFG

Q. E. D.

Corollary. Let ABC be a circle, and AB a straight line meeting it in A, B; let C be a point in the circumference, let CA, CB, be joined; let EF be a straight line meeting CA, CB in E, F; in AB produced let there be taken any point D so that $DB : DA :: CE^2 : CF^2$; let EF be bisected in T, and through D let there be drawn any straight line meeting the circle in H, K, and CH, CK be joined meeting the straight line EF in L, M. then we shall have $LT = TM$.

Let $FG : FC :: DB : DA$

Then rect. CFG : $CF^2 :: FG : FC$

$:: DB : DA$

and $DB : DA :: CE^2 : CF^2$

therefore rect. CFG : $CF^2 :: CE^2 : CF^2$

$\therefore CE^2 = \text{rect. CFG} :$

But, (by preceding Proposition.)

rect. LEM : rect. LFM : : CE^2 : rect. CFG.

wherefore rect. LEM = rect. LFM.

$\therefore EL : LF :: FM : ME$

and componendo, $EF : FL :: EF : EM$.

$\therefore EM = FL$. And since $ET = TF$, then $LT = TM$.

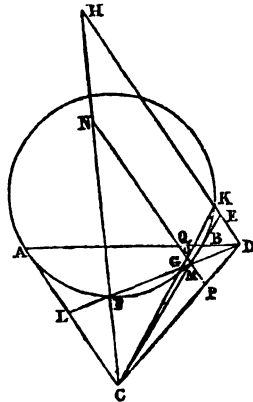
Q. E. D.

PROP. IV.

Let AB be a straight line meeting the circle in A, B; let AC, BC be tangents to the circle at A and B, through any point D in AB let DE be drawn parallel to AC, meeting BC in E. If through the point D be drawn any straight line meeting the circle in F, G; and CF, CG be joined, meeting DE in H, K; then the rect. HEK = DE^2 .

Analysis. Join CD. Let DF meet the straight lines AC, BC in L, M; through M draw a parallel to AC, DE, meeting CH, CE, CD, and AD in the points, N, O, P, Q.

Fig. 7.



Since $\text{rect. HEK} = \text{DE}^2$ and

$$\text{rect. NMO} : \text{MP}^2 :: \text{rect. HEK} : \text{DE}^2,$$

then $\text{rect. NMO} = \text{MP}^2$; and $\text{rect. FMG} = \text{MB}^2$;

$$\text{wherefore } \text{rect. FMG} : \text{rect. NMO} :: \text{MB}^2 : \text{MP}^2.$$

$$\text{and since } \text{FL} : \text{LC} :: \text{FM} : \text{MN}$$

$$\text{and } \text{GL} : \text{LG} :: \text{GM} : \text{MO},$$

then (by Lemma, Proposition 2.)

$$\text{rect. FLG} (= \text{AL}^2) : \text{LC}^2 :: \text{rect. FMG} : \text{rect. NMO}$$

$$\text{but } \text{MQ}^2 : \text{MP}^2 :: \text{AL}^2 : \text{LC}^2$$

$$:: \text{MB}^2 : \text{MP}^2$$

wherefore $\text{MQ} = \text{MB}$.

$$\text{But } \text{AC} : \text{CB} :: \text{MQ} : \text{MB}.$$

therefore $\text{AC} = \text{CB}$, which is true, since AC , BC are tangents to the circle at A and B .

Synthesis. Since AC , BC , are tangents at A and B , $\text{AC} = \text{BC}$.

$$\text{and } \text{MQ} : \text{MB} :: \text{AC} : \text{CB}$$

$$\text{wherefore } \text{MQ} = \text{MB} \text{ and } \text{MQ}^2 = \text{MB}^2 = \text{rect. FMG}.$$

$$\therefore \text{rect. FMG} : \text{rect. NMO} :: \text{MQ}^2 : \text{rect. NMO};$$

$$\text{and since } \text{FL} : \text{LC} :: \text{FM} : \text{MN}$$

$$\text{and } \text{GL} : \text{LG} :: \text{GM} : \text{MO},$$

$$\text{then } \text{rect. FLG} (= \text{AL}^2) : \text{LC}^2 :: \text{rect. FMG} : \text{rect. NMO}$$

$$:: \text{MQ}^2 : \text{rect. NMO}.$$

$$\text{But } \text{MQ}^2 : \text{MP}^2 :: \text{AL}^2 : \text{LC}^2$$

$$:: \text{MQ}^2 : \text{rect. NMO};$$

therefore $\text{rect. NMO} = \text{MP}^2$.

$$\text{and since } \text{rect. HEK} : \text{DE}^2 :: \text{rect. NMO} : \text{MP}^2,$$

$$\text{rect. HEK} = \text{DE}^2.$$

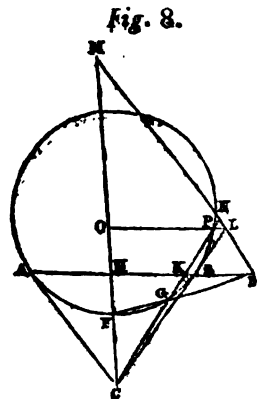
Q. E. D.

PROP. V.

Let AB be a straight line, meeting the circle in A , B ; let AC , BC be tangents at A , B ; through any point D in AB let there be drawn a straight line meeting the circle in F , G ; let CF , CG be joined, meeting AB in H , K ; then

$$\text{rect. HAK} : \text{rect. ABC} :: \text{AD}^2 : \text{DB}^2.$$

Analysis. Let this be granted. Draw DL parallel to the straight line AC , meeting CB , CF , CG in L , M , N ; and draw LO parallel to AB , meeting CF , CG in O , P .



Since rectangle $HAK : \text{rect. HBK} :: AD^2 : DB^2$
 $:: CL^2 : LB^2$;

and $\text{rect. HBK} : \text{rect. OLP} :: CB^2 : CL^2$,
 then, $\text{rect. HAK} : \text{rect. OLP} :: CB^2 : BL^2$
 $:: AC^2 : BL^2$

and alternando, $\text{rect. HAK} : AC^2 :: \text{rect. OLP} : BL^2$.

Again, because $OL : LM :: HA : AC$

and $PL : LN :: KA : AC$

(Lemma, Prop. 2) $\text{rect. OLP} : \text{rect. MLN} :: \text{rect. HAK} : AC^2$
 $:: \text{rect. OLP} : BL^2$.

Therefore $\text{rect. MLN} = BL^2 = DL^2$. Which, (by preceding Proposition,) we know to be true.

Synthesis. Since (preceding Prop.) $\text{rect. MLN} = DL^2 = BL^2$;

$\text{rect. OLP} : \text{rect. MLN} :: \text{rect. OLP} : BL^2$;

but since $HA : AC :: OL : LM$

and $KA : AC :: PL : LN$,

(by Lemma, Prop. 2.)

$\text{rect. HAK} : AC^2 :: \text{rect. OLP} : \text{rect. MLN}$
 $:: \text{rect. OLP} : BL^2$

and alternando, $\text{rect. HAK} : \text{rect. OLP} :: AC^2 : BL^2$
 $:: BC^2 : BL^2$;

But, $\text{rectang. OLP} : \text{rect. HBK} :: CL^2 : CB^2$;

wherefore $\text{rect. HAK} : \text{rect. HBK} :: CL^2 : BL^2$
 $:: AD^2 : BD^2$.

Q. E. D.

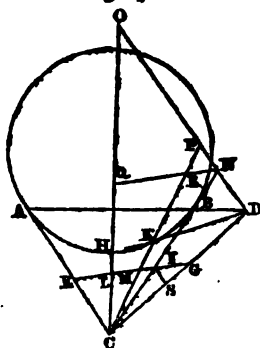
PROP. VI.

Let AC, BC be tangents at A, B , let D be a point in the straight line AB : let EF be a straight line meeting AC, BC , in E, F , and CD , when joined, in G : if through the point D any straight line be drawn meeting the circle in H, K ; and CH, CK be joined, meeting EF in L, M : then,

$\text{rect. LEM} : \text{rect. LFM} :: GE^2 : GF^2$

Analysis. Let this be granted. Through D draw a parallel to AC , meeting the straight lines CB, CH, CK , in N, O, P ; through the point N draw a parallel to EF , meeting CH, CK in Q, R : through the point F draw FS parallel to AC , meeting CD in S .

Fig. 9.



Since rectangle $LEM : \text{rect. LFM} :: EG^2 : FC^2$

$:: EC^2 : FS^2$;

alternando, $\text{rect. LEM} : EC^2 :: \text{rect. LFM} : FS^2$.

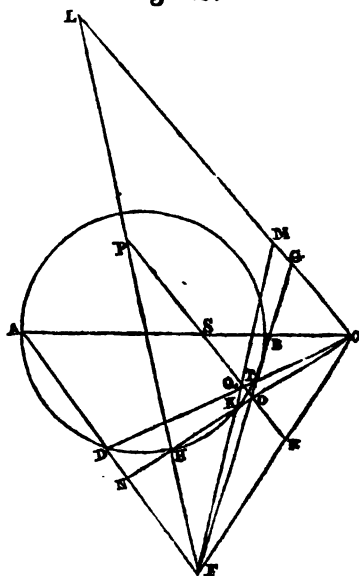
Also since rectangle QNR : $CN^2 :: \text{rect. LFM} : CF^2$
 and $CN^2 : ND^2 :: CF^2 : FS^2$
 therefore rect. QNR : $ND^2 :: \text{rect. LFM} : FS^2$
 $:: \text{rect. LEM} : EC.$
 But since LE : EC :: QN : NO
 and ME : EC :: RN : NP therefore
 (Lemma, Prop. 2) rect. LEM : $EC^2 :: \text{rect. QNR} : \text{rect. ONP}$
 $\therefore \text{rect. QNR} : \text{rect. ONP} :: \text{rect. QNR} : ND^2.$
 $\therefore \text{rect. ONP} = ND^2$ which (Proposition 4.) we know to be true.
Synthesis. Since rect. $ONP = ND^2$,
 rect. QNR : rect. ONP :: rect. QNR : ND^2
 and since LE : EC :: QN : NO
 and ME : EC :: RN : NP,
 rect. LEM : $EC^2 :: \text{rect. QNR} : ONP$
 $:: \text{rect. QNR} : ND^2.$
 But since rect. QNR : $CN^2 :: \text{rect. LFM} : CF^2$
 and $CN^2 : ND^2 :: CF^2 : FS^2$; then
 rect. QNR : $ND^2 :: \text{rect. LFM} : FS^2$;
 $\therefore \text{rect. LEM} : EC^2 :: \text{rect. LFM} : FS^2$, and
 alternando, rect. LEM : rect. LFM :: $EC^2 : FS^2$,
 $:: EG^2 : FG^2.$
 Q. E. D.

PROP. VII.

Let AB be a straight line meeting the circle in A, B, and in
 it take any point C; draw CD, meeting the circle in D, E,
 and let AD, BE be joined, meeting in F; draw CG
 parallel to the straight line AF, meeting BF in G. If
 through the point C there be drawn any straight line meet-
 ing the circle in H, K, and FH, FK be joined, meeting
 CG in L, M: then

rect. LGM = CG^2 .

Analysis. Join CF, and let CH meet AB, BF, in N, O; through the point O draw a parallel to CG, or AF, meeting the straight lines FL, FM, FC, in P, Q, R, and AB, DE, in S, T.



Since rect. $LGM = CG^2$ and

rectangle $POQ : OR^2 :: \text{rect. } LGM : CG^2$

then rect. $POQ = OR^2$

and rect. $HOK = \text{rect. } EOB$;

$\therefore \text{rect. } HOK : \text{rect. } POQ :: \text{rect. } EOB : OR^2$;

and since $HN : NF :: HO : OP$.

and $KN : NF :: KO : OQ$

Lemma, Proposition 2.)

rect. $HNK : FN^2 :: \text{rect. } HOK : \text{rect. } POQ$
 $:: \text{rect. } EOB : OR^2$;

But $FN^2 : \text{rect. } AND :: OR^2 : \text{rect. } SOT$;

wherefore rect. $HNK : \text{rect. } AND :: \text{rect. } EOB : \text{rect. } SOT$;

and since rect. $HNK = \text{rect. } AND$, rect. $EOB = \text{rect. } SOT$;

$\therefore OB : OS :: OT : OE$

that is, $BG : GC :: GC : GE$.

therefore $\angle GEC = \angle GCB = \angle BAD$, which is true,

since A, B, E, D are points in a circle.

Synthesis. Since A, B, E, D , are points in a circle,

$\angle GEC = \angle BAD = \angle GCB$.

$\therefore BG : GC :: GC : GE$

that is $OB : OS :: OT : OE$.

$\therefore \text{rect. } SOT = \text{rect. } EOB = \text{rect. } HOK$; wherefore

rect. $HOK : OR^2 :: \text{rect. } SOT : OR^2$

$:: \text{rect. } AND : FN^2$

$:: \text{rect. } HNK : FN^2$

But since $HN : NF :: HO : OP$

and $KN : NF :: KO : OQ$, then

rect. $HNK : FN^2 :: \text{rect. } HOK : \text{rect. } POQ$.

$\therefore \text{rect. } HOK : \text{rect. } POQ :: \text{rect. } HOK : OR^2$

and rect. $POQ = OR^2$

and since rect. $LGM : GC^2 :: \text{rect. } POQ : OR^2$,

then also rect. $LGM = GC^2$.

Q. E. D.

PROP. VIII.

Let AB be a straight line, meeting the circle at A, B ; in AB , let there be taken a point C ; let CD be drawn, meeting the circle in D, E , let AD, BE be joined, meeting in F ; through the point C let there be drawn any straight line, meeting the circle in G, H ; let FG, FH be joined meeting AB in K, L ; then

rect. KAL : rect. KBL :: AC^2 : BC^2 .

Analysis. Let this be granted.

Draw CM parallel to AF , meeting the straight lines FB, FK, FL in M, N, O , through the point M draw a parallel to the straight line AB , meeting FK, FL in P, Q .

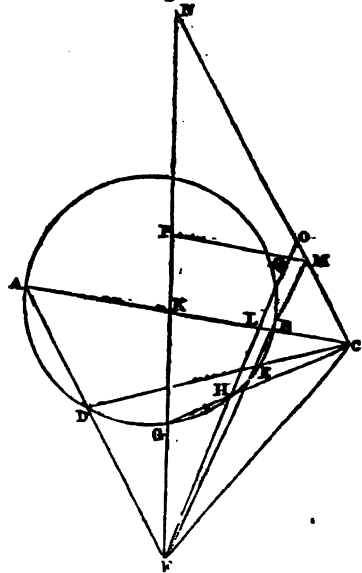


Fig. 11.

Since rect. KAL : rect. KBL :: AO^2 : BO^2
 :: FM^2 : BM^2 ;
 and rect. KBL : rect. PMQ :: FB^2 : FM^2 ; therefore
 rect. KAL : rect. PMQ :: FB^2 : BM^2
 :: AF^2 : CM^2

and, alternando,

rect. KAL : AF^2 :: rect. PMQ : MC^2 .

But since PM : MN :: KA : AF ,
 and QM : MO :: LA : AF , then

(by Lemma, Prop. 2.)

rect. PMQ : rect. NMO :: rect. KAL : AF^2
 :: rect. PMQ : CM^2

∴ rect. NMO = CM^2 . Which (by preceding Prop.) is known to be true.

Synthesis. Since rect. NMO = MC^2 .

rect. PMQ : rect. NMO :: rect. PMQ : MC^2 .

But since KA : AF :: PM : MN
 and LA : AF :: QM : MO , then
 rect. KAL : AF^2 :: PMQ : NMO
 :: PMQ : MC^2 ;

and alternando,

$$\text{rect. KAL} : \text{rect. PMQ} :: AF^2 : MC^2$$

$$:: FB^2 : BM^2$$

$$\text{and rect. PMQ} : \text{rect. KBL} :: FM^2 : FB^2$$

$$\text{therefore rect. KAL} : \text{rect. KBL} :: FM^2 : BM^2$$

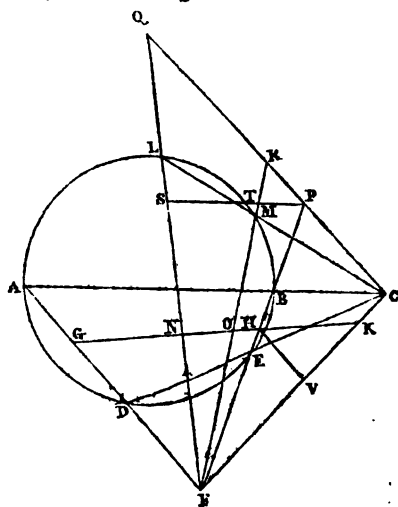
$$:: AC^2 : BC^2.$$

Q. E. D.

PROP. IX.

Let AB be a straight line meeting the circle in A, B, and in point C, through C let there be drawn a straight line meeting the circle in D, E, let AD, BE be joined meeting in the point F; and let the straight line GH meet AF, BF in G, H, and FC, (when joined), in K; if through the point C be drawn any straight line meeting the circle in L, M; and EL, FM be joined meeting GH in N, O, then rect. OGN : rect. OHN :: GK² : HK²

Fig. 12.



Analysis. Let this be granted. Through the point C draw a parallel to AF meeting the straight lines FB, FL, FM, in P, Q, R; through

the point P draw a parallel to GH meeting FL, FM in S, T; and draw HV parallel to the straight line PC, meeting FC in V.

$$\text{Since rectangle OGN} : \text{rect. OHN} :: GK^2 : HK^2$$

$$:: FG^2 : HV^2 \text{ then}$$

$$\text{alternando, rect. OGN} : FG^2 :: \text{rect. OHN} : HV^2$$

$$\text{since rect. SPT} : PF^2 :: \text{rect. OHN} : HF^2$$

$$\text{and } PF^2 : PC^2 :: HF^2 : HV^2, \text{ then}$$

$$\text{rect. SPT} : PC^2 :: \text{rect. OHN} : HV^2$$

$$:: \text{rect. OGN} : PG^2.$$

$$\text{But since NG} : GF :: SP : FQ$$

$$\text{and OG} : GF :: TP : FR, \text{ then}$$

(by Lemma, Prop. 2.)

$$\text{rect. OGN} : FG^2 :: \text{rect. SPT} : \text{rect. QPR.}$$

$$\therefore \text{rect. SPT} : \text{rect. QPR} :: \text{rect. SPT} : PC^2.$$

Therefore rectangle $QPR = PC^2$. Which (by Prop. 7) we know to be true.

Synthesis. Since rect. $QPR = PC^2$.

$$\text{rectangle SPT} : \text{rect. QPR} :: \text{rect. SPT} : PC^2.$$

$$\text{and since } NG : FG :: SP : PQ$$

$$\text{and } OG : GF :: TP : PR$$

$$\text{then, rect. OGN} : FG^2 :: \text{rect. SPT} : \text{rect. QPR} \\ :: \text{rect. SPT} : PC^2$$

$$\text{But since rect. SPT} : PF^2 :: \text{rect. OHN} : HF^2$$

$$\text{and } FP^2 : PC^2 :: HF^2 : HV^2,$$

$$\text{therefore rect. SPT} : PF^2 :: \text{rect. OHN} : HV^2$$

$$\therefore \text{rect. OGN} : FG^2 :: \text{rect. OHN} : HV^2$$

$$\text{and, alternando, rect. OGN} : \text{rect. OHN} :: FG^2 : HV^2 \\ :: GK^2 : HK^2.$$

Q. E. D.

Scholium. The preceding propositions obtain also in the Conic Sections. And in considering this, the following theorems occurred to me, from which many others may be deduced.

Theor. 1. Let AB , CD be two straight lines, tangents to a Conic Section in A , C ; and let A , C be joined, let there be any straight line meeting AB , CD , AC in B , D , E , and the Section in F , G ; then

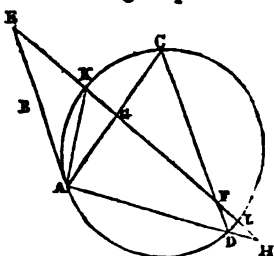
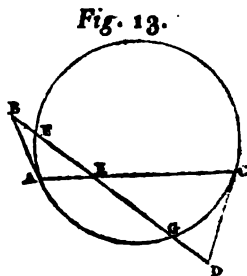
$$\text{rect. BFD} : \text{rect. BGD} :: EF^2 : EG^2.$$

Theor. 2. Let AB be a tangent to a conic section at A ,

Fig. 14.

and CD a straight line meeting the section in C , D , let AC , AD be joined; let any straight line meet AB , CD , AC , AD in E , F , G , H and the section in K , L : then

$$\text{rect. EKF} : \text{ELF} :: \text{rect. GKH} : \text{rect. GLM}.$$



Theor. 3. Let two straight lines AB, CD be inscribed in a conic section, and AC, BD be joined, let any straight line meet AB, CD, AC, BD in E, F, G, H and the section in K, L ; then

$\text{rect. } EKF : \text{rect. } ELF :: \text{rect. } GKH : \text{rect. } GLH.$

From these we have also the following theorem.

If in a conic section any quadrilateral figure be inscribed, three sides of which pass through given points in a straight line; the remaining side will either converge to a given point, or will be parallel to a straight line whose position is given.

In conclusion, I will add one more theorem, not altogether undeserving of notice; which, with a very slight change, obtains also in conic sections; whence others may be derived, which have not been mentioned by any writers on Geometry, with whose works I am acquainted.

Fig. 16.

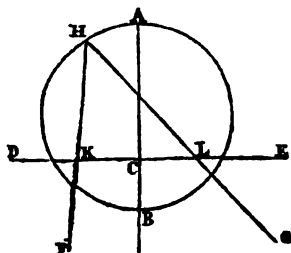
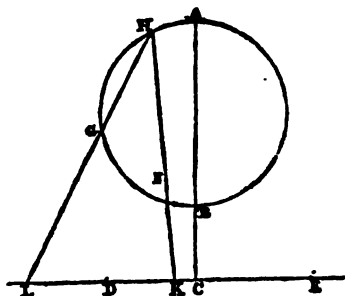


Fig. 17.



Let AB be a diameter of a circle, whose position is given; let the line CD meet it perpendicularly in C , (C not being the centre of the circle) and let the point D be given in CD . If C be within the circle, let CD^2 be greater than rectangle ACB , but if C be without the circle, let CD^2 be less than $\text{rect. } ACB$. Produce DC to E , so that $CE = CD$; then two points F, G will be given without the line DE ; such that if we draw two lines FH, GH to any point H in the circle, meeting DE in K, L ; the sum of DK^2 , and EK^2 will have a given ratio to the rectangle DLE .

ARTICLE V.

ON CERTAIN PROPERTIES OF PLANE TRIANGLES WHICH
ARE NOT GENERALLY KNOWN. BY C. F. A. JACOBI.

(Communicated by Mr. DAVIES.)

A short time ago, a curious tract fell into my hands, written by Mr. Jacobi, containing a considerable number of interesting properties of plane triangles. As they belong to a class that has not been much considered in this country, I thought I should be affording a gratification to British Geometers, by sending a copy of it to the Editor of the Mathematical Repository; and I am sure, devoted as his whole life has been to the cultivation and diffusion of mathematical science, and having contributed by the publication of that work, in so eminent a degree to the formation of that elegance of taste, which is now so generally diffused amongst English mathematicians—that the Editor will cheerfully accord in my views by printing it in his valued mathematical annual. I shall add a few notes at the end—and merely remark in this place, that all, or most, of the properties have beautiful analogies in *sphæra*.

T. S. D.

De triangulorum rectilineorum proprietatibus quibusdam
nondum satis cognitis,

Memoriam anniversariam inauguratæ ante hos CCLXXXII
annos Scholæ Provencialis Portensis.

A. D. I. Novembr MDCCCXXV pie celebrandam indicit
et ad recitationes et orationes discipulorum ex omnibus classibus
electorum audiendas invitat.

CAROLUS FREDERICUS ANDREAS JACOBI,
Math. et Phys. Prof.

Præstat Lipsiæ apud E. B. Schwickertum, 1825.

Geometry has often been praised for the clearness and cogency of the reasonings employed, which are on this account admirable exercises for the mind.

They who have endeavoured to weaken the opinion commonly entertained of geometry on this head, have generally shown a want of sufficient acquaintance with the nature and objects of that science, so that its honour has rather been more firmly established by these attempts than at all injured.

Another merit of geometry, and one which, in my opinion,

is equally important, is the daily increase it admits of,* not only by the introduction of new matter, (by which it is exceedingly useful in different studies,) but also by the discovery of easier and more elegant methods of accomplishing and amplifying problems, which had been already discovered.

Geometry gives, as it were, a vast field to our view—one part (which was cultivated by the nations of antiquity, and more especially by the Greeks), has through many ages yielded fruit plenteously to mankind, and indeed it has never been neglected by mathematicians. Another portion which was first cultivated in the last century has engaged the attention of many great men, some of whom indeed seem almost more than mere mortals: another portion again waits for the ingenuity and diligence of posterity. The recent publications of two distinguished men, A. L. Crelle† and C. G. Feuerbach‡ have confirmed me in the opinion of the indefinite extent of geometry.

These mathematicians by publishing new properties of that simple figure, the rectilineal triangle, and by giving new demonstrations of theorems already known, have clearly shown the errors of those who conceived that the geometry of the triangle is altogether limited; and their propositions are capable of extension and generalization.

It seemed, therefore, proper to publish the new demonstrations I had found of Crelle's theorems, and also properties which I had discovered similar to those of Feuerbach both for the sake of those students, for whom this book was more especially designed, and because I remembered the opinion given on this subject by that great geometer Laplace.§

* "Die eroberungen der Mathematik wachsen täglich, wiewohl ohne Geräusch" Herbart über die Möglichkeit Mathematik auf Psychologie anzuwenden. Königsberg 1822, p. 47.

† Über einige Eigenschaften des ebenen geradlinigen Dreiecks etc. Berlin, 1816.

‡ Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks etc. Nürnberg, 1822.

§ Laplace. Exp. Sys. Mon. 4me. ed. Par. 1813, p. 403. Tout se tient dans la chaîne immense des vérités de la géométrie pure; et souvent une seule observation a suffi pour féconder les plus stériles en apparence etc.

CHAPTER I.

On transversals intersecting in one point.

I.

Fig. 1.

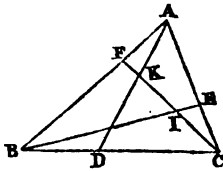
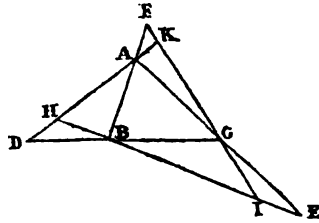


Fig. 2.



The sides of the triangle ABC we shall call a, b, c , and the opposite angles A, B, C in the usual way.

Right lines from the vertices of a triangle ABC as AD, BE, CF which intersect the opposite sides produced if necessary are called transversals. We shall call

BD	..	..	$a,$	IE	..	..	$e_{''}$
CE	..	..	$b,$	CI	..	.	$f_{''}$
AF	..	..	$c,$	FK	..	..	$f_{''}$
CD	..	..	$a_{''}$	KH	..	..	i
AE	..	..	$b_{''}$	HI	..	..	k
BF	..	..	$c_{''}$	KI	..	..	h
AD	..	..	d	$\angle BAD$	..	..	$A,$
BE	..	..	e	$\angle CBE$	..	..	$B,$
CF	..	..	f	$\angle ACF$	..	..	$C,$
AK	..	..	$d,$	$\angle CAD$	..	..	$A_{''}$
HD	..	..	$d_{''}$	$\angle ABF$	..	..	$B_{''}$
BH	..	..	$e,$	$\angle BCF$	..	..	$C_{''}$

Then by a well known property of triangles

$$a, : d :: \sin A, : \sin B$$

$$d : a_{''} :: \sin C : \sin A_{''}$$

$$b, : e :: \sin B, : \sin C$$

$$e : b_{''} :: \sin A : \sin B_{''}$$

$$c, : f :: \sin C, : \sin A$$

$$f : c_{''} :: \sin B : \sin C_{''}$$

$$\therefore a, . b, . c, : a_{''} . b_{''} . c_{''} :: \sin A, . \sin B, . \sin C, : \sin A_{''} . \sin B_{''} . \sin C_{''}$$

Or the rectangular parallelopipeda contained by three non-contiguous segments of the sides have the same ratio to one another as the sines of the opposite angles when multiplied together.

Cor. In the same way

$$d \cdot e \cdot f :: (d + i)(f + k)(f + h) :: \sin A \cdot \sin B \cdot \sin C :: \sin A'' \cdot \sin B'' \cdot \sin C'' \\ \therefore a \cdot b \cdot c \cdot a'' \cdot b'' \cdot c'' :: d \cdot e \cdot f :: (d + c)(e + k)(f + h)$$

II.

It is evident that the triangle GHI formed by transversals is not constant but varies with the transversals themselves and that its area vanishes when G, H, I coalesce or the three transversals pass through the same point. Then

$k = h = c = 0$. Hence the proportions will become

$$d \cdot e \cdot f :: d \cdot e \cdot f :: \sin A \cdot \sin B \cdot \sin C :: \sin A'' \cdot \sin B'' \cdot \sin C''$$

$$d \cdot e \cdot f :: d \cdot e \cdot f :: a \cdot b \cdot c :: a'' \cdot b'' \cdot c''$$

$$\therefore \sin A \cdot \sin B \cdot \sin C = \sin A'' \cdot \sin B'' \cdot \sin C''$$

$$\text{and } a \cdot b \cdot c = a'' \cdot b'' \cdot c''$$

Or, besides the equality of the two parallelopipeda formed of the alternate segments—the products of the sines of the opposite angles are equal.*

III.

Fig. 3.

The converse of (II) is true. For if it be not it follows that, when we draw the right line AGH .

$$BH \cdot b \cdot c = CH \cdot b'' \cdot c''$$

But as by hyp. $a \cdot b \cdot c = a'' \cdot b'' \cdot c''$.

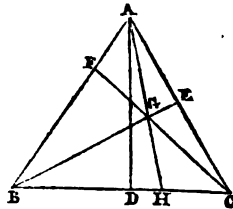
It follows.

$$a : BH :: a'' : CH.$$

$$\text{Comp. } a + a'' : BH + CH : a'' : CH.$$

$\therefore a'' = CH$, which is absurd: therefore if $a \cdot b \cdot c = a'' \cdot b'' \cdot c''$, the transversals will pass through one point G .

Cor. From this proposition and from (II) it follows that transversals making



* John Bernoulli first demonstrated this theorem, vid. Opp. Tom. iv. p. 22; other demonstrations have been given. Carnot "Geometrie de Position;" Von Schumacher, Tom ii. p. 329. Crelle in the book mentioned above, page 7 and others.

$\sin A, \sin B, \sin C, = \sin A'', \sin B'', \sin C''$ intersect in one point.

Another demonstration may be given of this theorem.

If it be possible that transversals making $\sin A, \sin B, \sin C, = \sin A'', \sin B'', \sin C''$, do not intersect in one point it follows (ΔGH being drawn and $\angle BAH$ and $\angle CAH$ being denoted by A''', A_{IV})

$$\sin A''', \sin B'', \sin C, = \sin A_{IV}, \sin B'', \sin C''$$

But by hyp.

$$\sin A, \sin B, \sin C, = \sin A'', \sin B'', \sin C''$$

$$\therefore \sin A, : \sin A'' :: \sin A''' : \sin A_{IV}.$$

Comp. et Divid.

$$\sin A, + \sin A'' : \sin A''' + \sin A_{IV} :: \sin A, - \sin A'' : \sin A'' - \sin A_{IV}$$

Or,

$$\sin \frac{1}{2} A \cdot \cos \frac{1}{2} (A - A'') : \sin \frac{1}{2} A \cdot \cos \frac{1}{2} (A''' - A_{IV}) :: \sin (\frac{1}{2} A, - A'') \cos \frac{1}{2} A : \sin \frac{1}{2} (A, - A_{IV}) \cos \frac{1}{2} A.$$

$$\therefore 1 : 1 :: \tan \frac{1}{2} (A, - A'') : \tan \frac{1}{2} (A''' - A_{IV}).$$

$$\therefore A'' + A_{IV} = A''' + A_{IV}.$$

Which since $A, < A'''$

$$A_{IV} < A''$$

is absurd; and therefore transversals so drawn will intersect in one point.

IV.

From these propositions others may easily be deduced,* viz.

Transversals bisecting either the angles or the sides of the triangles pass through the same point.

Transversals which are perpendicular to the opposite sides pass through the same point—†

$$\text{For then } A, = C, B, = A'' C, = B''$$

Transversals which cut the opposite sides in the same points as the inscribed circle—

$$\text{For then } a, = c, b, = a'' c, = b''$$

* Vid. Crelle in the above mentioned work p. 10.

† Gauss has given a very elegant synthetical demonstration of this property, which seems not to be commonly known by geometers, vid. Carnot in his "Geometrie de Position," Tom. ii. p. 363.

When any two sides AB , AC of a triangle ABC are each divided into two parts having the same ratios as the sides themselves; then the third intersecting line AGD being drawn, it shall bisect the third side BC . For

$$c_1 : b_{11} :: c : b$$

$$b_1 : c_{11} :: b : c$$

$$\text{whence } b_1 c_1 = b_{11} c_{11}$$

But also $a_1 b_1 c_1 = a_{11} b_{11} c_{11}$
from which it is manifest that

$$a_1 = a_{11} = \frac{1}{2} a.$$

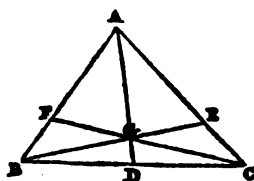


Fig. 4.

On the other hand the line by which the base is bisected meets the lines BE , CF by the other lines are divided in the same point G : and hence it is manifest that a straight line can be drawn from a given point to a given line which shall bisect a third.

But if we take on either side b , c any number of parts

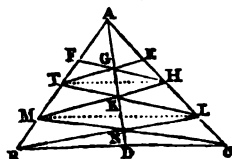
AE , AF , AH , AI , AL , AM , &c. (Fig. 5) so that

$$b : c :: AE : AF :: AH : AI :: AL : AM, \text{ etc}$$

and draw the lines EI , FH , HM , IL , LB , MC , etc:

then the straight line which bisects the third side, will pass through the points G , K , N , etc. in which these pairs of lines mutually intersected each other. For if the two lines HI , LM are drawn, they will be parallel to one another, and be bisected by the same transversal. Therefore, if $BD = DC$ the transversal AD , from what has been already shewn, must pass through N , and also through K since it also bisects the straight line ML .

Fig. 5.



V.

In all triangles innumerable *triads* of transversals may be drawn to pass through the same point besides those which have been already mentioned.

For the sake of brevity let us call the triad in which the transversals bisect the opposite sides—

The first triad :

where the angles are bisected.

The second triad :

where the transversals are perpendicular to the opposite sides.

The third triad :

where $A_1 = B_1 = C_1$.

The fourth triad :

where $A_{11} = B_{11} = C_{11}$

The fifth triad :

where $a_1 = c_{11}$, $b_1 = a_{11}$, $c_1 = b_{11}$

The sixth triad :

where $a_1 = b_{11}$, $b_1 = c_{11}$, $c_1 = a_{11}$

The seventh triad :

a, b, c , etc will still be used in the same way as before, but with a number over them, to show to which triad they refer.

VI.

With regard to the first triad we may observe : First, The point of intersection is the centre of gravity of the triangle. Secondly,

$$d_1 = \frac{(1)}{2}d_{11}; e_1 = \frac{(1)}{2}e_{11}; f_1 = \frac{(1)}{2}f_{11}$$

Thirdly

$$d_1^2 + e_1^2 + f_1^2 = \frac{(1)}{4}(a^2 + b^2 + c^2)$$

Fourthly

$$d_1 \cdot d_1 + e_1 \cdot e_1 + f_1 \cdot f_1 = \frac{(1)}{4}(a^2 + b^2 + c^2)$$

Fifthly

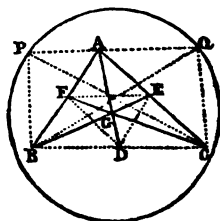
$$\begin{aligned} d_1 \cdot d_{11} + e_1 \cdot e_{11} + f_1 \cdot f_{11} &= \frac{(1)}{4}(a^2 + b^2 + c^2) \\ &= a_1^2 + b_1^2 + c_1^2 = \overline{DE}^2 + \overline{DF}^2 + \overline{EF}^2 \end{aligned}$$

Sixthly, if any circle be described from G as a centre and in its circumference any points P and Q be taken,

$$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = \overline{QA}^2 + \overline{QB}^2 + \overline{QC}^2.$$

And many other properties of the same kind may be found which I shall pass over, as they are sufficiently known.

Fig. 6.



VII.

The second triad.

It is evident that

whence $a,^{(2)} b = a_{,,}^{(2)} \cdot c$; $b,^{(2)} c = b_{,,}^{(2)} \cdot a$; $c,^{(2)} a = c_{,,}^{(2)} \cdot b$

$$\begin{aligned} & a,^{(2)} (b, + b_{,,}) + b,^{(2)} (c, + c_{,,}) + c,^{(2)} (a, + a_{,,}) \\ &= a_{,,}^{(2)} (c, + c_{,,}) + b_{,,}^{(2)} (a, + a_{,,}) + c_{,,}^{(2)} (b, + b_{,,}). \\ & a, \cdot b, + b, \cdot c, + a, \cdot c, + a, \cdot b_{,,} + b, \cdot c_{,,} + a_{,,} \cdot c, \\ &= a_{,,} c_{,,} + a_{,,} b_{,,} + b_{,,} c_{,,} + a_{,,} c, + a_{,,} b, + b, c_{,,} \\ & a, \cdot b, + a, \cdot c, + b, \cdot c, = a_{,,} \cdot b_{,,} + a_{,,} \cdot c_{,,} + b_{,,} \cdot c_{,,}. \end{aligned}$$

That is, the sum of the three rectangles contained under the alternate segments of the sides are equal to the sum of the three other rectangles contained under the remaining segments.

VIII.

In every triangle, (if we denote the radii of four circles such that each touches the sides of the triangle produced if necessary by $r, r_{,,}, r_{,,,}, r_{,,,,}$) $ab + ac + bc$ will equal

$$rr, + rr_{,,} + rr_{,,,} + r_{,,}r_{,,,} + r_{,,}r_{,,,,} + r_{,,,}r_{,,,,}$$

This was demonstrated by Feurbach,* and from this and the preceding theorem it easily follows that any two triangles, the sides of one of which equal a, b, c , each to each, and those of the other equal $a_{,,} b_{,,} c_{,,}$ have always this property, that the sum of the rectangles under the radii (of the circles touching the sides) taken by twos in the one triangle shall be equal to the sum of the similar rectangles in the other triangle.

IX.

By elementary geometry

$$\begin{aligned} d^{(2)} &= bc - a_{,,}^{(2)}; e^{(2)} = ac - b_{,,}^{(2)}; f^{(2)} = ab - c_{,,}^{(2)} \\ \therefore d^2 + e^2 + f^2 &= ab + ac + bc - (a_{,,} + b_{,,} + c_{,,}) \end{aligned}$$

that is, the sum of the squares of the transversals equals the difference between the three rectangles under the sides and the three rectangles of their segments.

Cor.

$$d^2 + e^2 + f^2 + a_{,,} + b_{,,} + c_{,,} = rr, + rr_{,,} + rr_{,,,} + r_{,,}r_{,,,} + r_{,,}r_{,,,,} + r_{,,,}r_{,,,,}$$

* Sec. 6 of the above-mentioned work.

X.

In every case

$$\therefore b : c :: a, : a,$$

$$\therefore b + c : a :: c : a,$$

$$\therefore (b + c)^2 : a^2 :: c^2 : a^2,$$

$$\therefore (b + c)^2 - a^2 : c^2 - a^2 :: (b + c)^2 : c^2$$

$$(b + c + a)(a - b + c) : c^2 - a^2 :: (b + c)^2 : c^2 \dots (1)$$

Now we have

$$\overset{(2)}{d^2} = b \cdot c - a, a, = b \cdot c - \overset{(2)}{a,^2} \cdot \frac{b}{c} = \frac{b}{c} \cdot (c^2 - a,^2).$$

$$\therefore \overset{(2)}{d^2} : b :: c^2 - a,^2 : c$$

$$\text{or } \overset{(2)}{d^2} : b \cdot c :: c^2 - a,^2 : c^2.$$

From this and the preceding expression (1) we obtain

$$\overset{(3)}{d^2} : bc :: (a + b + c)(-a + b + c) : (b + c)^2.$$

In like manner

$$\overset{(2)}{e^2} : ac :: (a + b + c)(a - b + c) : (a + c)^2$$

$$\overset{(2)}{f^2} : ab :: (a + b + c)(a + b - c) : (a + b)^2;$$

$$\therefore \overset{(2)}{d^2 e^2 f^2} : a^2 b^2 c^2 :: (a + b + c)^3 (-a + b + c)(a - b + c)(a + b - c)$$

$$: (a + b)^2 (b + c)^2 (b + c)^2 :: (a + b + c)^3 \cdot 16 \Delta^2 : (a + b)^2 (a + c)^2 (b + c)^2$$

$$\therefore d \cdot e \cdot f : a \cdot b \cdot c :: (a + b + c) \cdot 4 \Delta : (a + b)(a + c)(b + c)$$

that is, the parallelopipeda, one of which is contained under the transversals and the other under the sides of the triangle are in the same ratio to one another, that a prism (whose base is four times the area of the triangle, and altitude equal to the sum of the three sides of the triangle) is to the parallelopipedon contained under the sides of the triangle taken two and two.

XI.

The perpendiculars GL, GM, GN (Fig. 7) being let fall from G upon the sides of the triangle, we have

$$\overset{(3)}{d,^2} = \overline{AM^2} + \overline{GM^2} : \text{but since}$$

$$AM = \frac{1}{2}(-a + b + c), \text{ and } GM = \frac{2\Delta}{a + b + c} \text{ it follows}$$

$$d_1^{(2)} = \frac{1}{2} (-a + b + c)^2 + \frac{4\Delta^2}{(a + b + c)^2}$$

$$\therefore d_1^2 (a + b + c)$$

$$= \frac{1}{2} (a + b + c)^2 (-a + b + c)^2 + 4\Delta^2$$

$$= \frac{1}{2} (a + b + c) (-a + b + c) \{ (a + b + c) (-a + b + c) + (a - b + c) (a + b - c) \}$$

$$= (a + b + c) (-a + b + c) bc; \text{ whence}$$

$$d_1^{(2)} : bc :: -a + b + c : a + b + c; \text{ similarly}$$

$$d_2^{(2)} : ac :: a - b + c : a + b + c \text{ and}$$

$$f_1^{(2)} : ab :: a + b - c : a + b + c; \text{ whence}$$

$$d_1^2 d_2^2 f_1^2 : a^2 b^2 c^2 :: (-a + b + c) (a - b + c) (a + b - c) (a + b + c)^3$$

$$:: 8r^6 \Delta : (a + b + c)^3$$

$$:: 4r^2 : (a' + b + c)^2$$

$$d_1 d_2 f_1 : abc :: 2r : a + b + c.$$

That is, the rectangular parallelopipeda one of which is contained under the segments of the transversals next their respective vertices or superior segments, and the other under the sides of the triangle, are to one another in the same ratio as the diameter of the inscribed circle is to the perimeter of the triangle.

XII.

From (10 and 11)

$$d_1^{(2)} . e . f : a . b . c :: (a + b + c) 4\Delta : (a + b) (a + c) (b + c)$$

$$a . b . c : d_1^{(2)} . e . f :: a + b + c : 2r$$

$$\therefore d_1^{(2)} . e . f : d_1^{(2)} . e . f :: (a + b + c)^2 4\Delta : 2r (a + b) (a + c) (b + c)$$

$$:: (a + b + c)^2 4\Delta : \frac{4\Delta}{a + b + c} (a + b) (a + c) (b + c)$$

$$:: (a + b + c)^3 : (a + b) (a + c) (b + c).$$

That is, the rectangular parallelopipeda under the whole transversals and the superior segments are in the same ratio as the cube whose side is the perimeter of the triangle, is to the rectangular parallelopipedon under the sums of the sides of the triangle taken two and two.

* r signifies the radius of the inscribed circle.

XIII.

It is easily seen that

$$\angle AGE = \frac{1}{2}(A + B) \text{ and } \angle AGM = \frac{1}{2}(B + C)$$

$$\therefore \angle EGM = \frac{1}{2}(A - C)$$

in like manner

$$\angle FGN = \frac{1}{2}(A - B)$$

$$\angle DGL = \frac{1}{2}(B - C)$$

$$\therefore d^{(2)} \sin \frac{1}{2} A = e^{(2)} \cos \frac{1}{2}(A - C)$$

$$e^{(2)} \sin \frac{1}{2} B = f^{(2)} \cos \frac{1}{2}(A - B)$$

$$f^{(2)} \sin \frac{1}{2} C = d^{(2)} \cos \frac{1}{2}(B - C)$$

$$\therefore d^{(2)} \cdot e^{(2)} \cdot f^{(2)} : d^{(2)} \cdot e^{(2)} \cdot f^{(2)} :: \cos \frac{1}{2}(A - B) \cdot \cos \frac{1}{2}(A - C) \cdot \cos \frac{1}{2}(B - C) \\ : \sin \frac{1}{2} A \cdot \sin \frac{1}{2} B \cdot \sin \frac{1}{2} C.$$

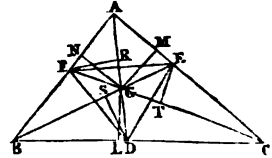


Fig. 7.

But since in every triangle

$$a + b : c :: \cos \frac{1}{2}(A - B) : \sin \frac{1}{2} C$$

$$a + c : b :: \cos \frac{1}{2}(A - C) : \sin \frac{1}{2} B$$

$$b + c : a :: \cos \frac{1}{2}(B - C) : \sin \frac{1}{2} A$$

we obtain

$$d^{(2)} \cdot e^{(2)} \cdot f^{(2)} : d^{(2)} \cdot e^{(2)} \cdot f^{(2)} :: (a + b)(a + c)(b + c) : abc$$

that is, the rectangular parallepipeda, one of which is contained under the superior segments of the transversals, and the other under the remaining or *inferior* segments are to one another in the same ratio as the parallelopipeda, one of which is contained under the sums of the sides of the triangle taken two and two, and the other under the sides of the triangle.

Cor. I. In any equilateral triangle

$$d^{(2)} : d^{(2)} :: 8a^3 : a^3$$

$$\therefore d^{(2)} : d^{(2)} :: 2 : 1.$$

That is, the superior segments of the transversals of this triad in an equilateral triangle are double the inferior segments, a fact already known from elementary geometry.

Cor. II. The angles EGM, FGN, DGL, have always this relation to one another, that one of them (that which is equal to half the difference between the greatest and least angle of the triangle) is always equal to the sum of the other two. For

$$\begin{aligned}\angle EGM &= \frac{1}{2}(A - C) = \frac{1}{2}(A - B) + \frac{1}{2}(B - C) \\ &= \angle FGN + \angle DGL\end{aligned}$$

and since in an isosceles triangle one of these angles vanishes, the other two are equal.

XIV.

In the triangles $\triangle DB, \triangle EC, \triangle FA$

$$a, : c :: d_{//} : d_{//}^{(2)}$$

$$b, : a :: e_{//} : e_{//}^{(2)}$$

$$c, : b :: f_{//} : f_{//}^{(2)}$$

$$\therefore a, . b, . c, : a . b . c :: d_{//} . e_{//} . f_{//} : d_{//} . e_{//} . f_{//}^{(2)}$$

and since we have

$$d_{//} . e_{//} . f_{//} : d_{//} . e_{//} . f_{//}^{(2)} : a . b . c : (a + b) (a + c) (b + c)$$

it follows

$$a, . b, . c, : a . b . c :: a . b . c : (a + b) (a + c) (b + c)$$

$$\text{and } a_{//} . b_{//} . c_{//} : a . b . c :: a . b . c : (a + b) (a + c) (b + c)$$

that is, the rectangular parallelopipedon contained under the three sides of the triangle is a mean proportional between the rectangular parallelopipeda, one of which is contained by the non-contiguous segments of the sides of the triangle, and the other by the sums of the sides taken two and two.

XV.

In these triangles

$$d_{//}^{(2)} = b . c, - f_{//} . f_{//}^{(2)}$$

$$e_{//}^{(2)} = c . a, - d_{//} . d_{//}^{(2)}$$

$$f_{//}^{(2)} = a . b, - e_{//} . e_{//}^{(2)}$$

$$\therefore d_{//}^2 + e_{//}^2 + f_{//}^2 = ab, + bc, + ca, - d_{//} d_{//}^{(2)} - e_{//} e_{//}^{(2)} - f_{//} f_{//}^{(2)}$$

$$\therefore d . d_{//} + e . e_{//} + f . f_{//} = ab, + bc, + ca,$$

that is, the sum of the three rectangles contained under each transversal, and its respective superior segment, is equal to the sum of the three rectangles contained under each side of the triangle and the farther segment of the succeeding side.*

* By the first side I mean the side denoted by (a), and by the second side that denoted by (b), and by the third side, that denoted by (c), so that the side b succeeds or comes after a, but precedes or goes before c.

Cor. Hence

$$d \cdot d_{//} + e \cdot e_{//} + f \cdot f_{//} = (a - b_{//}) b_{//} + (b - c_{//}) c_{//} + (c - a_{//}) a_{//}$$

XVI.

Subtracting the two preceding equations from one another, we obtain the following equation

$$\begin{aligned} & d(d_{//} - d_{//}) + e(e_{//} - e_{//}) + f(f_{//} - f_{//}) = ab_{//} + bc_{//} + ca_{//} \\ & + (b_{//} - a) b_{//} + (c_{//} - b) c_{//} + (a_{//} - c) a_{//} \\ & (d_{//} + d_{//})(d_{//} - d_{//}) + (e_{//} + e_{//})(e_{//} - e_{//}) + (f_{//} + f_{//})(f_{//} - f_{//}) = \\ & (a_{//} + a_{//}) b_{//} + (b_{//} + b_{//}) c_{//} + (c_{//} + c_{//}) a_{//} + (b_{//} - a_{//} - a_{//}) b_{//} + \\ & (c_{//} - b_{//} - b_{//}) c_{//} + (a_{//} - c_{//} - c_{//}) a_{//} \end{aligned}$$

$$\begin{aligned} \therefore d_{//}^2 + e_{//}^2 + f_{//}^2 - (d_{//}^2 + e_{//}^2 + f_{//}^2) &= a_{//} (a_{//} + b_{//}) \\ &+ b_{//} (b_{//} + c_{//}) + c_{//} (c_{//} + a_{//}) - ab_{//} - bc_{//} - ca_{//} \\ &= (a_{//} + b_{//} + c_{//}) (a_{//} + b_{//} + c_{//}) - 2(ab_{//} + bc_{//} + ca_{//}) \end{aligned}$$

that is, the difference between the sums of the squares of the superior and inferior segments of the transversals, is equal to that between the rectangle under the two sums of the non-contiguous segments of the sides, and twice the sum of the rectangles under the segments taken two and two.

XVII.

The extremities D, E, F, (Fig. 7) of the transversals being joined, and perpendiculars DS, FR, ET, being let fall from these three points on the succeeding transversal, this equation will be found

$$\begin{aligned} \Delta DEF &= \Delta DGF + \Delta DGE + \Delta EGF \\ &= \frac{1}{2} (e_{//} \cdot DS + d_{//} \cdot FR + f_{//} \cdot ET) \\ &= \frac{1}{2} r^2 \{ DS \cdot \sec \frac{1}{2} (A - C) + FR \cdot \sec \frac{1}{2} (B - C) + ET \cdot \sec \frac{1}{2} (A - B) \} \\ &= \frac{1}{2} r^2 \{ \sec \frac{1}{2} (A - C) \cdot (\sec \frac{1}{2} (B - C) \cdot \sin \frac{1}{2} (A + B) + \\ &\quad \sec \frac{1}{2} (A - B) \cdot \sec \frac{1}{2} (B - C) \cdot \sin \frac{1}{2} (A + C) + \\ &\quad \sec \frac{1}{2} (A - B) \cdot \sec \frac{1}{2} (A - C) \cdot \sin \frac{1}{2} (B + C)) \} \\ \therefore \Delta DEF &= \frac{r^2}{2 \cos \frac{1}{2} (A - B) \cdot \cos \frac{1}{2} (A - C) \cos \frac{1}{2} (B - C)} \times \end{aligned}$$

$$\{ \sin \frac{1}{2} (A + C) \cos \frac{1}{2} (A - C) + \sin \frac{1}{2} (B + C) \cdot \cos \frac{1}{2} (B - C) + \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B) \};$$

and since

$$\sin \frac{1}{2} (A + C) \cos \frac{1}{2} (A - C) = \frac{1}{2} (\sin A + \sin C) \&c.$$

$$\Delta DEF = \frac{1}{2} r^2 \frac{\sin A + \sin B + \sin C}{\cos \frac{1}{2} (A - B) \cos \frac{1}{2} (A - C) \cos \frac{1}{2} (B - C)}$$

from which since

$$\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cdot \cos \frac{1}{2} B \cdot \cos \frac{1}{2} C$$

we get

$$\Delta DEF = \frac{2 r^2 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} (A - B) \cdot \cos \frac{1}{2} (A - C) \cos \frac{1}{2} (B - C)}.$$

XVIII.

It is well known by trigonometry, that

$$\cos \frac{1}{2} (A - B) = \cos \frac{1}{2} A \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B$$

$$\cos \frac{1}{2} (A - C) = \cos \frac{1}{2} A \cdot \cos \frac{1}{2} C + \sin \frac{1}{2} A \sin \frac{1}{2} C, \text{ etc.}$$

$$\therefore \cos \frac{1}{2} (A - B) \cdot \cos \frac{1}{2} (A - C) \cos \frac{1}{2} (B - C) =$$

$$(\cos \frac{1}{2} A \cdot \cos \frac{1}{2} B + \sin \frac{1}{2} A \sin \frac{1}{2} B) (\cos \frac{1}{2} A \cos \frac{1}{2} C + \sin \frac{1}{2} A \sin \frac{1}{2} C) (\cos \frac{1}{2} B \cos \frac{1}{2} C + \sin \frac{1}{2} B \sin \frac{1}{2} C)$$

$$= \cos^3 \frac{1}{2} A \cdot \cos^2 \frac{1}{2} B \cos^2 \frac{1}{2} C +$$

$$\frac{1}{2} \{ \sin A \sin B + \sin A \sin C + \sin B \sin C \}$$

$$+ \sin^3 \frac{1}{2} A \sin^2 \frac{1}{2} B \sin^2 \frac{1}{2} C$$

$$= 1 - \cos^3 \frac{1}{2} A - \cos^3 \frac{1}{2} B - \cos^3 \frac{1}{2} C$$

$$+ \cos^3 \frac{1}{2} A \cos^3 \frac{1}{2} B + \cos^3 \frac{1}{2} A \cos^2 \frac{1}{2} C + \cos^3 \frac{1}{2} B \cos^2 \frac{1}{2} C$$

$$+ \frac{1}{2} [\sin A \sin B + \sin A \sin C + \sin B \sin C]$$

$$\therefore \Delta DEF =$$

$$\frac{2 r^2 \cos \frac{1}{2} A \cdot \cos \frac{1}{2} B \cos \frac{1}{2} C}{\left\{ \begin{array}{l} 1 - \cos^3 \frac{1}{2} A - \cos^3 \frac{1}{2} B - \cos^3 \frac{1}{2} C \\ + \cos^3 \frac{1}{2} A \cos^3 \frac{1}{2} B + \cos^3 \frac{1}{2} A \cos^2 \frac{1}{2} C + \cos^3 \frac{1}{2} B \cos^2 \frac{1}{2} C \\ + \frac{1}{2} [\sin A \cdot \sin B + \sin A \sin C + \sin B \sin C] \end{array} \right\}}$$

Substituting in this expression for $\cos \frac{1}{2} A$, $\cos \frac{1}{2} B$, $\cos \frac{1}{2} C$ and putting $s = \frac{1}{2} (a + b + c)$ we get

$$\Delta DEF =$$

$$\frac{\frac{a+b+c}{abc} r^2 \Delta}{\left\{ \begin{aligned} &1 - \frac{s}{abc} [a(s-a) + b(s-b) + c(s-c)] + \\ &\frac{s}{abc} \left(\frac{s(s-a)(s-b)}{c} + \frac{s(s-a)(s-c)}{b} + \frac{s(s-b)(s-c)}{a} \right) + \\ &\frac{s(s-a)(s-b)(s-c)}{abc} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \end{aligned} \right\}}$$

And hence since

$$a(s-a) + b(s-b) + c(s-c) = 2[(s-a)(s-b) + (s-a)(s-c) + (s-b)(s-c)] = (s-a)(s-b) + (s-a)(s-c) + (s-b)(s-c) + (s-a)(s-b) + (s-a)(s-c) + (s-b)(s-c)$$

it follows that $\Delta DEF =$

$$\frac{2r^2 \Delta}{\left\{ \begin{aligned} &\frac{abc}{s} + (s-a)(s-b)(s-c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \\ &(s-a)(s-b) - (s-a)(s-c) - (s-b)(s-c) + \\ &(s-a)(s-b)(s-c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \end{aligned} \right\}}$$

$$\therefore \Delta DEF =$$

$$\frac{2r^2 \Delta}{\left\{ \begin{aligned} &\frac{abc}{s} + 2(s-a)(s-b)(s-c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \\ &(s-a)(s-b) - (s-a)(s-c) - (s-b)(s-c) \end{aligned} \right\}}$$

from which since

$$s[(s-a)(s-b) + (s-a)(s-c) + (s-b)(s-c)] = (s-a)(s-b)(s-c) + abc \text{ we obtain}$$

$$\begin{aligned} \Delta DEF &= \frac{2sr^2 \Delta}{\left\{ \begin{aligned} &abc + 2s(s-a)(s-b)(s-c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - \\ &(s-a)(s-b)(s-c) - abc \end{aligned} \right\}} \\ &= \frac{(a+b+c)r^2 \Delta}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) 2\Delta^2 - \frac{2\Delta^2}{a+b+c}} \\ &= \frac{(a+b+c)r^2 \Delta}{[(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 1] 2\Delta^2} \end{aligned}$$

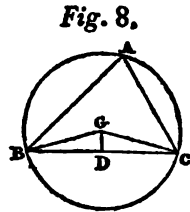
$$= \frac{2\Delta}{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 1}$$

which remarkable property is not capable of an easy enunciation in words.

XIX.

Although the third triad has some remarkable properties, yet as they have been admirably discussed by Feurbach, I shall not dwell upon them. One however, there is, which I do not remember ever to have seen and I shall therefore give it a place here. Since

$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$,* it follows that



$$\begin{aligned} \frac{ab+ac+bc}{\sin A \sin B \sin C} &= \frac{4R^2}{\sin A} + \frac{4R^2}{\sin B} + \frac{4R^2}{\sin C} \\ &= 4R^2 \left(\frac{\sin A \sin B + \sin A \sin C + \sin B \sin C}{\sin A \sin B \sin C} \right) \end{aligned}$$

$$\begin{aligned} \therefore ab+ac+bc &= 4R^2 (\sin A \sin B + \sin A \sin C + \sin B \sin C) \\ &= 4R^2 \left(\frac{4\Delta^2}{abc^2} + \frac{4\Delta^2}{ab^2c} + \frac{4\Delta^2}{a^2bc} \right) \\ &= 4R^2 \frac{2\Delta}{abc} \left(\frac{2\Delta}{c} + \frac{2\Delta}{b} + \frac{2\Delta}{a} \right) \\ &= 2R \left(\overset{(3)}{d} + \overset{(3)}{e} + \overset{(3)}{f} \right) \end{aligned}$$

that is, the sum of the three rectangles contained under the sides of the triangle (combined by twos) is equal to the rectangle contained under the diameter of the circumscribed circle and the sum of the three perpendiculars from the angles on the opposite sides.

* The following is an easy demonstration of this proposition:

$$\frac{CD}{CG} = \frac{a}{2R} = \sin CGD = \sin \frac{1}{2} BGC = \sin A;$$

$$\therefore 2R = \frac{a}{\sin A} \text{ etc (fig. 8).}$$

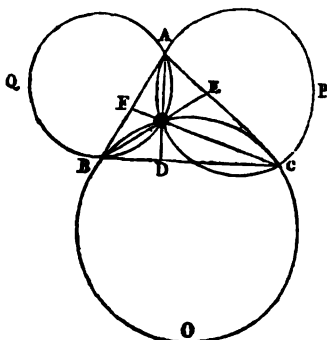
XX.

The fourth triad is that in which $\Delta = B = C$.

In the first place let us investigate the mode of drawing the transversals so as to form this triad; for this is not obvious, nor does it appear from the elementary parts of geometry.

Let three circles be drawn, each of which has one side of the triangle for its chord and the succeeding side for its tangent; these circles will intersect one another in the same point as the transversals of this triad. It is evident that any two of the circles will intersect in some two points. Let G be the point through which, (besides the other point c) the circles ACP and BCO pass, then will the third circle pass through G . For the circle passing through ABC will evidently have c for a chord; and since from the construction $\angle BAG = \angle ACG = \angle BCG$ it will also touch AB , whence it coincides with the third circle, which fulfils the conditions.

Fig. 9.



XXI.

The angles under which the transversals cut one another are easily seen to be equal to Δ , B and C , that is

$$\alpha = \Delta$$

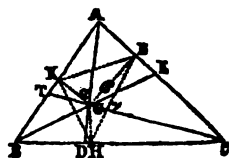
$$\beta = B$$

$$\gamma = C$$

If perpendiculars be let fall from G on the sides of the triangle ABC and the points H, I, K be joined, the triangle HIK will be similar to ABC ; and if transversals be drawn in the triangle HIK so as to form the third triad, the point of their intersection will be G .

Since the quadrilaterals $AI GK BH GK$ and $CH GI$ are inscribable in a circle

Fig. 10.



$$\angle KIG = \angle KAG = \angle ACF$$

$$\angle HIG = \angle GCH \text{ and therefore}$$

$$\angle HIK = C \text{ and in the same way}$$

$$KHI = B$$

$$HKI = A$$

$$\angle KIG = \angle BAH = \angle CBE = \angle HKG = \angle ACF \\ = \angle GHI$$

from which the truth of the proposition stated above is evident. If the same thing be done in the triangle HIK as in the original triangle ABC , a third triangle will be formed similar to the two others; and in this triangle the transversals of the fourth triad will also pass through the same point G . And so on with all succeeding triangles.

XXII.

In every triangle perpendiculars from any point being let fall on the sides as GH, GI, GK , from G on BC, AC , and AB , we have by the elementary part of geometry

$$a.BH + b.CI + c.AK = a.CH + b.AI + c.BK \\ = \frac{1}{2} (a^2 + b^2 + c^2)$$

and therefore

$$\frac{1}{2} (a^2 + b^2 + c^2) \tan A \stackrel{(4)}{=} \frac{1}{2} [a.BH + b.CI + c.AK] \tan A \stackrel{(4)}{=} \\ = \frac{1}{2} [a.GH + b.GI + c.GK] \\ = \Delta$$

whence since by trigonometry

$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta} \text{ we get}$$

$$\cot A \stackrel{(4)}{=} \frac{a^2 + b^2 + c^2}{4\Delta} = \cot A + \cot B + \cot C =$$

that is, the cotangent of the angle made in this triad by each of the transversals with the side of the triangle equals the sum of the cotangents of the three angles of the triangle.

Cor. I.

$$(1) \operatorname{cosec}^2 A \stackrel{(4)}{=} \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

* This elegant theorem is due to Crelle, who has given several demonstrations of it widely different from mine in the above-mentioned work.

(2) If one of the angles A, B, C, be a right angle, as for instance A

$$\begin{aligned} \cot A, &= \cot B + \cot C = 2 \operatorname{cosec} 2B = 2 \operatorname{cosec} 2C \\ &= \operatorname{cosec} 2B + \operatorname{cosec} 2C. \end{aligned}$$

XXIII.

The triangles BGC and HGI are similar : whence

$$a : GC :: HI : GI$$

$$\therefore HI = a \cdot \frac{GI}{GC} = a \cdot \sin A, \quad (4)$$

$$\begin{aligned} \therefore \Delta : \Delta HIK :: a^2 :: a^2 \sin^2 A, &:: 1 : \sin^2 A, \\ &:: \operatorname{cosec}^2 A, : 1 \end{aligned}$$

$$:: \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C : 1.$$

Corol. If the triangles are both isosceles and right angles, we obtain this proportion :

$$\Delta : \Delta HIK :: 1 + 2 \operatorname{cosec}^2 \frac{1}{2} B : 1 :: 5 : 1$$

that is, the primary triangle is to the secondary as 5 to one.

XXIV.

Since the triangles BGD, CGE and AGF are similar respectively to BAD, CBE, ACF

$$a^2, = d \cdot d'', \quad (4)$$

$$b,^2 = e \cdot e'', \quad c,^2 = f \cdot f'', \quad (4) \text{ whence}$$

$d \cdot e \cdot f : a, \cdot b, \cdot c, :: a, \cdot b, \cdot c, : d, \cdot e, \cdot f, \quad (4)$ that is, the rectangular parallelopipedon contained under the three non-contiguous segments of the sides is a mean proportional between the parallelopipeda, one of which is contained under the transversals, and the other under their inferior segments.

XXV.

In the same way

$$a, : e, :: d : c \quad (4)$$

$$b, : f, :: e : a$$

$$c, \cdot d, :: f : b$$

$$\therefore a, b, c, : d, e, f, :: d^{(4)}e^{(4)}f^{(4)} : abc$$

and $\therefore d, e, f, : d, e, f, :: abc : a, b, c, (24)$. But Crelle has shown, that

$$a, = a \frac{c^2}{a^2 + c^2}, b, = b \frac{a^2}{a^2 + b^2}, c, = c \frac{b^2}{a^2 + c^2}$$

whence

$$d, e, f, : d, e, f, :: abc : \frac{a^3 b^3 c^3}{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)}$$

$$d, e, f, : d, e, f, :: (a^2 + b^2)(a^2 + c^2)(b^2 + c^2) : a^3 b^3 c^3 :$$

this expression I have given chiefly, that it might be compared with those obtained above (Art, 13), And it is worth while also to compare the following expressions :

$$d, e, f, : d, e, f, :: abc : a, b, c, (1)$$

$$d, e, f, : d, e, f, :: abc : a, b, c, (14)$$

$$d, e, f, : d, e, f, :: abc : a, b, c, *$$

$$d, e, f, : d, e, f, :: abc : a, b, c, (4)$$

$$d, e, f, : d, e, f, :: abc : a, b, c, (28)$$

XXVI.

If a circle be circumscribed about the triangle ABC, and if the transversals produced meet the circumference in P, Q, R, and AR, BP, &c. being joined

$$AR = BP = CQ;$$

and also the triangles AGQ, AGR, BGP, BGR, CGP, CGQ will be similar to each and to the original triangle ABC; and therefore

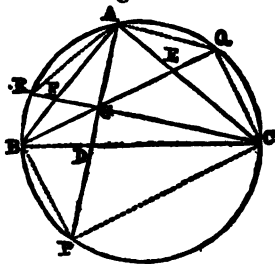
$$AG : BG :: AQ : BP$$

$$BG : CG :: BR : CQ$$

$$CG : AG :: CP : AR$$

$$\therefore AQ \cdot BR \cdot CP = AR \cdot BP \cdot CQ = AR^3,$$

Fig. 11.



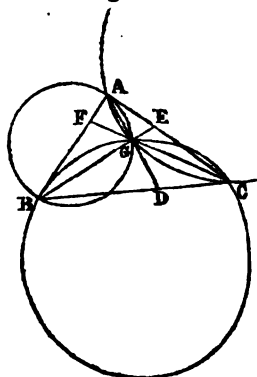
* The truth of this is obvious from the similarity of the triangles AGF, and ABD etc.

Hence we find that if in a triangle whose sides are $\Delta R + \Delta Q$, $\Delta R + BR$ and $\Delta R + CP$ transversals be drawn, so that $a, = b, = c, = \Delta R$, or that $a_{//} = b_{//} = c_{//} = \Delta R$, these transversals will intersect one another in the same point.

XXVII.

The fifth triad in which $A_{//} = B_{//} = C_{//}$ is very similar to the last. In the first place, the construction by which the transversals are drawn so as to fulfil the conditions, is very similar. Let three circles be described, having one side of the triangle for a chord, and the *preceding* side for a tangent; then the point of intersection of the three circles will be that through which the transversals of this triad pass. In this triad, not only will the angles at which the transversals cut one another be respectively equal to A, B and C , but also will

Fig. 12.



⁽⁵⁾
 $\cot A_{//} = \cot A + \cot B + \cot C$
 as is easily seen; it may be demonstrated in a way similar to that of Crelle : that

$$\begin{aligned} a, &= a \frac{a^2}{a^2 + b^2}, \quad b, = b \frac{b^2}{b^2 + c^2}, \quad c, = c \frac{c^2}{a^2 + c^2} \\ a_{//} &= a \frac{b^2}{a^2 + b^2}, \quad b_{//} = b \frac{c^2}{b^2 + c^2}, \quad c_{//} = c \frac{a^2}{a^2 + c^2}. \end{aligned}$$

XXVIII.

From these expressions we easily obtain the following :

$$\begin{aligned} (1) \quad A, &= B, = C, = A_{//} = B_{//} = C, \\ (2) \quad a, b, c, &= a_{//} b_{//} c_{//} = \frac{a^2 b^2 c^2}{(a^2 + b^2)(a^2 + c^2)(b^2 + c^2)} \\ &= a_{//} b_{//} c_{//} \end{aligned}$$

that is, the rectangular parallelopipeda under the non-contiguous segments of the sides in the fourth triad, are equal to the similar parallelopipeda in the fifth triad.

(3) $def = def$. For we have

$$\therefore def : a, b, c :: \sin A \sin B \sin C : \sin^3 A,$$

$$\text{and } def : a'', b'', c'' :: \sin A \sin B \sin C : \sin^3 A'',$$

that is, the rectangular parallelopipeda under the transversals of these two triads are equal to one another.

$$(4) d_{f''} f'' = d_{f''} f'' \text{ for}$$

$$def : a, b, c :: a, b, c : d_{f''} f'' \text{ and}$$

$$def : a'', b'', c'' :: a'', b'', c'' : d_{f''} f''$$

$$(5) d_{f'} f' = d_{f'} f'$$

For if AD, BE, CF be transversals of the fourth triad, and AH, BI, CK be those of the fifth, we necessarily have these proportions :

$$d_f : b :: \sin A' : \sin A$$

$$d_f : c :: \sin A'' : \sin A$$

$\therefore d_f : d_f :: b : c$ and in the same way,

$$e_f : e_f :: c : a$$

and

$$\text{and } f_f : f_f :: a : b$$

$$\therefore d_{f'} f' = d_{f'} f',$$

therefore the two rectangular parallelopipeda contained the one under the superior, and the other under the inferior segments of the transversals of the fourth triad, are equal to the similar parallelopipeda in the fifth triad.

Cor. From the proportion just given :

$$d_f : d_f :: b : c,$$

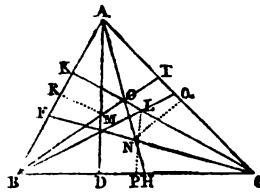
it follows that the superior segments of the *same* transversals in the fourth and fifth triads are, when the triangle is isosceles, equal to one another.

XXIX.

If $\triangle ABC$ and $\triangle A'B'C'$ be two similar triangles, having
VOL. VI. PART III.

μ

Fig. 13.



$d_{e,f}^{(4)} = d_{e,f}^{(5)}$, they will necessarily be equal to one another.
For

$$d_{e,f}^{(4)} = abc \frac{\sin^3 A}{\sin A \sin B \sin C}$$

$$d_{e,f}^{(5)} = abc \frac{\sin^3 B}{\sin A \sin B \sin C}$$

$\therefore$ Since by hyp.

$$d_{e,f}^{(4)} = d_{e,f}^{(5)} \text{ and (28)}$$

$$A = B$$

$$\text{we get } abc = abc$$

$$\therefore a = a, b = b, c = c.$$

for if it were possible that

$$a > a \quad \text{it would follow that}$$

$$b > b$$

$$c > c \quad \text{because the triangles are}$$

by hypothesis, similar. Since, therefore, the three sides of the one are equal to the three sides of the other, the triangles must be equal.

XXX.

If the transversals of the fourth and fifth triads be drawn, three isosceles triangles, AMB, BLC, ANC are formed, the sum of which equals the original triangle. For

$$\triangle AMB = \frac{1}{2} c \cdot MR = \frac{1}{2} c^2 \cdot \tan A,$$

$$\triangle BLC = \frac{1}{2} a^2 \cdot \tan A,$$

$$\triangle ANC = \frac{1}{2} b^2 \cdot \tan A,$$

$$\therefore \triangle AMB + \triangle BLC + \triangle ANC = \frac{1}{2} (a^2 + b^2 + c^2) \tan A, = \triangle (28.)$$

Cor. Whence we find that

$$\triangle CGL = \triangle AGN + \triangle BGM$$

$$\triangle CON = \triangle AOM + \triangle BOL.$$

XXXI

The sixth triad is that in which the transversals cut the sides so that

$$a, = c_{//}, b, = a_{//}, c, = b_{//}$$

In the first place, since from elementary geometry it is clear that the transversals cut the sides in the same points as the inscribed circle touches them, it is easy to construct the triad. It is also evident that

$$\begin{aligned} a, = c_{//} &= \frac{1}{2} (a - b + c); & b, = a_{//} &= \frac{1}{2} (a + b - c) \\ c, = b_{//} &= \frac{1}{2} (-a + b + c) \end{aligned}$$

$$\begin{aligned} \therefore a, b, c, &= a_{//}, b_{//}, c_{//} = \frac{1}{8} (-a + b + c) (a - b + c) (a + b - c) \\ &= \frac{2\Delta^2}{a + b + c} = r. \Delta \end{aligned}$$

that is, the rectangular parallelopipedon contained under the non-contiguous segments of the sides, is equal to a prism whose base is the triangle, and altitude, the radius of the inscribed circle.

XXXII.

The extremities D, E, F, of the transversals being joined, the triangle thus formed will have this property,

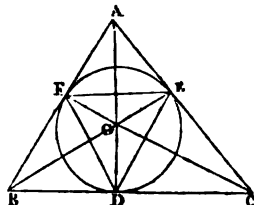
$$r. \Delta = 2R. \Delta DEF.$$

This was demonstrated by Feuerbach,* and from it, it follows that

$$a, b, c, = a_{//}, b_{//}, c_{//} = 2R. \Delta DEF$$

that is, the parallelopipedon contained (as above) by the non-contiguous segments of the sides is equal to the prism whose base is the triangle inscribed in the original triangle, and altitude the radius of the inscribed circle.

Fig. 14.



XXXIII.

Since, in every case

$$(-a + b + c) (a - b + c) + (-a + b + c) (a + b - c) + (a - b + c) (a + b - c) =$$

$$a (-a + b + c) + b (a - b + c) + c (a + b - c),$$

we get the following equation :

* See Feuerbach, in the above-mentioned work, (§. 6.)

$$\begin{aligned}
 a, b, + a, c, + b, c, &= a, h, + a, c, + b, c, = \\
 \frac{1}{2} (-a + b + c) (a - b + c) + \frac{1}{2} (-a + b + c) (a + b - c) + \\
 \frac{1}{2} (a - b + c) (a + b - c) &= \\
 \frac{1}{2} [a (-a + b + c) + b (a - b + c) + c (a + b - c)] &= \\
 \frac{1}{2} [ac, + b, a, + cb,] &
 \end{aligned}$$

that is, the sum of the rectangles contained under the non-contiguous segments of the sides is equal to the half of the sum of the three rectangles, each of which is contained under one side of the triangle, and the preceding non-contiguous segments of a side.

XXXIV.

The following equation is evidently true :

$$\begin{aligned}
 (-a + b + c) (a - b + c) + (-a + b + c) (a + b - c) + \\
 (a - b + c) (a + b - c) &= 2 (ab + ac + bc) - (a^2 + b^2 + c^2) \\
 \text{whence} &
 \end{aligned}$$

$$\begin{aligned}
 a, b, + a, c, + b, c, &= a, b, + a, c, + b, c, = r (r, + r, + r,) \\
 \text{that is, the sum of the three rectangles contained under the non-} & \\
 \text{contiguous segments of the sides taken two and two, is equal} & \\
 \text{to the rectangle, one side of which is the radius of the inscribed} & \\
 \text{circle, and the other the sum of the radii of the three circles} & \\
 \text{which touch the sides of the triangle.} &
 \end{aligned}$$

XXXV.

Since in every case

$$\begin{aligned}
 a (a - b + c) + b (a + b - c) + c (-a + b + c) &= \\
 a (a + b - c) + b (-a + b + c) + c (a - b + c) &= \\
 = a^2 + b^2 + c^2 & \quad \text{we find that}
 \end{aligned}$$

$$\begin{aligned}
 aa, + bb, + cc, &= na, + bb, + cc, = \frac{1}{2} (a^2 + b^2 + c^2) \\
 \text{that is, the sum of the rectangles contained under each side of} & \\
 \text{the triangle and one of its segments is equal to the sum of the} & \\
 \text{other three rectangles contained under each side of the triangle} & \\
 \text{and its segment.} &
 \end{aligned}$$

Cor. Since it is evident from the thing itself that

$$\begin{aligned}
 aa, + bb, + cc, &= aa, + bb, + cc, = \frac{1}{2} (a^2 + b^2 + c^2) \\
 \text{and since from the elementary proposition, quoted (22) we have} &
 \end{aligned}$$

$aa, + \overset{(2)}{bb,} + cc, = aa,, + \overset{(2)}{bb,,} + cc,, = \frac{1}{2} (a^2 + b^2 + c^2)$
it follows that

$$\begin{aligned} aa, + \overset{(1)}{bb,} + cc, &= aa,, + \overset{(1)}{bb,,} + cc,, \\ &= aa, + \overset{(2)}{bb,} + cc, = aa,, + \overset{(2)}{bb,,} + cc,, \\ &= aa, + \overset{(3)}{bb,} + cc, = aa,, + \overset{(3)}{bb,,} + cc,, \\ &= aa, + \overset{(4)}{bb,} + cc, = aa,, + \overset{(4)}{bb,,} + cc,, \\ &= aa, + \overset{(5)}{bb,} + cc, = aa,, + \overset{(5)}{bb,,} + cc,, \dots (38) \end{aligned}$$

XXXVI.

It is easily seen that

$$\begin{aligned} abc - a(-a + b + c)^2 - b(a - b + c)^2 - c(a + b - c)^2 \\ = (-a + b + c)(a - b + c)(a + b - c) \end{aligned}$$

wherefore

$$\begin{aligned} ac,^2 + \overset{(6)}{ba,^2} + cb,^2 &= abc - \frac{1}{4} (-a + b + c)(a - b + c)(a + b - c) \\ &= 4R \Delta - \frac{4 \Delta^2}{a + b + c} = (2R - r) 2 \Delta \end{aligned}$$

that is, the sum of the three rectangular parallepipeda, each of which has one side of the triangle for its altitude, and the square of the non-contiguous segment of the preceding side for its base, is equal to the prism whose base is twice the triangle, and altitude is the difference between the diameter of the circumscribed and the radius of the inscribed circle.

Cor. Hence

$$\begin{aligned} ac,^2 + \overset{(6)}{ba,^2} + cb,^2 + 2a,b,c, + 2aa,,^2 + 2bb,,^2 + 2cc,,^2 \\ + a,,b,,c,, = 4R \cdot \Delta = abc. \end{aligned}$$

XXXVII.

If a circle DEF be inscribed in the original triangle (Fig. 14.) and the triangle DEF be inscribed in this circle, then necessarily,

$$DE \cdot DF \cdot EF = 4r \Delta$$

and therefore

$$DE \cdot DF \cdot EF = 4a, \cdot b, \cdot c, (31)$$

that is, the parallelopipedon contained under the lines joining the ends of the transversals is equal to four times the parallelopipedon under the three non-contiguous segments of the sides.

XXXVIII.

The seventh and last triad is that where

$$a_{,} = b_{,,}, \quad b_{,} = c_{,,}, \quad c_{,} = a_{,,}$$

and is intimately connected with the preceding. For it is easily seen that

$$a_{,} = \frac{1}{2} (a + b - c), \quad b_{,} = \frac{1}{2} (-a + b + c), \quad c_{,} = \frac{1}{2} (a - b + c)$$

wherefore

$$a_{,} = a_{,,}, \quad b_{,} = b_{,,}, \quad c_{,} = c_{,,}$$

that is, each of these triads becomes the other, when the segments of the sides are interchanged. Wherefore, I shall not dwell longer on this triad.

XXXIX.

Let me be permitted to add in this, a proposition which has reference to transversals intersecting one another in the same point.

If from the vertices ABC of any triangle, AI, AK, HB, BK, CH, CI, be drawn, so that

$$\angle BAK = \angle CAI; \quad \angle ABK = \angle CBH$$

$$\angle BCH = \angle ACI:$$

then the lines AH, BI, CK will pass through the same point. For it is evident that

$$AI : IF :: \sin AEI : \sin \phi$$

$$IE : CI :: \sin \phi_{,,} : \sin IEC$$

$$CH : HD :: \sin CDH : \sin \phi_{,,}$$

$$HD : BH :: \sin \phi_{,} : \sin BDH$$

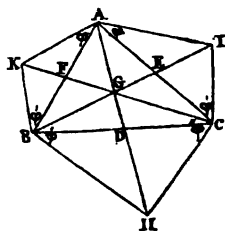
$$BK : FK :: \sin BFK : \sin \phi_{,}$$

$$FK : AK :: \sin \phi : \sin AFK$$

$$\therefore AI \cdot BK \cdot CH = AK \cdot BH \cdot CI$$

and therefore since

Fig. 15.



$$BI : CI :: \sin (C + \phi_{//}) : \sin B$$

$$AI : BI :: \sin B_{//} : \sin (A + \phi_{//})$$

$$CH : AH :: \sin A_{//} : \sin C + \phi_{//})$$

$$AH : BH :: \sin (B + \phi_{//}) : \sin A,$$

$$CK : AK :: \sin (A + \phi) : \sin C,$$

$$BK : CK :: \sin C_{//} : \sin (B + \phi_{//})$$

$\therefore AI \cdot BK \cdot CH : AK \cdot BH \cdot CI :: \sin A_{//} \sin B_{//} \sin C_{//} : \sin A \sin B \sin C$, whence since

$$AI \cdot BK \cdot CH = AK \cdot BH \cdot CI$$

$$\sin A, \sin B, \sin C, = \sin A_{//} \sin B_{//} \sin C_{//}$$

and therefore the transversals AH, BI, CK will pass through the same point. (3 Cor.)

Sch. The demonstration shews that the proposition will hold good if the lines AI, AK , &c. be drawn to the interior of the triangle.

Cor. 1. Since the angles $\phi, \phi, \phi_{//}$ are arbitrary, it is plain that it is possible to make

$$\phi + \frac{1}{2} A = \phi, + \frac{1}{2} B = \phi_{//} + \frac{1}{2} C = 90^\circ$$

and then $AI, AK; BH, BK; CH, CI$ will either form (by twos) straight lines, or there will be a triangle HIK .

But from the hypothesis

$$\phi + \frac{1}{2} A + \phi, + \frac{1}{2} B + \phi_{//} + \frac{1}{2} C = 3R$$

$$\therefore \phi + \phi, + \phi_{//} = 2R.$$

Besides

$$\phi + \phi, + K = \phi + \phi_{//} + I = \phi, + \phi_{//} + H = 2R$$

$$\therefore \phi = H, \phi, = I, \phi_{//} = K$$

and therefore circles may be inscribed in the quadrilaterals $\Delta BHI, \Delta CHK$ and ΔCIK . Wherefore

$$\angle CAH = \angle HKC = \angle BIC = \angle BAH = \frac{1}{2} A$$

$$\therefore \angle HAK = \angle HAI = 90^\circ$$

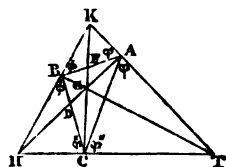
Similarly

$$\angle IBH = \angle IBK = \angle KCH = \angle KCF$$

that is, the transversals meet the sides of the triangles ABC, HIK at right angles.

Cor. 2. Since it makes no difference which we consider as the original triangle, we easily obtain this theorem: If from the angles of any acute angled triangle, perpendiculars be let fall on

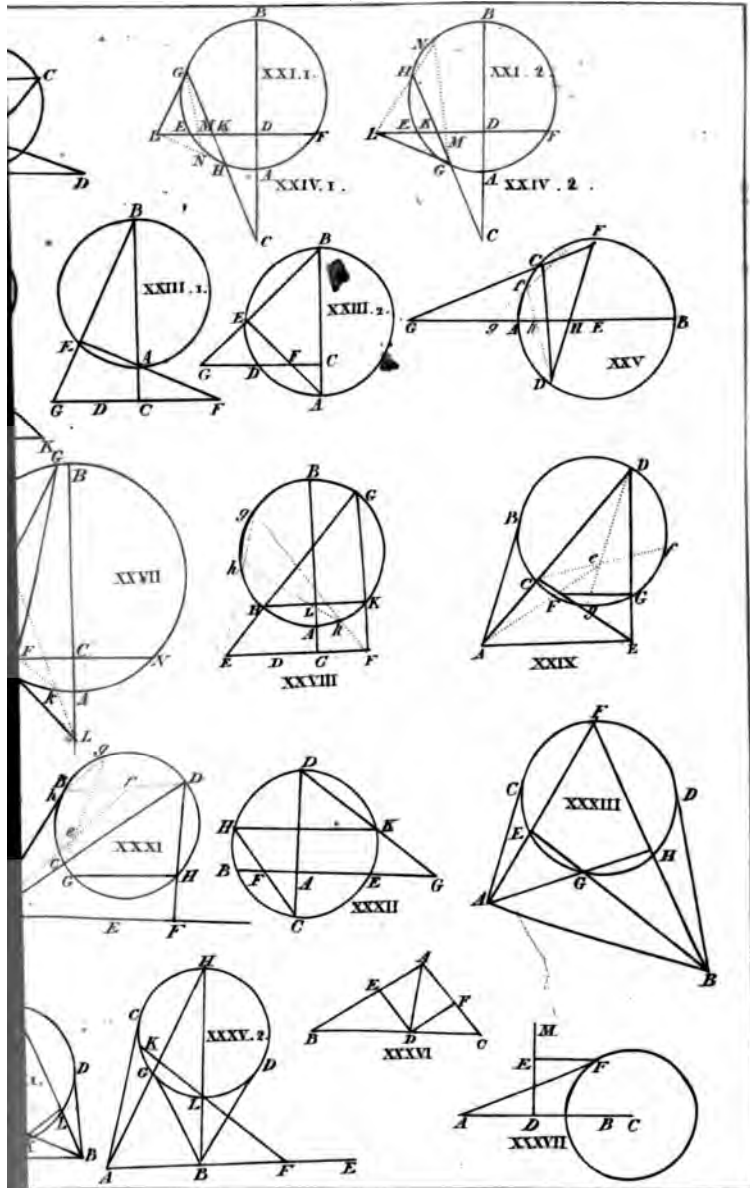
Fig. 16.



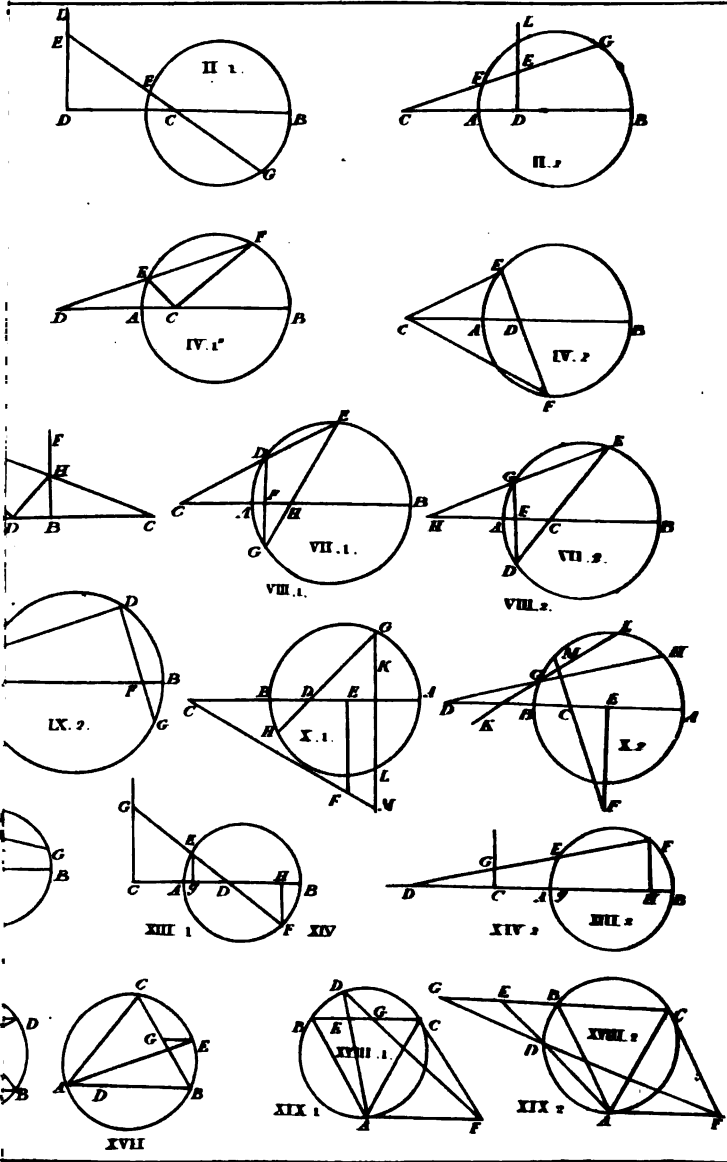
the sides, the common point of the perpendiculars will be the centre of the circle inscribed in the triangle formed by joining the feet of the perpendiculars.

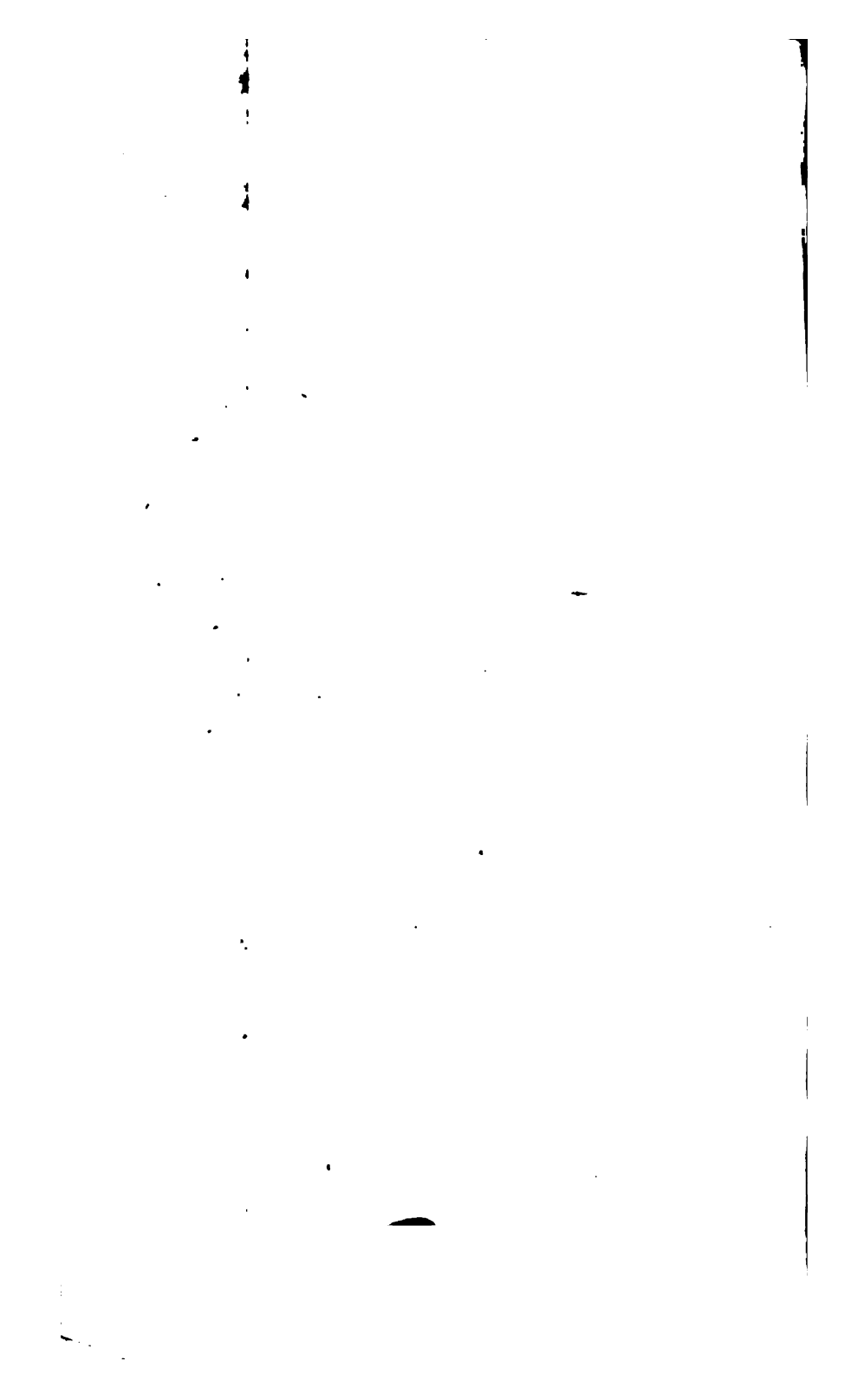
There are many other properties of the same kind, which are too well known to induce me to dwell on them.

END OF PART III. VOL. VI.

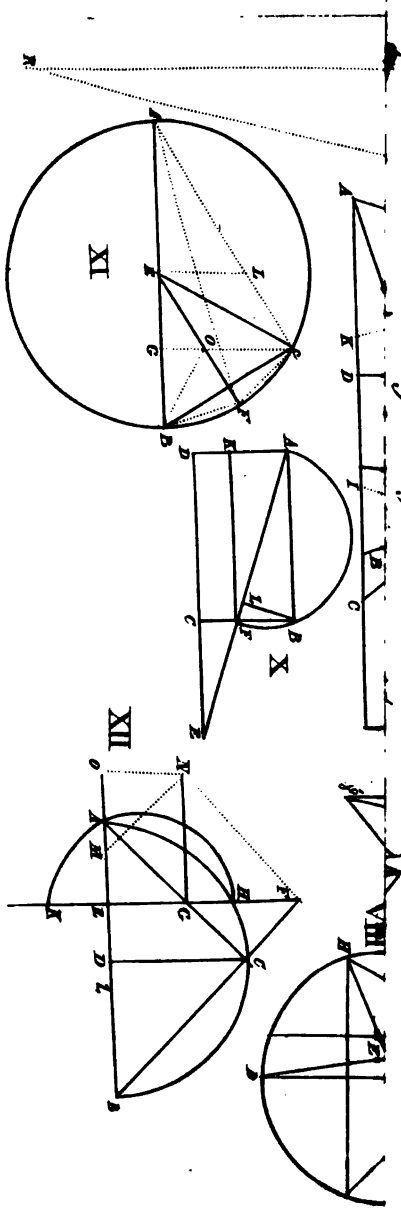


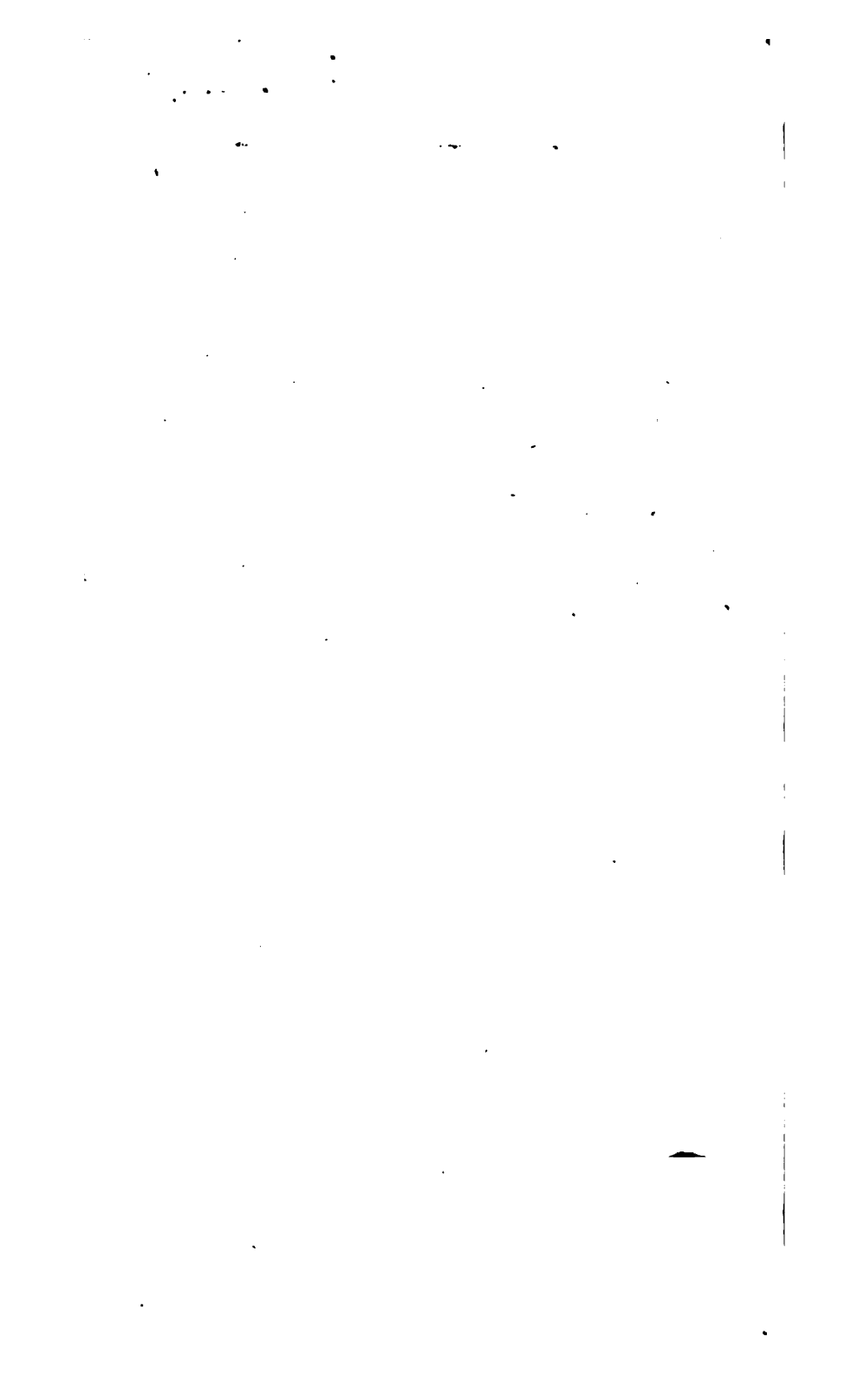
Theorems — to face Page 26, Part 3, Vol. VI. Repository.





*Figures to Willbours' Demonstrations of Lawsons' Geom. Theorems.
 to page Page 26, Part 3, Vol VI, Repository.*





CAMBRIDGE PROBLEMS.

THE SENATE-HOUSE PROBLEMS.

*Given to the Candidates for Honors during the Examination
for the degree of B. A.*

BY THE TWO MODERATORS.

MODERATORS,

Mr. HANSON, CAIUS. Mr. KING, Queen's.

EXAMINERS,

Mr. MILLER, St. John's. Mr. BOWSTEAD, Corpus.

FRIDAY MORNING.—*January 15, 1830.*

9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. IN any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the sum of the squares described upon the sides containing the right angle.

2. The sides about the equal angles of equiangular triangles are proportional, and those which are opposite to the equal angles are homologous sides.

3. Divide 1538 feet $9\frac{1}{2}$ inches by 81 feet 9 inches.

4. Find the amount of an annuity (A) in n years, and the present worth of £140. per annum for ever at 5 per cent.

5. In any number, if a point be placed over every third digit, beginning with the one on the right hand, shew that the number of digits in the cube root is equal to the number of such points; and extract the cube root of 318.61199 to 2 places of decimals.

6. Investigate a rule for finding the greatest common measure

of any two algebraical quantities, and shew that a factor of any divisor, which is contained in the corresponding dividend, may be removed without affecting the result.

7. Solve the following equations:

$$(1) \sqrt{x+16} = 2 + \sqrt{x},$$

$$(2) x^2 + y^2 = 1001,$$

$$(3) x + y = 11,$$

$$(3) \sqrt{(1+x)^2} = \sqrt{(1-x)^2} = \sqrt{1-x^2}.$$

8. The coefficients of the terms of an expanded binomial are whole numbers, when the index is a whole number; and the coefficients of the terms equidistant from the extremes are equal.

9. If two magnitudes, when substituted for the unknown quantity in an equation, give results affected with different signs, an odd number of roots lie between them; but if they give results with the same sign, either no root or an even number of roots lies between them.

10. An equation of n dimensions has n equal roots; shew how to find them, and solve the equation

$$x^4 + 13x^3 + 33x^2 + 81x + 10 = 0,$$

which has three equal roots.

11. Prove that in any system of logarithms

$$\log MN = \log M + \log N, \log \frac{M}{N} = \log M - \log N,$$

$$\text{and } \log M^r = r \log M.$$

Explain the advantages of Briggs's system, and having a table constructed for one system, give the method of constructing a table for a different system.

12. Find the sines of the sum and difference of two arcs in terms of the sines and cosines of the arcs themselves.

13. Given the sides of a plane triangle, find the cosine of an angle; investigate formulae adapted to logarithmic computation for the solution of the triangle, and explain which of the methods is best in particular cases.

14. Given $\tan A, \tan B, \dots$ find $\tan (A \pm B \pm \dots)$ and thence deduce $\tan 7A$.

15. Prove Napier's rule for the solution of right-angled triangles when one of the sides is the middle part; and having given one side and an angle opposite to it, solve the triangle, and explain whether there is any ambiguity.

16. If the distance of any point P from a fixed point S be in a given ratio to its distance PM from a fixed line, find the rect-

angular equation to the locus of P , according as SP is equal to, less, or greater than PM .

17. In the ellipse, $\frac{PV \cdot VG}{QV^2} = \frac{CP^2}{CD^2}$.

18. Investigate a rule for transforming numbers from one scale of notation to another, and transform 42.36 from the denary scale to the scale of 5.

19. In the expansion of $(a + b + c + \dots)^n$, find the coefficient of $a^p \cdot b^q \cdot c^r \cdot \dots$.

20. If m be a prime number, and a a number not divisible by m , then $a^{m-1} - 1$ is divisible by m .

21. Find the equation to a straight line passing through a given point, and cutting a given straight line at a given angle. Required also the co-ordinates of the point of intersection.

FRIDAY AFTERNOON.... Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. A cord passing round a fixed point is drawn in different directions by two equal forces acting at a given angle; find the pressure on the point.

2. Explain the method of graduating the common steelyard.

3. Find the relation between the power and the weight when there is an equilibrium on the inclined plane.

4. Two bodies whose common elasticity is e , moving with given velocities, impinge directly upon each other; it is required to determine their velocities after impact.

5. Define *specific gravity*: what is the relation between the weight, volume, and specific gravity of any substance? Explain the meaning of the numbers given in tables of specific gravity.

6. Describe Nicholson's Hydrometer, and shew how the specific gravity of a fluid may be found by it.

7. Explain the experiment by which it is ascertained that the density of air is proportional to the force which compresses it.

8. Describe the common pump, and find the least play of the piston which will enable the pump to work, the lower valve being at the surface of the water.

9. Having given the position of an object placed between two plane mirrors, inclined to each other at a given angle, find the number and position of the images.

10. Diverging rays are incident nearly perpendicularly upon

a given spherical refracting surface; having given the focus of incidence, find the focus after refraction, and prove that the conjugate foci move in the same direction upon the axis of the surface.

11. Explain the construction of the divided object glass micrometer, and shew its use.

12. Enunciate and prove Newton, Lemma 9.

13. A body is projected round a centre of force varying as the distance, with a given velocity, in a given direction; find the magnitude and position of the axes of the orbit described, and also the periodic time. Newton, Prop. x. Cors. 1 and 2.

14. A body revolves in a parabola, find the law of the force tending to the focus. Newton, Prop. XIII.

15. In different conic sections described round the same centre of force, situated in the focus; the latera recta are as the squares of the areas described in a given time. Newton, Prop. XIV.

16. Explain what is meant by the *error of the line of collimation*, and shew how it may be avoided.

17. Having given the sidereal time of any phenomenon, find the corresponding mean solar time.

18. Find the latitude of the place of observation, from two equal altitudes of the sun before and after noon, and the time between.

19. Determine the precession in north polar distance and right ascension, and shew when it is additive and when subtractive.

EVENING PROBLEMS.

1. Find the value of $\frac{1}{(\sin \theta)^2} - \frac{1}{\theta^2}$, when $\theta = 0$.

2. A person spends in the first year m times the interest of his property, in the second am times that of the remainder, in the third $3m$ times that at the end of the second, and so on; and at the end of $2p$ years he has nothing remaining; shew that in the p^{th} year he spends as much as he has left at the end of that year.

3. Given $\tan \theta + \tan \phi + \tan \psi = 1 + \frac{4}{\sqrt{3}}$
 $\tan \theta \cdot \tan \phi + \tan \theta \cdot \tan \psi + \tan \phi \cdot \tan \psi = 1 + \frac{4}{\sqrt{3}}$
 $\tan \theta \cdot \tan \phi \cdot \tan \psi = 1$

find θ , ϕ , and ψ ; and sum the series.

$$(\sec \theta)^2 + \left(\frac{1}{2} \sec \frac{\theta}{2}\right)^2 + \left(\frac{1}{2^2} \sec \frac{\theta}{2^2}\right)^2 + \left(\frac{1}{2^3} \sec \frac{\theta}{2^3}\right)^2 + \dots \text{ad inf.}$$

4. In any polygon with n sides $A_1 A_2$, $A_2 A_3$, ... respectively represented by a_1 , a_2 , ... prove that

$$a_1 \sin A_1 - a_2 \sin (A_1 + A_2) + a_3 \sin (A_1 + A_2 + A_3) - \dots + a_{n-1} \sin (A_1 + A_2 + A_3 + \dots + A_{n-1}) = 0.$$

5. A ray of light is refracted through a prism, the angle of which is 60° and index of refraction $\sqrt{2}$, so as to undergo the least possible deviation; determine that deviation. Shew also that no ray can be directly transmitted through a prism of the same refracting power when the angle exceeds 90° .

$$\left. \begin{aligned} 6. \text{ If } aX + bY + cZ = 0 \\ a_1X + b_1Y + c_1Z = 0 \end{aligned} \right\} \text{ where } \begin{aligned} X &= ax + a_1x_1 + a_2, \\ Y &= bx + b_1x_1 + b_2, \\ Z &= cx + c_1x_1 + c_2, \end{aligned}$$

then $X^2 + Y^2 + Z^2$

$$= \frac{\{a_2bc_1 - b_1c + b_2a_1c - ac_1 + c_2ab_1 - a_1b\}^2}{(bc_1 - b_1c)^2 + (a_1c - ac_1)^2 + (ab_1 - a_1b)^2}.$$

7. Trace the curve the equation to which is $y = e^{\sin x}$, and express in a series the area which recurs.

8. A perfectly smooth rod in a vertical plane revolves uniformly round a vertical axis, and a ring placed on it is attracted to a horizontal plane by a force varying as the distance in addition to the uniform force of gravity; required the form of the rod that the ring may remain on whatever point it is placed.

9. An ellipse may be constructed so that if any abscissa be taken to represent the aberration in longitude of a given star, the corresponding ordinate will represent the aberration in latitude, co-ordinates being measured from the centre along the axes; prove this and determine the axes.

10. The equation of a projectile is

$$y = ax + \frac{g}{k^2} \text{hyp. log } (1 - bx),$$

gravity ($=g$) acting parallel to the axis of y ; shew that the resistance $= k \cdot \text{velocity}$.

11. Find the volume of a solid the equation to which is

$$z = e^{-\frac{y}{a}(a^2 + x^2)}$$

$$\text{between } \left\{ \begin{aligned} x &= 0 \\ x &= \infty \end{aligned} \right\} \text{ and } \left\{ \begin{aligned} y &= 0 \\ y &= \infty \end{aligned} \right\}$$

and integrate $\frac{dx}{(1-x^2)^{\frac{3}{2}}}$, $d\theta \cdot \frac{a+b \tan \theta}{A+B \tan \theta}$.

$$\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + (2x + a) \frac{d^2y}{dx^2} + a \frac{dy}{dx} + 4x + 2a = 0.$$

12. A circular sector revolves through any angle round one of its extreme radii; find the centre of gravity of the solid generated, its density varying as the n^{th} power of the distance from the centre of the circle.

13. In the above case, supposing the angle of the sector and the angle through which it has revolved to remain the same, prove that as the radius varies the motion of the centre of gravity will be in a plane passing through the centre of the circle; find the line of motion, and the equation to the plane.

14. Water issues from the horizontal surface of a fountain, at an angle α , with a velocity due to h , through a circular annulus of which the radius is r ; V is the volume contained by the surface of the fountain, the ascending, and the descending stream; and V' by the surface, the ascending stream and a plane touching it at the highest point: prove that $\frac{V}{V'}$ is constant when

$$\sin \alpha \propto \frac{r}{h}.$$

15. A body acted on by gravity oscillates in a curve, and a chain of given length, suspended from the horizontal ordinate where the motion commences, is divided by the ordinate at each point into two parts proportional to the two parts of the tension at that point arising from the centrifugal force and from gravity. What is the curve.

16. A paraboloid revolving round its axis strikes a body P in a direction perpendicular to the radius, and P , being attracted to the intersection of the radius and axis by a force varying as $\frac{1}{D^2}$, after impact describes a parabola of the same dimensions as the generating one. Determine the velocity of rotation and the point of impact.

17. A given opaque sphere and a given luminous paraboloid of revolution have their axes in the same line; the distance between them being known, deduce the equation to the surface of the shadow, and find the form of the shadow thrown on a given plane.

18. In the series of quantities $A_1, A_2, A_3, \dots$

$$\text{if } A_1 = r \tan \left(\sin \frac{2\pi}{3} + \alpha \right), \quad A_2 = r \tan \left(\sin \frac{4\pi}{3} + \alpha \right),$$

and the remaining ones be derived according to the following law :

$$A_1 \cdot A_2 \cdot A_3 = r^2 (A_1 + A_2 + A_3),$$

$$A_2 \cdot A_3 \cdot A_4 = r^2 (A_2 + A_3 + A_4), \dots$$

prove that $A_n = r \tan \left(\sin \frac{2n\pi}{3} + \pi \right)$.

19. If $Ay^2 + Bxy + Cxy + Dx^2 + Ey + Fx = 0$ be the equation to a curve; $By^2 + Cy + 2Dx + F = 0$ is the equation to a parabola which bisects all the chords parallel to the axis of x .

$2By^2 + Cy + 2Dx + F = 0$ and $Cy + 2Dx + F = 0$ are equations to a parabola and straight line which are asymptotes to the curve.

The two parabolas and the straight line have a common point of contact in the bisection of that chord which passes through the origin.

20. If a pendulum of length l vibrate in a small circular arc in a medium of which the resistance $\propto kv^2$ to velocity v , and if s be the arc described from the commencement of a vibration to the point where the velocity is greatest when the friction at the axis of suspension is taken into account, and s' the corresponding arc when the friction is neglected, prove that

$$s - s' = f \cdot \frac{2kl}{g},$$

f being the constant effect of friction, and g gravity.

21. Two planets P_1, P_2 revolve in circular orbits at the distances r_1, r_2 from the sun, and when they appear stationary to one another got P_2 's elongation seen from $P_1 = \frac{1}{2} \tan \theta$; shew that $\frac{r_1}{r_2} = \frac{1}{2} \tan \frac{\theta}{2} \cdot \tan \theta$.

22. $Ax + By + Cz = 0$
 $A'x + B'y + C'z = 0$ are the equations to the planes

in which two planets move. Apply them to find the inclination of the orbits to one another, in terms of their inclinations to the ecliptic and of the longitudes of their ascending nodes, the ecliptic being in the plane of x and y .

23. If the earth be an oblate spheroid of small ellipticity with semi-axis a and b , the ratio of the mean density to that at the surface is

$$\frac{3}{\kappa^2 a^2} \left(1 - \frac{k(4a - b)}{3 \tan kb} \right) \text{ very nearly,}$$

assuming the density to be uniform throughout each spheroidal stratum at the same distance from the earth's surface, and to vary

as $\frac{\sin kr}{r}$ at different distances, where k is a constant quantity and r the polar semi-axis of the surface of equal density.

24. Explain the theory of the interferences of light, and determine the colour, origin, and intensity of a ray resulting from the interference of two similar rays, differing in origin and intensity.

SATURDAY MORNING 9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. Similar triangles are to one another in the duplicate ratio of their homologous sides.

2. If two straight lines meeting one another, be parallel to two straight lines which meet one another, but are not in the same plane with the first two, the plane which passes through these, is parallel to the plane passing through the others.

3. If $y = ax + b$, and $y = a'x + b'$, be the equations to two straight lines in the same plane; prove that the cosine of the angle contained between them is equal to

$$\frac{1 + aa'}{\sqrt{(1 + a^2) \cdot (1 + a'^2)}}.$$

4. If $\frac{A_1}{B_1}, \frac{A_2}{B_2}$ &c. $\frac{A_n}{B_n}$ be a series of fractions approximating to $\frac{A}{B}$ shew that

$$\frac{A}{B} - \frac{A_n}{B_n} < \frac{1}{B_n^2} \text{ and } > \frac{1}{B_n(B_n + B_{n+1})}.$$

5. Resolve $x^m - 1$ into its simple and quadratic factors.

6. In a spherical triangle,
 $\cot a \cdot \sin b = \cos b \cdot \cos C + \sin C \cdot \cot A.$

7. If CP and CD be semi-conjugate diameters of an hyperbola, prove that

$$CP^2 - CD^2 = AC^2 - BC^2.$$

8. Investigate Napier's analogies: shew for what cases in the solution of spherical triangles they are applicable; shew also how these cases may be solved by the aid of Napier's rules alone.

9. Shew how to find the number corresponding to a given logarithm not found exactly in the tables.

10. If a, b and c be three whole numbers taken in succession, prove that $\text{Nap. log. } b =$

$$\frac{1}{2} \text{ Nap. log. } a + \frac{1}{2} \text{ Nap. log. } c + \left(\frac{1}{2ac+1} + \frac{1}{2ac+1} + \&c \right);$$

and shew the peculiar use of this formula in finding the logarithms of prime numbers.

11. From the equation

$$y^3 - ay + x = 0,$$

find y in a series ascending by the powers of x , by the *Reversion of Series*.

12. Having given the equation to a plane, and the co-ordinates of a given point without it, find the equations to a straight line drawn from the point perpendicular to the plane; and determine its length.

13. Define the differential co-efficient of any function, and from that definition find the differential co-efficient of uv , u and v being functions of x .

14. Find the differential of the surface of a solid of revolution.

15. Prove Maclaurin's theorem, and thence expand $\tan^{-1} x$ to 5 terms.

16. Define the radius of curvature; and shew that in curves

referred to rectangular co-ordinates is $= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}$: shew

also that in general the circle of curvature at once *touches* and *cuts* the curve.

17. Integrate the following differentials:

$$x^3 \cdot (x^2 + 4), \quad \frac{x^3 dx}{\sqrt{1-x^2}}, \quad \sqrt{x} \cdot dx \cdot \sqrt{1-x^2}, \quad \frac{dx}{\cos x}, \quad e^x \cdot x^m dx.$$

18. Trace the curve whose equation is

$$a^2 y = x^3 - \frac{b^4}{x},$$

and determine the number and nature of its singular points.

SATURDAY AFTERNOON, Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. Find the centre of gravity of any system of points whatever.

2. Find the resultant of any number of forces acting in the same plane upon a rigid body, and the equation to the line in which it acts.

3. Investigate the equation to the catenary between the arc and abscissa; and shew that the tension at the vertex is equal to the weight of a portion of the catenary of the same length as the radius of curvature at the vertex.

4. If two bodies A and B of elasticity e impinge directly on each other with velocities a and b respectively, and if u and v be their respective velocities after impact, and p and q the velocities lost and gained respectively,

$$\text{then } Aa^2 + Bb^2 = Au^2 + Bv^2 + \frac{1-e}{1+e} \cdot (Ap^2 + Bq^2).$$

5. Define the *centre of pressure*, and find it in the case of a semi-parabola immersed in a fluid with its base contiguous to the surface.

6. Determine all the positions of equilibrium of an equilateral triangle floating on a fluid with one angle immersed.

7. The compressing force of the air varying as the density, and the force of gravity varying inversely as the square of the distance from the centre of the earth, find the relation between the density of the air at any altitude and the density at the earth's surface.

8. Determine the form of a surface which shall refract a pencil of rays proceeding from a given point accurately to another given point.

9. Explain the formation of the primary rainbow, and shew that the breadth of the bow = radius of the red arc — radius of the violet arc + the apparent diameter of the sun's disc.

10. Construct the astronomical telescope, and having given the focal lengths of the object and eye glasses, find the position of the eye when the field of view is greatest.

11. When a body descends from a point A towards a centre of force S , the force varying as the distance, shew that the space described, the velocity, and the time of motion, are respectively proportional to the versed sine, sine, and arc of the circle of radius SA ; and find the time to the centre.

12. Explain Newton's method of finding the angle between apsides in orbits nearly circular, and apply it when the force $= m/A + \frac{m_1}{A^2}$.

13. Newton, Sect. XI. Prop. 66.

14. In the case of the sun, moon, and earth, find the whole force on the moon in the direction of the radius vector, the orbits being considered in the same plane.

15. Shew that the centres of oscillation and suspension are reciprocal, and explain the use of this property in finding practically the length of a pendulum.

16. Explain fully the *equation of time*, and shew at what seasons that part of it arising from the obliquity of the ecliptic is positive, and at what seasons negative.

17. Shew that the inclination of the ecliptic to the horizon is a minimum when Aries rises, and a maximum when it sets; and explain the phenomenon of the harvest moon.

EVENING PROBLEMS.

1. If $x = m \cdot \tan z - nx$ where x is small compared with z ,
 prove that $x = \frac{m}{2} \cdot \frac{\sin 2z}{mn + \cos^2 z}$ very nearly.

2. If $-P_{m-p} \cdot x^{m-p}$, $-P_{m-q} \cdot x^{m-q}$, $-P_{m-r} \cdot x^{m-r}$, &c.,
 be the negative terms of an equation of m dimensions, then will
 the greatest root of this equation be less than the sum of the two
 greatest of the quantities $P_{m-p}^{\frac{1}{p}}$, $P_{m-q}^{\frac{1}{q}}$, &c.

3. a and b are respectively the first term and common difference of an arithmetic series,

S_n the sum of n terms,

$S_{n+1} \dots \dots (n+1)$ terms,

&c. &c.

prove that $S_n + S_{n+1} + S_{n+2} + \dots$ to n terms

$$= (3n-1) \cdot n \cdot \frac{a}{2} + (7n-2) \cdot (n-1) \cdot n \cdot \frac{b}{6}.$$

4. A vertical prismatic column the horizontal section of which is an equilateral and equiangular pentagon is cut by a given plane; find the sides and angles of the section.

5. Eliminate by differentiation $f\left(\frac{y}{x}\right)$ and $\phi(xy)$ from the equation

$$z = xf\left(\frac{y}{x}\right) + \phi(yx).$$

6. Find the locus of the intersections of the tangent of an hyperbola with the perpendiculars upon them from the centre: determine its maximum ordinate, its area, and the angles at which it intersects the axis.

7. Having given the two first terms of the expansion of

$(a^2 + b^2 + 2ab \cdot \cos \theta)^{-\frac{1}{2}}$ in a series of the form

$$A_0 + A_1 \cdot \cos \theta + A_2 \cdot \cos 2\theta + \dots$$

shew how from them the two first terms of the expansion of

$(a^2 + b^2 + 2ab \cos \theta)^{-\frac{1}{2}}$ may be determined.

8. If S_1 represent the sum of the ordinates in the quadrant of a circle whose radius is 1,

S_2 represent the sum of their squares,

$S_3 \dots \dots \dots$ cubes,

&c.

&c.

$$\text{prove that } S_{n-1} \cdot S_n = \frac{3}{n+1} \cdot S_1 \cdot S_2.$$

9. A straight line revolving in its own plane about a given point intersects a curve line in two points; find the curve when the rectangle of the lines intercepted between the given point and the points of intersection is constant.

10. Prove that if the tangent plane to any curve surface make with the three co-ordinate planes the least possible volume, the distance of the intersection of the plane and axes from the origin are respectively gx , gy , and gz , x , y , and z being the co-ordinates of the point of contact.

11. A plane is so moved as always to cut off from a given paraboloid of revolution equal volumes; determine the equation to the surface to which it is always a tangent.

12. Integrate the following differentials and differential equations:

$$\frac{dx}{x^2 + 1}, \quad \frac{dx}{x^2 \sqrt{1 - x^2}}, \quad \frac{dx}{\sqrt{a - x} - \sqrt{x}}$$

$$(x^2 + y^2) dx + x^2 y dy = 0, \quad (1 + x) \cdot \frac{d^2 y}{dx^2} + a \cdot \frac{dy}{dx} = 0,$$

$$1 + p^2 + q^2 = m^2,$$

and also the following equations of differences:

$$f(x^2) - f(x) = m, \text{ and } u_n \cdot u_{n+k} = k^2$$

13. There are two urns *A* and *B*, the former containing three white and the latter three black balls. a ball is taken from each at the same time and put into the other, and this operation is repeated three times; what is the probability that *A* will contain three black and *B* three white balls?

14. A uniform rod rests with one of its extremities in a semi-circle whose axis is vertical, find the nature of the line supporting its other extremity so that it may rest in every position.

15. Having given the variation of the obliquity of the ecliptic; find the corresponding variations in right ascension and declination.

16. Determine the latitude of the place of observation from observing the times of the rising of two known stars.

17. The axis of a given cone filled with fluid is inclined at a given angle to the horizon; find how much of the fluid will flow out and determine the pressure exercised by the remainder upon the conical surface.

18. A body attracting with a force varying directly as the distance moves uniformly in a straight line; determine the motion of another body situated in the same plane and subject to its influence.

19. Determine the orbit described and the time of describing any angle when a body is projected round a centre of force

varying as $\frac{1}{D^2}$, at an angle whose tangent $= \frac{3}{2}$, and with a velocity which is to the velocity in a circle at the same distance $:: \sqrt{2} : \sqrt{9}$.

20. A body descends down the arc of a vertical catenary having its vertex at the lowest point; find the curve of ascent when the oscillations are isochronous, the two curves being so united at the lowest point as to have a common tangent.

21. A corpuscle is attracted by two straight lines at right angles to each other, the particles of which attract with forces varying as $\frac{1}{D}$, having given the position of the corpuscle and the length of one of the lines, find the length of the other when the direction in which the corpuscle begins to move is towards their common intersection.

22. A body descends in a straight line in a medium whereof the density varies as the square root of the distance from a given point, and is urged by a constant force tending to that point: find the velocity and time corresponding to a given space, supposing the resistance to vary as the density and velocity jointly.

23. A body describes a circle of given radius uniformly, acted upon by two forces each varying as the distance and without the plane of the circle; find the velocity of the body and the position of the plane of its orbit.

24. If a body revolve in an ellipse round the focus, prove that a progressive motion of the apse will be the effect of any continual addition of force in the direction of the radius vector during the progress of the body from the higher to the lower apse, and point out the effect on the eccentricity.

25. Two balls connected together by an inflexible and inextensible line are constrained to move the one on a horizontal plane, the other on an inclined plane which is at liberty to move freely on the horizontal plane; find the motion of the balls and of the plane, supposing the motion of the rod to be in a vertical plane.

MORNING PROBLEMS.

MONDAY MORNING, 9 o'clock to 11.

1. If n be a whole number, prove that $\frac{n^3 + 5n}{6}$ is also a whole number.

2. The ratio between the area of an equilateral and equiangular decagon described about a circle, and that of another within the same circle is equal to

$$\frac{8}{7 + \frac{1}{4 + \frac{1}{4 + \dots}}}$$

3. If a and b be the sides of a plane triangle, A and B their opposite angles, then will

$$\text{hyp. log. } b - \text{hyp. log. } a = \cos 2A - \cos 2B \\ + \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{2}(\cos 6A - \cos 6B) + \dots$$

4. Of all spherical triangles which have the same base and equal perpendiculars from the vertex to the base, shew that the isosceles has the greatest vertical angle; and from the result prove that the same is true in plane triangles.

5. Having given

$$\begin{aligned} \log 8801 &= 3.9445320 \\ \log 8802 &= 3.9445814 \\ \log 8804 &= 3.9446800 \\ \log 8805 &= 3.9447294 \\ \text{find } \log 8803. \end{aligned}$$

6. Two straight lines, which are always tangents to a given parabola, are so inclined to the axis of x that the sum of the cotangents of the angles which they make with that axis is constant; prove that the locus of their intersection is a straight line parallel to the axis.

7. Find the surface in which a tangent plane always cuts the axis of z at distances from the origin proportional to $\frac{1}{x^2}$; and when $n = 1$ give to the arbitrary function that particular form which will produce the equation to the ellipsoid.

8.
$$\begin{aligned} x &= az \} \quad \text{and} \quad \begin{aligned} x^2 + y^2 &= 2cx \} \\ y &= bz \} \quad \begin{aligned} x^2 + y^2 &= m^2 z^2 \} \end{aligned} \end{aligned}$$

are the equations to a straight line and curve of double curvature, find the equation to the surface generated by a straight line moving always parallel to the plane of xy , and passing through the straight line and the curve.

9. Integrate

$$\frac{d^2y}{dx^2} + a^2y = e^x \cos ax,$$

$$z - px - qy = m(x + y + z).$$

10. Solve the following equations of differences:

$$\Delta^2 x + \Delta^2 x + \Delta x = x^3,$$

$$u_x \cdot u_{x+1} + u_x \cdot u_{x+2} + u_{x+1} \cdot u_{x+2} = m^2.$$

11. Sum the following series :

$$\frac{1}{1 \cdot 3} + \frac{1}{4 \cdot 6} + \frac{1}{7 \cdot 9} + \frac{1}{10 \cdot 12} + \dots \text{ad infinitum,}$$

$$\sec \theta \cdot \cos \theta + 4(\sec \theta)^2 \cos 2\theta + 13(\sec \theta)^3 \cos 3\theta \\ + 40(\sec \theta)^4 \cos 4\theta + \&c. \text{ to } n \text{ terms.}$$

12 Find $\sin x$ from the equation

$$\sin x \cdot \cos \cos x + a \sin^2 x = b,$$

and shew its use in the solution of the following problem. To determine how much the azimuth of a known star on the horizon is affected by refraction.

13. A right-angled triangle vibrates in its own plane about an axis passing through its vertex, find the length of the isochronous simple pendulum; and if one of the sides be slightly diminished and the other as much increased, determine the variation of the pendulum.

14. If a hemisphere and paraboloid of equal bases and similar materials have their bases cemented together, the whole solid will rest on a horizontal plane on any point of the spherical surface if the altitude of the paraboloid $= a \sqrt{\frac{2}{3}}$, (a) being the radius of the hemisphere.

15. Prove that the eye cannot be achromatic for objects at all distances.

16. A body is acted on by two forces, the one repulsive and varying as the distance from a given point, the other constant and acting in parallel lines. Determine the motion of the body.

17. A body falls towards a centre of force which varies as $\frac{1}{D^3}$, in a medium of which the density varies as $\frac{1}{D^3}$, and the resistance varies as $(\text{vel.})^2$. Prove that at any distance (r) from the centre,

$$(\text{vel.})^2 = \frac{m}{h} \left\{ 1 - e^{-2} \left(\frac{1}{r^2} - \frac{1}{a^2} \right) \right\},$$

where m = force at distance 1, h = density at distance 1, and a = distance from centre at the beginning of motion.

18. A uniform rod vibrates in a medium the resistance of which varies as the velocity; find the time of one of its small oscillations.

MONDAY AFTERNOON, 1 o'clock to 3.

First and Second Classes.

1. Expand $f(x + h, y + k)$ in a series ascending by powers of h and k .

2. Reduce $\frac{u_{x+1}}{u_x \cdot u_{x+2} \cdot u_{x+3}}$ to an integrable form when $u_x = a + bx$: and sum the series

$$\frac{2^2}{1 \cdot 3 \cdot 4 \cdot 5} + \frac{3^2}{2 \cdot 4 \cdot 5 \cdot 6} + \frac{4^2}{3 \cdot 5 \cdot 6 \cdot 7} + \dots \text{to } n \text{ terms.}$$

3. Prove Lagrange's theorem.

4. A bag contains m white balls and n black balls; find the probability of taking out a white ball at least p times in r trials, the ball being replaced after each trial.

5. State the general nature of developable surfaces. Investigate the partial differential equation of the second order which belongs to them.

6. Find the longitude of the perihelion, and the time of the earth's passing through it.

7. Prove that the centre of gravity of the earth and moon describes about the sun very nearly an ellipse in one plane, and that the area described by its radius vector is very nearly proportional to the time.

8. Find the horary variation of the inclination of the moon's orbit. (Newton, Vol. III. Prop. 34.)

9. Find the attraction of an oblate spheroid on a particle in its equator.

10. Having given the declination of the moon, compare the magnitudes and duration of the superior and inferior tides in any latitude, the effect of the sun on the tide being neglected.

MONDAY AFTERNOON, 1 o'clock to 3.

Third and Fourth Classes.

1. Find the equation to a plane, and determine the constants when the distances of the intersections of the plane with the co-ordinate axes from the origin are given.

2. In a spherical triangle, the sides of which are small compared with the radius of the sphere, having given two sides and the included angle, find the angle between the chords of those two sides.

3. Explain Newton's method of approximating to the roots of an equation; and shew that its accuracy does not depend upon the ratio of the quantity assumed to the root, but upon its being nearer to one root than to any other.

4. Integrate $\frac{x^4 \cdot dx}{\sqrt{(2ax - x^2)}}$ between the limits $x = 0$, and $x = a$,

$$\frac{dx}{x^2 + 1} \cdot (7x + 3y + 3) \cdot \frac{dy}{dx} + 28x + 20y - 7 = 0.$$

5. A body moving on a curve is acted on by forces X and Y parallel to the axes of the curve, find the re-action; and apply

it to find the tension of a string, at the lowest point, when a body oscillates in a circle through an arc of 120° .

6. As the line of nodes of the moon's orbit moves from syzygy to quadrature, the inclination of the orbit to the ecliptic is diminished; and as the line of nodes moves from quadrature to syzygy, the inclination is increased. (Newton, Prop. LXVI. Cor. 10.)

7. A ladder rests with one end on a smooth horizontal plane, and the other against a smooth vertical wall; find the horizontal force at its foot which will keep it at rest; and when the force is removed determine its motion.

8. Find experimentally the refracting power of any transparent substance.

9. Find the sun's right ascension by Flamsteed's method. Why must the observations be made near the equinox?

10. Explain Mercator's projection of the sphere, and find the length of the projection of an arc of the meridian included between the latitudes 30° and 60° .

TUESDAY MORNING, 9 o'clock to 11.

First Class.

1. If from a point two straight lines be drawn, and their extremities be joined by a curve; find its nature when its length is a maximum, the area contained by the two lines and the curve being given.

2. Explain the method of integrating the partial differential equation

$$\frac{d^2z}{dx^2} + P \frac{d^2z}{dx dy} + Q \frac{d^2z}{dy^2} = R,$$

where P, Q, R are functions of $x, y, z, \frac{dz}{dx}, \frac{dz}{dy}$.

3. Prove that

$$\Sigma u_x = \int u_x dx - \frac{u_x}{2} + \frac{1}{2 \cdot 1 \cdot 2 \cdot 3} \frac{du_x}{dx} - \frac{1}{6 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \frac{d^2u_x}{dx^2} + \text{\&c. the coefficients being the same as those of } t \text{ in the expansion of } \frac{1}{e^t - 1}.$$

4. Find the general equation to conical surfaces; and if a conical surface be described about a surface of the second order, shew that the curve of contact will be in one plane.

5. Find the horary increment of the area which the moon describes about the earth in a circular orbit. (Newton, Vol. III. Prop. 26.)

6. A pile of weight w , is driven by a hammer H impinging with a velocity v , the friction being represented by F ; find the
VOL. VI. c

motion, and when the velocity given to the pile is small, approximate to the whole space through which it is driven.

7. Find the two parts of solar nutation, and prove that they are connected by the equation to an ellipse, the axes of which are in the ratio of $\cos I : 1$, where I is the obliquity of the ecliptic.

8. A body is acted on by gravity, find the tautochronous curve in a medium in which the resistance varies partly as the velocity and partly as the square of the velocity: and from the result prove that it is a cycloid when the resistance vanishes, or varies as the velocity.

$$9. \quad s = k \left\{ \sin (g\theta - \gamma) + \frac{3m}{8} \sin (2 - 2m - g)\theta + 2\beta + \gamma \right\}$$

where $s = \tan.$ of moon's latitude, $\gamma =$ longitude of node, $-\beta$ the sun's mean longitude when $\theta = 0$. Explain the effect of these terms, and thence shew that the inclination of the orbit is greatest when the line of nodes is in syzygies, and least when it is in quadratures.

TUESDAY MORNING, 9 o'clock to 11.

Second and Third Classes.

1. Find the present value of an annuity of £1 to be continued during the life of an individual of a given age, allowing compound interest for the money.

2. In the surface whose equation is

$$Ax^2 + By^2 + Cz^2 + Kx = 0,$$

shew in what cases the surface will be respectively an ellipsoid, hyperboloid, elliptic paraboloid, hyperbolic paraboloid, and a paraboloid of revolution.

3. Prove that

$$\Delta^n u_x = u_{x+n} - n \cdot u_{x+n-1} + n \cdot \frac{n-1}{2} \cdot u_{x+n-2} - \&c.$$

and thence shew that

$$1 \cdot 2 \cdot 3 \cdot \&c. n = n^n - \frac{n}{1} \cdot \overline{n-1}^n + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \overline{n-2}^n - \&c.$$

4. Integrate the following differential equations:

$$\frac{dy}{dx} + y = xy^3, \quad \frac{d^2y}{dx^2} - n^2y = \cos mx, \quad m^2 \cdot \left(\frac{dy}{dx} \right)^2 = y - x \cdot \frac{dy}{dx} :$$

explain also the relation which subsists between the particular solution and complete integral of a differential equation.

5. Find the moment of inertia of an ellipse revolving in its own plane about any axis.

6. Determine the centre and diameter of the least circle of chromatic aberration, in a given lens.

7. Explain the method of determining altitudes above the earth's surface by the barometer, account being taken of the variation of temperature.

8. Find the ratio of the diameters of the lunar orbit, supposing it to have been originally without eccentricity. Newton, B. III. Prop. 28.

9. Having given the length of a degree of latitude, and also the length of a degree in a direction perpendicular to the meridian, in a given latitude; find the ellipticity of the earth.

TUESDAY MORNING, 9 o'clock to 3.

Fourth Class.

1. The roots of the equation $x^3 - px^2 + qx - r = 0$ are a , b and c , transform it into one the roots of which are

$$\frac{a}{b} + \frac{b}{a}, \frac{a}{c} + \frac{c}{a}, \frac{b}{c} + \frac{c}{b}.$$

2. In each of the conic sections, the radius of curvature (normal)²
 $\sqrt{\frac{4}{3} \text{ lat. rect.}}^{\frac{1}{2}}.$

3. Find the differential of a solid of revolution, and apply it to find the content of a segment of a paraboloid, the radii of the greater and smaller ends of which are (a) and (b) respectively, and the distance between them (c) .

4. Find $\int \frac{dx}{\sqrt{x \cdot (1-x)}}, \int \frac{dx}{(1+x^2)^{\frac{3}{2}}}.$

$$\int \frac{A(\sin \theta)^n + B \cdot (\cos \theta)^n}{(\cos \theta)^{n+2}} \cdot d\theta, \int \cos(a+b\theta) \cos(A+B\theta) \cdot d\theta.$$

5. State the principle of virtual velocities, and prove it when two bodies are in equilibrium on a bent lever.

6. Prove that the image of a straight line placed before a spherical reflector is a conic section, and determine in what cases it is each particular conic section.

7. Find the specific gravity of a body lighter than the fluid in which it is weighed.

8. Shew how to determine whether a planet is a superior or an inferior one; and having given the synodic period of a planet, and the length of a year, find the planet's period.

9. Construct a horizontal dial, and find the limits beyond which it is unnecessary to graduate it.

10. If two bodies S and P attract each other mutually, the orbit which P appears to describe about S in motion may be described about S fixed, by the action of the same force. (Newton, Sect. XI. Prop. 58.)

TUESDAY AFTERNOON, 1 o'clock to 3.

First Class.

1. At any point in any curve surface, the sections of greatest and least curvature are at right angles to each other.

2. Approximate to the following integrals:

$$dx \cdot \log 1 + e \cdot \cos x, \text{ and } \frac{d^2y}{dx^2} - ax^my = 0,$$

and solve the following differential equation,

$$\frac{d^2y}{dt^2} = a^2 \cdot \frac{d^2y}{dx^2}.$$

3. Explain fully the mode of applying the calculus of variations to cases wherein it is required to determine one function (u) a maximum or minimum, the value of another function (v) being given: and exemplify it by finding the curve of quickest descent from one given point to another, the length of the curve being given: shew also how the constants introduced by integration may be determined in this case.

4. When a small pencil of diverging rays is incident obliquely on a concave refracting surface, the ultimate intersection of two refracted rays in a normal plane is determined from the equation

$$\frac{\cos \phi_1}{u} \cdot \frac{1}{v} - \frac{\cos \phi}{\mu} \cdot \frac{1}{u} = \left(\cos \phi_1 - \frac{\cos \phi}{\mu} \right) \cdot \frac{1}{r}.$$

5. If a system move in any manner whatever, prove that $\int \Sigma m v ds$ is a minimum.

6. Determine the circular orbit of a planet from two observations.

7. Find the mean horary motion of the nodes of the lunar orbit, supposed to be elliptical. Newton, B. III. Prop. 31.

8. When the force at the pole of a revolving fluid spheroid is to the force at the equator as the equatoreal radius is to the polar radius, any two canals drawn from any points in the surface and meeting within it, will balance each other.

9. Eliminate t from the differential equations

$$\frac{d \cdot \left(\varrho^2 \cdot \frac{d\theta}{dt} \right)}{dt} = T\varrho,$$

$$\text{and } \frac{d^2 \cdot (\varrho^2)}{dt^2} = -S.$$

TUESDAY AFTERNOON, 1 o'clock to 3.

Second and Third Classes.

1. Given the two sides and the included angle of a spherical triangle, required its area; and from the expression obtained, find the area of a plane triangle in corresponding terms.

2. Find the magnitude and position of the principal axes of the curve of the second order, the equation of which is

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0.$$

3. Explain the method of finding whether a curve has multiple points, and find the number and nature of multiple points of the curve the equation of which is

$$y^4 - 2a^2y^2 - 2ax^3 - 3a^2x^2 + a^4 = 0.$$

4. Explain the transformation of the independent variable, and transform the equation

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2} \cdot \frac{dy}{dx} = \frac{y}{1-x^2} = 0,$$

where x is the independent variable, into one where θ is the independent variable, θ being equal to $\cos^{-1}x$.

5. Find the conditions of the stable and unstable equilibrium of a floating body; and if it revolve about a horizontal axis, shew that it passes alternately through positions of stable and unstable equilibrium.

6. Find the time of an oscillation in a hypocycloid, the body being acted upon by a force varying as the distance from the centre of the globe.

7. When a chain fixed at two points is acted upon by a central attractive or repulsive force, the tension at any point is inversely as the perpendicular from the centre of force upon the tangent at that point.

8. Find the moment of inertia of a system about any axis passing through the origin of the co-ordinates, and the moment of inertia about any axis in terms of the moments about the *principal axes*.

9. If a body acted upon by gravity be projected in a medium the resistance of which varies as the square of the velocity, find the equation to the curve described; and when the resistance vanishes, shew that it is the equation to a parabola.

TUESDAY AFTERNOON, 1 o'clock to 3.

Fourth Class.

Investigate Waring's Rule for the solution of a biquadratic equation.

2. x, y and X, Y are the co-ordinates of a point referred to two systems of rectangular co-ordinates having a common origin and inclined to each other at a given angle: find the relation subsisting between $x, y, X,$ and Y .

3. The radius of curvature of a spiral $= \frac{rdr}{dp}$.

4. Trace the curve whose equation is $ay^3 = \frac{x^4}{x-a}$, and find its area.

5. Having given the quantity of air in the tube of a barometer, determine the depression below the standard altitude.
6. Having given the radius of the arc of any colour in the primary rainbow, find the index of refraction for that colour out of air into water.
7. The difference of the forces at corresponding points in the fixed and revolving orbit varies inversely as the cube of the distance. Newton, Prop. XLIV.
8. Compare the axis major of the ellipse apparently described by P round T , with that of the ellipse described by P round T fixed in the same periodic time. Newton, Prop. LX.
9. In a given latitude find the sun's azimuth; his declination and the time of the day being given; and adapt the trigonometrical formula to logarithmic computation.

Questions Proposed by PROFESSOR AIRY, *at the*
EXAMINATION FOR SMITH'S PRIZES.

1. The expression for the radius of curvature is sometimes given.

$$\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{-dx dy^2}, \text{ and sometimes } \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dy dx^2 - dx dy^2}.$$

Explain the different suppositions to which these expressions correspond; and give the expression in terms of differential coefficients only.

Find the expression for the radius of curvature in terms of $\frac{dU}{dx}$, $\frac{dU}{dy}$, &c. the equation to the curve being $U = 0$, where U is a function of x and y .

2. Investigate the corrections in the A. R. and declination of a star for precession, nutation, and aberration: and explain what is meant by the equation of the equinoxes in right ascension.

Shew that the sum of the corrections in A. R. and in declination may be expressed by $Aa + Bb + Cc + Dd$ and $Aa' + Bb' + Cc' + Dd'$, where A, B, C, D , depend only on the day and year, and $a, a', b, b', c, c', d, d'$, only on the place of the star.

3. Explain generally the optical theory of interferences. Mention the circumstances under which the union of two streams of light will produce black stripes.

Shew that, *ceteris paribus*, the breadth of the stripes is inversely proportional to the distance between the sources of light.

Explain why the stripes farthest from the centre become coloured, and finally disappear.

4. Prove that the variations of the axis and excentricity of a planet's orbit are defined by these expressions :

$$\frac{da}{dt} = -2a^2 \cdot n \frac{dR}{d\varepsilon},$$

$$\frac{de}{dt} = \frac{na}{c} \sqrt{1-e^2} (1 - \sqrt{1-e}) \frac{dR}{d\varepsilon} + \frac{na}{c} \sqrt{1-e^2} \frac{dR}{d\pi},$$

where

$$R = m' \cdot \frac{xx' + yy' + zz'}{r^2} - \frac{m'}{\sqrt{(x-x')^2 + y-y'^2 + z-z'^2}},$$

the mass of the sun being 1.

5. $\log a = 1,2649236$; $\log b = 1,7436282$: find $\log a + b$ by means of a subsidiary angle, without the use of any natural numbers

6. If

$$\frac{1}{\sqrt{a^2 - 2aa' \cos \theta + a'^2}} = A_0 + A_1 \cos \theta + A_2 \cos 2\theta + \&c.,$$

find the general relation between A_n, A_{n+1}, A_{n+2} . And if

$$\frac{1}{(a^2 - 2aa' \cos \theta + a'^2)^{\frac{3}{2}}} = B_0 + B_1 \cos \theta + B_2 \cos 2\theta + \&c.$$

express B_n in terms of A_n and A_{n+1} .

7. Prove that $\overline{a-b} \times \overline{c-d} = ac - bc - ad + bd$: and explain how this equation can be said to have any meaning if $b > a$.

8. In the Lunar Theory, given that

$$\frac{T}{u^2} = -m' \cdot \frac{3}{2} \cdot \frac{u'^3}{u^2} \sin 2 \cdot \overline{\theta - \theta'},$$

$$\frac{S-Ps}{u^3} = m's \cdot \frac{u'^3}{u^2} \left(\frac{3}{2} + \frac{3}{2} \cos 2 \cdot \overline{\theta - \theta'} \right),$$

$$\frac{d^2 s}{d\theta^2} + s + \frac{\frac{S-Ps}{u^3} + \frac{T}{u^2} \cdot \frac{ds}{d\theta}}{h^2 + 2 \int \theta \frac{T}{u^2}} = 0.$$

Solve the last equation, and shew that

1. The node of the moon's orbit regresses, the mean inclination remaining unaltered.
2. The motion of the node, and the inclination, are subject to periodical inequalities, the length of the periods being the same, and the maximum of one inequality corresponding to the disappearance of the other.

9. The elasticity of the air being supposed to vary as the n^{th} power of the expansion, find the density of the air at any distance above the earth's surface.

Investigate the quantity of atmospheric refraction: $\overline{\text{velocity}}$ of light on the Newtonian hypothesis, being represented by $a + b$ density of air.

10. Distinguish between the *force* and the *power* of an agent: shew that the former may be increased by the intervention of machinery, but that the latter cannot.

11. Explain the principle and construction of the hydraulic ram.

12. Given the attraction of a homogeneous spheroid on a point within it, shew how the attraction of a homogeneous spheroid on a point without it may be found. And from this expression shew how the attraction of a heterogeneous spheroid on a point without it may be found.

13. Given $u_x, u_{x+1}, u_{x+2}, u_{x+3}$, prove that

$$u_{x+\frac{1}{2}} = \frac{9(u_{x+1} + u_{x+2}) - (u_{x+3})}{16}.$$

14. In a spherical triangle, prove that

$$\tan \frac{C}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cot \frac{A+B}{2}.$$

15. Solve the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = X,$

and the equations $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + x + 2y = t.$

$$4\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} - 3x + y = et.$$

16. Explain the method of calculating tables by differences: and calculate $\log \frac{\sin \theta}{\theta}$ (taking the terms only as far as θ^2) for every $30'$ to 20° .

17. A small pencil of light falls obliquely on a spherical refracting surface: investigate the distances at which rays converge in the primary and secondary planes.

18. A rod acts with a uniform force in the direction of its length on the fly-wheel-crank of a steam-engine, and a uniform resistance is to be overcome by the fly-wheel: find the velocity of the wheel at any time. And find the expression, supposing the force and resistance to be so adjusted that after half a revolution the velocity may be unaltered.

THE SENATE-HOUSE PROBLEMS,

*Given to the Candidates for Honors during the Examination—
for the degree of B. A.*

BY THE TWO MODERATORS.

MODERATORS,

Mr. CHALLIS, Trinity. Mr. BOWSTEAD, Corpus.

EXAMINERS.

Mr. KING, Queen's. Mr. HANSON, Caius.

FRIDAY MORNING.—*January 14, 1831.*

9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

2. Describe an equilateral and equiangular pentagon about a given circle.

3. What sum must be paid for £439 12s. 5d. 3½ per cent. stock, at 92½ per cent.?

4. Shew that $\frac{3\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}} = 16.746$, &c. also prove that the cube root of any number containing $3n$, $3n - 1$, or $3n - 2$ digits, contains n digits.

5. Reduce the fraction $\frac{30a^2x^4 - 5a^3x^3 + 5a^5x}{9ax^3 - a^3x + 2a^4}$ to its lowest terms.

6. Solve the following equations:

$$x + y - 8 = 0.$$

$$\frac{x - y}{2} + \frac{2x - y}{3} + \frac{4}{3} = 0,$$

$$\frac{x + 3}{x} + 4x = 14.$$

$$x^2 + \sqrt{x^3} + 11 = 31.$$

7. Investigate the binomial theorem, and apply it to extract the square root of $\frac{1}{a^2 - b^2}$ to four terms.

8. Prove that $\log (m + n)$

$$= \log m + 2M \left\{ \frac{n}{4m+n} + \frac{1}{3} \left(\frac{n}{2m+n} \right)^3 + \frac{1}{5} \left(\frac{n}{2m+n} \right)^5 + \&c. \right\}$$

9. The c^{th} root of a binomial, one or both of whose factors are possible quadratic surds, may sometimes be expressed by a binomial of that description.

10. Investigate an expression for the sine of any angle of a plane triangle in terms of the sides.

11. Prove that

$$\cos n\theta \pm \sqrt{-1} \sin n\theta = (\cos \theta \pm \sqrt{-1} \sin \theta)^n.$$

whether n be integral or fractional.

12. Impossible roots enter equations by pairs.

13. Investigate Waring's rule for the solution of a biquadratic equation, and shew that the reducing cubic equation is soluble by Cardan's rule, only in the case where two roots of the given equation are possible and two impossible.

14. In the hyperbola prove that $CD \cdot PF = AC \cdot BC$.

15. The radius of curvature in the ellipse $= \frac{CD^2}{PF}$.

16. Shew how the fraction $\frac{A}{B}$ may be represented in the form of a continued fraction, and apply the process to find a fraction which shall be nearly equal to $\frac{1593}{7667}$, and in lower terms.

17. Having given the equations to two straight lines in the same plane, find the tangent of the angle contained between them.

18. Obtain the equation to a straight line which passes through two given points of a parabola, and find what the equation becomes when the points are supposed to coincide.

19. The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.

20. Having given two sides and the included angle of a spherical triangle, obtain the third side in a formula convenient for logarithmic computation.

FRIDAY AFTERNOON Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. If two equal weights act perpendicularly on a straight lever, they may be kept in equilibrium round any fulcrum by the same force as if they were collected at the middle point between them.

2. Define the centre of gravity, and find it in a plane triangle.
 3. Two weights keep each other in equilibrium on a bent lever:

(1) Compare them.

(2) Prove that if an indefinitely small motion be given, the centre of gravity will neither ascend nor descend.

4. Define *inertia*, *mass*, *weight*, *moving force*, *accelerating force*.

5. A body acted on by gravity descends in a straight line: find the space described in a given time, and prove that it is equal to half the space which would be described in the same time with the last acquired velocity continued uniform.

6. A body descends on an inclined plane: find the accelerating force; and prove that at any point in the descent, the velocity is equal to that which would be acquired down the perpendicular height.

7. Rays fall upon a plane reflecting surface: prove that the foci of incident and reflected rays are on opposite sides of the reflector, and equi-distant from it.

8. A small pencil of rays diverging from a point in the axis of a spherical reflector is incident nearly perpendicularly upon it: find the focus of reflected rays. When the foci are on opposite sides of the reflector, the divergence of the reflected is less than that of the incident pencil by a constant quantity.

9. Find the principal focus of a double convex lens, the thickness of which is inconsiderable.

10. If given volumes of two fluids of known specific gravities be mixed together, find the specific gravity of the compound, supposing its volume to be equal to the sum of the volumes of the parts.

11. If the particles of an elastic fluid repel each other with forces varying inversely as the n^{th} power of their distances, prove

that the compressing force varies as (density) $\frac{n+2}{3}$.

12. Find the density of air in the air-pump after n turns.

13. The limiting ratio of the arc, chord, and tangent of a curve of continued curvature is a ratio of equality.—(NEWTON, *Lemma VII.*)

14. A body being acted upon by a force tending to a centre, the areas described are in one plane, and proportional to the times of description.—(NEWTON, *Prop. I.*)

15. Investigate an expression for the force by which a body may be made to describe any orbit whatever round a fixed centre in the same plane with it: and apply it to find the law of force tending to the focus of an ellipse.—(NEWTON, *Props. VI. & XI.*)

16. State the arguments from which it is inferred that the earth revolves round its axis and round the sun.

17. Shew how to draw a meridian line by observing the shadow of a vertical gnomon on a horizontal plane, and correct for the change in the sun's declination between the observations.

18. Explain the nature of the five astronomical corrections: Refraction, Parallax, Aberration, Precession, and Nutation.

EVENING PROBLEMS.

1. Find all the solutions in positive whole numbers of the equation $11x + 15y = 1031$.

2. Determine which is the greatest term of the expansion of $(a + b)^n$.

3. Shew that $\sqrt{5}$ is greater than $\frac{682}{305}$ and less than $\frac{2889}{1292}$, and that it differs from the latter fraction by a quantity less than

$$\frac{1}{2 \times 305 \times 1292}.$$

4. BC, CD are two consecutive arcs of a parabola, the sagittæ of which, bisecting the chords, and parallel to the axis, are equal; prove that the chord of BCD is parallel to the tangent at C .

5. If a, b, c , be the lengths of the chords of three arcs of a circle, which together make up a semi-circumference, and r the radius of the circle, then $4r^3 - (a^2 + b^2 + c^2)r - abc = 0$.

6. If two roots of the cubic $x^3 - qx + r = 0$,

$$\text{be } a + b\sqrt{-3} \text{ and } a - b\sqrt{-3},$$

$$\text{then will } -\frac{r}{2} + \sqrt{\frac{r^3}{4} - \frac{q^3}{27}} = (b-a)^2.$$

7. Required the conditions to be satisfied that in the division of $x^4 + qx^3 + rx + s$ by $x^2 + ax + b$, the remainder may be $= 0$ independently of the value of x , after three terms of the quotient are obtained; and when these conditions are satisfied, obtain an equation for finding b from q, r, s , given, and shew how to solve it.

8. Assuming that if $\delta p, \delta q, \delta r$, be the virtual velocities of three forces P, Q, R , which keep a point at rest,

$$P\delta p + Q\delta q + R\delta r = 0,$$

in whatever direction the virtual motion of the point takes place; prove that the forces are proportional to the sides of a triangle drawn in their directions.

9. A ladder rests with its foot on a horizontal plane, and its upper extremity against a vertical wall; having given its length, the place of its centre of gravity, and the ratios of the friction to the pressure both on the plane and on the wall, find its position when in a state bordering upon motion.

10. Find the content of the solid generated by the revolution of the curve the equation to which is

$$(a^2 + x^2)(y^2 - x^2(a^2 - x^2)) = 0$$

about the axis of x ,

11. The straight line joining any points P and Q of the surface of an ellipsoid is bisected by a plane passing through the centre of the ellipsoid, and through the line of intersection of the tangent planes at P and Q .

12. A curve surface is described by a straight line always passing through two straight lines, the equations to which are $x = a$, $y = b$; and $x = a'$, $z = b'$; and through a curve, $z = f(y)$, in the plane zy ; shew that the equation to the surface is $\frac{xb' - a'z}{x - a'} = f\left(\frac{xb - ay}{x - a}\right)$.

13. When a very small conical pencil of rays is reflected at a spherical surface, the transverse section of the reflected pencil will be a circle, at a distance from the point of reflection, which is an harmonic mean between the distances of the focal lines in the primary and secondary planes from the same point.

14. Integrate the differential equations, $\frac{d^2y}{dx^2} + m \frac{dy}{dx} + n^2y = A \sin nx$, and $\frac{z - b'}{x - a'} = \frac{dz}{dy} + \frac{dz}{dx} \frac{y - b}{x - a}$; and obtain a particular integral of

$$\frac{d^2z}{dy^2} + \frac{2dz}{dx} \cdot \frac{d^2z}{dx dy} + (dx^2 - a^2) \frac{d^2z}{dx^2} = 0.$$

15. Eliminate arbitrary functions by differentiation from the equation $\frac{z}{x} = \varphi\left(\frac{y}{x}, \frac{x}{t}\right)$; and prove that the integral of every differential equation of the first order and degree between four variables, is of the form $P = f(Q, R)$.

16. Prove that the integral of

$$\Delta^3 u_x + a \Delta^2 u_x + b \Delta u_x + cu_x = 0,$$

is $u_x = Ca^x + C'\beta^x + C''\gamma^x$, if $a - 1$, $\beta - 1$, $\gamma - 1$, be the roots of $z^3 + az^2 + bz + c = 0$. Also by varying the parameters, obtain the integral of

$$\Delta^3 u_x + a \Delta^2 u_x + b \Delta u_x + cu_x = A_x.$$

17. Supposing the earth to be spherical, and the matter in

its interior being compressed according to the law $p = k(\rho^2 - \delta^2)$, p being the pressure and ρ the density at any distance r from the centre, and δ the density at the surface; shew that $\rho \propto \frac{\sin qr}{r}$, q being a certain constant.

18. The path of a ship in a horizontal plane, is referred to rectangular axes, that of x being in the direction of the wind, and its velocity in sailing from one given point to another is assumed to be $\int \left(\frac{dy^2}{dx^2} \right)^{\frac{1}{2}}$; required the nature of the brachystochronous path between the given points.

19. If the material particles $m, m', \dots$ of any system in motion, pass from the positions $a, a', \dots$ to $b, b', \dots$ during the very small time δt , and $c, c', \dots$ be the positions they would have had, if during δt only the impressed forces had acted; then forces proportional to and in the directions of $\overline{bc}, \overline{b'c'}, \dots$ will produce equilibrium with the pressures on the fixed points and axes of the system.

20. Also, if $e, e', \dots$ be the places $m, m', \dots$ would have had, if the impressed motions had been compounded with uniform motions during δt along the actual paths with the velocities at $a, a', \dots$ then forces proportional to and in the directions of $\overline{be}, \overline{b'e'}, \dots$ will keep the system at rest; and if $\gamma, \gamma', \dots$ be any other positions compatible with the conditions of the system $\Sigma m \overline{eb^2} < \Sigma m \overline{e\gamma^2}$.

21. An elastic chord $AabcB$ is stretched between two fixed points A, B ; the portion abc is made to assume the form of two straight lines ab, bc , the points a and c being in the straight line joining A, B , and b at a small distance from it: when the chord is suddenly left to itself, what motion will take place, and what will happen when the motion reaches the fixed points?

22. Let τ be the mean interval between successive passages of the moon over the meridian of a place in lat. λ , and δ be her declination at a time t reckoned from the high-water of a known spring-tide; let τ', δ' be corresponding quantities for the sun: prove, on the supposition that each luminary causes vertical isochronous oscillations of the ocean, which vary slowly in extent, that the height of the tide at that time above its mean height, is nearly

$$M \cos^2(\lambda - \delta) \cos \frac{4\pi t}{\tau} + S \cos^2(\lambda - \delta') \cos \frac{4\pi t}{\tau'}.$$

23. Prove that a column of air in a cylindrical pipe of length l , closed at one end, may be made to vibrate so that at distances

$2\lambda, 4\lambda \dots 2k\lambda$ from the closed end, the air will be stationary, and at distances $\lambda, 3\lambda, 5\lambda, \dots (2k+1)\lambda$ the last of which $= l$, the density will be constant.

24. What properties of a medium in which the density varies as the pressure, correspond to and serve to explain the following observed properties of light :—Its rectilinear and uniform transmission; the different intensity of different rays; the difference of intensity of the same ray at different distances from its origin; the difference which the eye distinguishes in rays by colour; their crossing in all possible ways without mutual disturbance; the interference of two rays, the paths of which nearly coincide in direction and differ in length by a multiple of a certain interval?

25. If any number of rays be incident parallel to the axis on the surface of an elliptic paraboloid, the reflected rays will all pass through each of two parabolas lying in the principal planes of the paraboloid.

SATURDAY MORNING 9 o'clock to 11.

First, Second, Third, and Fourth Classes.

1. Triangles and parallelograms of the same altitude are to one another as their bases.

2. Planes to which the same straight line is perpendicular are parallel to one another.

3. Express $(\sin A)^{n+1}$ in terms of the sines of the multiple arcs.

4. Having given the sides of a spherical triangle, find the sine of one of its angles.

5. Construct a table of proportional parts and explain its use;
and if $\log 67833 = 4.8314410$,
and $\log 67832 = 4.8314346$,
find $\log 678328$.

6. Find the equation to the ellipse,

(1) referred to rectangular co-ordinates measured from the centre.

(2) polar focus.

7. Find the equation to the section of a right cone made by a plane, and determine the position of the plane when the section is a parabola.

8. Prove that in any equation the greatest negative coefficient increased by unity is greater than the greatest root. Find also a limit less than the least positive root.

9. Find the sum of the m^{th} powers of the roots of an equation of n dimensions in terms of the sums of inferior powers.

10. Determine in what cases the equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

belongs to each conic section.

11. Find the equation to a plane, having given the distances of its intersections with the three co-ordinate axes from the origin.

12. If there be (a) chances of an event happening, and (b) of its failing in one trial, find the probability of its happening (t) times at least in (n) trials.

13. Investigate the differential coefficients of

$$\sqrt{a^2 + x^2}, \log x, \text{ and } \sin x.$$

14. If u be a function of y , and y a function of x , then

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

15. Find the equation to a straight line touching a given curve at a given point; and apply it to draw a tangent to the ellipse at the extremity of the latus rectum.

16. Find $\int (a + bx)^3 \cdot x^2 dx$, $\int \frac{x^3 dx}{1 + x^2}$, $\int x^2 dx \sqrt{1 - x^2}$,

$$\int \frac{dx}{x\sqrt{ax^2 - b}}, \int d\theta \cdot \cos \theta \cdot \cos m\theta.$$

17. Find the differential of a surface of revolution, and apply it to find the portion of the surface of the sphere included between two given parallel planes.

18. Shew generally how to resolve a rational fraction into factors, and apply your method in the integration of

$$\frac{dx}{(x^3 - 4x^2)(x^2 + 1)}.$$

SATURDAY AFTERNOON, Half-past 12 to 3.

First, Second, Third, and Fourth Classes.

1. Find the relation of the power to the weight when there is equilibrium on the screw.

2. Having given the direction and velocity of projection of a body acted upon by gravity, find the equation of the trajectory.

3. A body is supported on a plane curve by forces X and Y acting in the directions of the rectangular axes of x and y ; prove that

$$Xdx + Ydy = 0.$$

4. Find the time of oscillation in a cycloidal arc.

5. If a body by the action of any centripetal force move in any manner, and another body ascend or descend in a straight line, and their velocities in any case of equal distance be equal, their velocities at all equal distances will be equal. NEWTON, *Prop.* XL.

6. Prove that the difference of the forces in the fixed and moveable orbits varies inversely as the cube of the distance. NEWTON, *Prop.* XLIV.

7. Required the part of the sun's disturbing force perpendicular to the plane of the moon's orbit.

8. A system of material points moveable about a horizontal axis, has all its parts acted on by gravity: it is required to determine the accelerative force, and to find a point of the system which shall be accelerated exactly as much as a single point in the same position.

9. Determine the pressure exercised by an incompressible fluid of uniform density against the surface of the vessel containing it.

10. Shew that if a material particle, moving in any manner in space, be solicited by the forces X, Y, Z , in the directions of three rectangular axes, and x, y, z , be the co-ordinates of its place at the time t ,

$$X = \frac{d^2x}{dt^2}, \quad Y = \frac{d^2y}{dt^2}, \quad Z = \frac{d^2z}{dt^2}.$$

11. Find the angle at which a small pencil of parallel rays must be incident on a sphere of water, that the rays after one reflexion within the sphere may emerge parallel in the plane of refraction.

12. The distance of the least circle of aberration from the approximate focus, is three-fourths of the longitudinal aberration of the extreme ray, and its diameter is half the lateral aberration.

13. Construct the astronomical telescope, and find its magnifying power.

14. When the vertical plane in which a transit instrument moves nearly coincides with the meridian, to find the deviation.

15. Investigate the two equations

$$nt = u - e \sin u,$$

$$\text{and } \tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2}.$$

the former expressing the relation between the eccentric and mean anomaly and the latter that between the eccentric and true anomaly.

16. What are the sidereal, the tropical and the anomalistic years? Which is longest and which shortest?

17. Determine the relative positions of the earth and an inferior planet, when the latter appears stationary.

EVENING PROBLEMS.

First, Second, Third, and Fourth Classes.

1. If S be the focus, and A the vertex of any conic section, and if LT the tangent at the extremity of the latus rectum L meet the axis in T ; shew that $\frac{AS}{AT} =$ the eccentricity.

2. From a station B at the base of a mountain its summit A is seen at an elevation of 60° ; after walking one mile towards the summit up a plane making 30° with the horizon, to another station C , the angle BCA is observed to be 135° . Find the height of the mountain in yards.

3. A mortgage is taken on an estate worth N acres of it; land rises n per cent. in price, and in consequence the mortgage is only worth N_1 acres, and it is then paid off. During the continuance of high prices another mortgage is taken, which is worth N acres as before; prices return to their former level, and the mortgage is worth N_2 acres;

$$\text{shew that } N - N_1 : N_2 - N = 1 : 1 + \frac{n}{100}.$$

4. The distance of a point P from the circumference of a circle : its distance from a fixed diameter $AB = n : 1$. Prove that the locus of P is a conic section.

5. An indefinite area is to be divided into similar and equal regular figures. Shew by what figures this can be done. Also if three equal areas be divided into the same number of equal regular figures which are respectively triangular, square, and hexagonal, shew that the sum of the lengths of the dividing lines in the cases of triangular, square, and hexagonal divisions, are to one another as $\sqrt{27} : \sqrt{16} : \sqrt{12}$; the whole area to be divided being very great in comparison of one of the divisions.

6. In a given equilateral parallelogram inscribe an ellipse of given eccentricity.

7. If A and a be the areas of horizontal sections of a waterfall at heights H and h above the horizon, find the height of the fall, the (velocity)² of the water at any point being as the height from which it has fallen.

8. In a given latitude a vertical rod is placed at a given distance from an east and west wall so as to cast a portion of its shadow upon it, find the equation to the extremity of the shadow traced upon the wall on a given day. Shew what the equation becomes when the sun is in the equator and the latitude of the place 45° .

9. Find the time of emptying a given prismatic vessel filled with water, by a cycloidal syphon of small bore placed with its base horizontal; the vertex of the syphon resting on the edge of the vessel.

10. A small pencil of diverging rays is incident obliquely on

a reflecting surface at its centre, and the section of the surface in the primary plane is a circle of given curvature; find the curvature of the section in the secondary plane, that the foci of rays reflected in the two planes may coincide.

11. If a uniform chain be suspended from two piers, the points of suspension being in the same horizontal line, shew that when the chain is nearly horizontal, the tension is nearly equal to

the weight of a length $\frac{s^2}{4(s-b)}$ of the same chain, where $s =$

length of the chain, and $b =$ the distance between the points of suspension. Shew also that such a length may be given to the chain as to render the tension at either pier a minimum, and investigate an equation for determining the minimum tension.

12. Find the quantities of matter of planets which have satellites. And these being given shew how to determine the quantities of matter of those which have not satellites.

13. The ages of a man and his wife are respectively $85 - m$ and $86 - n$, find the present worth of an annuity A to be paid to the wife after the death of her husband, supposing one male out of every 85 and one female out of every 86 to die annually; and n greater than m .

14. If AP be any curve referred to a pole S , and if s be the solid generated by the revolution of the area ASP about AS , $SP = r$, and the angle $ASP = \theta$,

$$\text{shew that } \frac{ds}{d\theta} = \frac{2}{3} \pi r^3 \cdot \sin \theta.$$

15. If two equal bodies which attract each other with forces varying as $\frac{1}{(\text{dist.})^2}$ are constrained to move in two straight lines

at right angles to one another, shew that they will arrive together at the point of intersection of the lines, from whatever points their motions commence. And having given their distance at the beginning of the motion, find the time to the point of intersection.

16. Shew that every differential equation of the n^{th} order has n first integrals.

$$\text{Integrate } \frac{dx}{x^2 \sqrt{2ax - x^2}}, \quad \frac{d\theta}{1 - e^2 \cdot \cos^2 \theta};$$

$$xy^2 dy + y^3 dx = a^3 dx,$$

$$\left. \begin{aligned} \frac{d^2 x}{dt^2} &= a - b \frac{dx^2}{dt^2} \\ \frac{d^2 y}{dt^2} &= b \left(c - \frac{dy}{dt} \right)^2 \end{aligned} \right\} \begin{aligned} &\text{eliminate } t \text{ supposing } x, y, t, \frac{dx}{dt}, \frac{dy}{dt} \text{ to} \\ &\text{vanish together.} \end{aligned}$$

$$\left(\frac{dz}{dx} \right)^2 + \left(\frac{dz}{dy} \right)^2 = a^2.$$

2 2

17. If a straight line PSP , revolve about a fixed point S ; and if perpendiculars PM, P_1M_1 be drawn upon a fixed straight line passing through S , find the equation to the curve in which $\frac{1}{PM^n} \div \frac{1}{P_1M_1^n} = \frac{2}{a^n}$.

18. Prove that $\frac{d^m u_x}{dx^m} = \{\log(1 + \Delta)\}^m \cdot u_x$, where any power of Δ as Δ^p , in the expansion of $\{\log(1 + \Delta)\}^m$, when u_x is joined to it, denotes the p^{th} difference of u_x .

And if $a_1, a_2, a_3, \dots, a_n$ be the coefficients of $\Delta^m u_x, \Delta^{m+1} u_x, \dots, \Delta^{m+n} u_x$ in the expansion of $\{\log(1 + \Delta)\}^m \cdot u_x$. Shew that a_n is found in terms of a_{n-1} &c. from the following equation

$$\begin{aligned} na_n &= \frac{1}{2} \{n - (m+1)\} a_{n-1} - \frac{1}{3} \{n - 2 \cdot (m+1)\} a_{n-2} \\ &\quad + \frac{1}{4} \cdot \{n - 3 \cdot (m+1)\} a_{n-3} - \&c. \\ &\quad \pm \frac{1}{n} \cdot \{n - (n-1) \cdot (m+1)\} a_1 \mp \frac{nm}{n+1}. \end{aligned}$$

19. Having given $u_0, u_\alpha, u_{2\alpha}$, three values of a function near its maximum, observed at times 0, $\alpha, 2\alpha$; find the time when the function will be a maximum. And if the declinations of the sun at noon on three successive days were $23^\circ.27', 23^\circ.27'.9, 23^\circ.27'.6$, find when the declination was greatest.

20. If the earth be an oblate spheroid, and from any point Q above it perpendiculars QM, QN be drawn to the axis and equator respectively, intersecting a meridian in P and p ; and if tangents PT, pt to this meridian meet the axis and equator in T and t , and if the straight line which joins T and t cuts the meridian in E and F , E and F are the extreme points of the meridian visible from Q .

21. A given hemisphere rests with its base upon a horizontal plane, and a given uniform rod, one end of which is moveable about a horizontal axis fixed in the plane, is placed against the hemisphere so as to be a tangent to a great circle of it; and the rod by its pressure puts the hemisphere in motion; find the equation for determining the motion of the rod when the friction of the plane varies as the pressure of the hemisphere upon it. And when there is no friction, find the angular velocity of the rod when it comes to the plane.

22. If normals be drawn at every point of the rhumb line, find the locus of their intersection with the equator, the earth being considered an oblate spheroid.

23. If x, y, z, r , and x', y', z', r' be the co-ordinates and distances of two planets m and m' from the centre of the sun supposed at rest, and if λ = the distance of m from m_1 ,

$$\text{and } Q = \frac{xx' + yy' + zz'}{r^3} - \frac{1}{\lambda},$$

prove that the axis-major of the ellipse of curvature at the point (x, y, z) of m 's orbit is

$$= \frac{1 + m}{2m_1 \int \left(\frac{dQ}{dx} dx + \frac{dQ}{dy} dy + \frac{dQ}{dz} dz \right)},$$

where 1 = mass of the sun.

How does it appear that the mean motions of the planets are subject to no *secular* variations?

MORNING PROBLEMS.

MONDAY MORNING, 9 o'clock to 11.

1. If n be any whole number, one of the three n^3 , $n^3 + 1$, $n^3 + 4$, is divisible by 5 without remainder.

2. A debt of $a\mathcal{L}$. accumulating at compound interest, is discharged in n years by annual payments of $\frac{a}{n}\mathcal{L}$.; prove that

$$n = - \frac{\log(1 - nr)}{\log(1 + r)}, \quad r \text{ being the interest of } 1\mathcal{L} \text{ for one year.}$$

3. If α be less than 45° , shew that

$$2 \sin \left(\frac{\pi}{4} \pm \alpha \right) = \sqrt{1 + \cos 2\alpha} \pm \sqrt{1 - \cos 2\alpha}.$$

$$\text{and } 2 \cos \left(\frac{\pi}{4} \pm \alpha \right) = \sqrt{1 + \cos 2\alpha} \mp \sqrt{1 - \cos 2\alpha}.$$

4. Prove that the locus of the points of bisection of any number of chords to an ellipse, which pass through the same point, is an ellipse; and find the magnitude and position of the axes, when the co-ordinates to the point are given.

5. Shew that three conditions must be satisfied by the constant coefficients in the general equation to lines of the second order, that it may be the equation to two straight lines.

6. If P, Q , be any two points in a curve referred to a pole S , and PM, QN perpendiculars on a fixed straight line TMN , find the equation of the curve, when the area SPQ is to the area $PQMN$ as 1 to n , and construct the curve when $n = 1$.

7. Trace the curve of which $a^m y = x(x - a)^m$ is the equation, m being an even number; find its maximum and minimum ordinates, point of contrary flexure, and the angle at which it cuts the axis of x .

8. Required the equation of the plane in which two given straight lines lie, which intersect each other in space.

9. Three uniform beams AB, BC, CD , of the same thickness, and of lengths $l, 2l, l$ respectively, are connected by hinges at B and C , and rest on a perfectly smooth sphere, the radius of which $= 2l$, so that the middle point of BC , and the extremities

A, D, are in contact with the sphere: shew that the pressure at the middle point of $BC = \frac{91}{100}$ of the weight of the beams.

10. Having given the latitudes of two places on the earth's surface, one of which is N.E. of the other, find the difference of their longitudes and their distance from each other, considering the earth a sphere.

11. If a body be attracted by a force which varies as $\frac{1}{(\text{dist.})^n}$, find the value of n when the velocity acquired from an infinite distance to a distance r from the centre, is equal to the velocity that would be acquired from r to $\frac{r}{4}$.

12. Water retained in a constant elevation, issues from one vessel into another through a cylindrical pipe of radius r , and thence into the air through another cylindrical pipe of radius r' ; having given the depth H of the mouth of the latter below the constant surface, find the velocity and pressure at a given depth h in the other pipe.

13. A reflector is formed of an indefinite number of faces, which are tangents to a parabola, and rays diverge from a given point in the axis of the parabola; find the locus of the images produced by reflection.

14. Three given quantities, $(a+z)$, $(a+z)+h$, $(a+z)+h'$, approximations to the root a of an equation, being substituted for the unknown quantity, give results n , $n+\delta$, $n+\delta'$; shew that z will be very nearly found from the equation,

$$z^2(h\delta' - h'\delta) + z(h^2\delta' - h'^2\delta) + nhh'(h' - h) = 0.$$

15. If a plumb line be drawn from the vertical by a small quantity of matter, at a small elevation m above the earth's surface, shew that the deviation will be a maximum, when

$$\sin \theta = \frac{m}{r\sqrt{2}}; \text{ where } \theta \text{ is the angular distance at the earth's}$$

centre of the plumb line and attracting point, and r the radius of the earth supposed spherical.

16. A body, drawn towards the origin of rectangular co-ordinates by a force mr , and from the plane yz by a force xs , is projected perpendicularly to the plane xz , from a point in it distant by D from each of the axes of x and z , with the velocity $\sqrt{m} \cdot D$; required the equation of the orbit, and the positions of the apses.

17. If the particles of a fluid mass, revolving with a given angular velocity, be attracted to a fixed centre by a force which is any function of the distance from the centre, the ellipticity, supposed small, will be half the ratio of the centrifugal force to the attraction at the equator.

18. Demonstrate the laws of reflexion and refraction on the undulatory theory of light.

MONDAY AFTERNOON, 1 o'clock to 3.

First and Second Classes.

1. Sum the following series :

(1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \dots \dots ad\ infinitum.$

(2) $1 + 5 + 21 + 85 + \dots \dots \dots$ to n terms.

2. Explain fully the mode of solving the equation

$$\frac{d^n y}{dx^n} + A \frac{d^{n-1} y}{dx^{n-1}} + \dots \dots + My = 0$$

where the coefficients are all constant, and where the equation

$$u^n + Au^{n-1} + \dots \dots + M = 0$$

contains p equal roots.

3. Find the variation of $\int V dx$, where V is a function of x, y, p, q , &c. and the conditions requisite for a maximum, or minimum value.

Apply the two resulting equations to find the shortest line that can be drawn from one given curve to another given curve.

4. Define points of simple and double reflexion in a curve of double curvature, and shew how to determine them.

5. Find the horary motion of the moon's nodes in a circular orbit. (NEWTON, Vol. III. *Prop.* xxx.)

6. A body being acted upon by any forces, prove that the motion of the centre of gravity will be the same as if all those forces acted at that centre, and that the motion of rotation will be affected as if the centre of gravity were fixed, and the same forces applied.

7. Having given the position of the sun and moon supposed to be in the equator, find the interval between high water at any place and the passage of the moon over the meridian.

8. Express the radius of curvature of the meridian in a spheroid of small ellipticity in terms of the latitude.

If λ, λ' , be the latitudes of two stations on the same meridian, prove that the length of the arc included between them is

$$b(\lambda' - \lambda) \left\{ 1 + \frac{e}{2} - \frac{3e}{2} \cos(\lambda' + \lambda) \cdot \frac{\sin(\lambda' - \lambda)}{(\lambda' - \lambda)} \right\} \text{ where } e = \frac{a-b}{a}.$$

9. Investigate the change in the inclination of the ecliptic produced by the sun's action on the earth.

MONDAY AFTERNOON, 1 o'clock to 3.

Third and Fourth Classes.

1. Find the present value of an annuity, to commence at the expiration of p years, and to continue q years; compound interest.

2. A recurring equation of an even number of dimensions

may be solved by means of an equation of half the number of dimensions; prove this, and solve the equation

$$x^4 + 4x^3 - 5x^2 + 4x + 1 = 0.$$

3. Prove Maclaurin's Theorem, and apply it to expand x as far as x^3 when $u^3 - 6ux - 8 = 0$.

4. Find generally an expression for the radius of curvature of a spiral curve, and apply it to determine the radius of curvature of the reciprocal spiral, the equation to which is $r = \frac{m}{\theta}$.

5. Integrate $\frac{x^2 dx}{(1+x^2)^3}$, $\int e^{mx} x^3 dx$,

$$\frac{dy}{dx} = x + y, \quad \text{and} \quad \frac{dy}{dx} = 1 + \frac{y^2}{x^2}.$$

6. The moment of inertia of any system with respect to any given axis, is equal to the moment about an axis parallel to this passing through the centre of gravity, together with the moment of the whole body collected in its centre of gravity about the given axis.

7. If the revolving orbit (NEWTON, *Sect. 9.*) be an ellipse, force in the focus and $= \frac{F^2}{A^2}$ at distance A , prove that the

$$\text{force on the body in fixed space} = \frac{F^4}{A^4} + \frac{RG^2 - RF^2}{A^3}.$$

If the orbit be nearly circular, shew that this expression approximates to $G^2 A \frac{F^2}{G^3 - 3}$ as its limit.

8. In what position of the node and moon is the inclination of the lunar orbit to the ecliptic respectively a maximum and minimum? (NEWTON, *Prop. LXVI. Cor. 10.*)

9. Investigate the precession in N.P.D. and R.A., and shew when they are additive and when subtractive.

TUESDAY MORNING, 9 o'clock to 11.

First Class.

1. Find the equation to a curve of the parabolic kind that will pass through four given points.

2. Investigate Lagrange's Theorem.

3. If each of the angles of a spherical triangle, the sides of which are very small compared to the radius of the sphere, be diminished by a third part of the spherical excess, the angles so diminished may be taken as the angles of a plane triangle, the sides of which are equal in length to those of the spherical triangle.

4. Integrate

$$\frac{dx}{\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}} + \frac{dy}{\sqrt{A+By+Cy^2+Dy^3+Ey^4}} = 0.$$

5. The mean horary variation of the inclination of the lunar orbit to the plane of the ecliptic, for a given position of the line of nodes, varies nearly as the sine of twice the angular distance of the nodes from syzygy. (NEWTON, *Book III. Prop. xxxiv. and Cors. 1, 2, 3.*)

6. Find the proportion of the axes of a homogeneous revolving fluid spheroid; shew that there are two forms of equilibrium, and find the eccentricity in each case when the centrifugal force is small.

7. If several angular velocities be impressed on a body at the same time, the resulting axis of rotation, and the angular velocity about it, will be found, by finding the direction and magnitude of the resultant of forces in the directions of the several axes of rotation and proportional to the angular velocities.

8. Prove that the vibrations propagated from any point of disturbance in a cylindrical column of air, are such that the velocity of the particles is proportional to the condensation, and the condensed particles move in the direction of propagation, the rarefied in the contrary direction.

9. Find the equation to a surface which envelopes a series of surfaces described after a given law.

TUESDAY MORNING, 9 o'clock to 11.

Second and Third Classes.

1. Transform the equation to a plane curve, from one system of co-ordinates to another inclined at a given angle to the former, and having a different origin, both systems being rectangular; and take the example $y^2 = mx + nx^2$.

2. The equations to a straight line and plane being given, find the conditions that they may be at right angles to one another.

3. Expand $f(x+h, y+k)$, and if $u = f(x, y)$, shew that each term involving differential coefficients of the p^{th} order will be of the form

$$\frac{1}{1.2.3..p} \cdot \frac{d^p n}{dx^p dy} \cdot (n+1)^{\text{th}} \text{ term of the expansion of } (h+k)^p$$

4. Find a multiplier which will render the homogeneous equation $Mdx + Ndy = 0$ an exact differential.

Integrate the equation $\frac{d^2 u}{d\theta^2} + u = k \cos \theta$.

5. Find the variation of the moon. (NEWTON, *Book III. Prop. 29.*)

6. A small pencil of rays being refracted obliquely at a convex spherical surface, determine the foci in the primary and secondary planes.

7. Find the general equation of the equilibrium of incompressible fluids. If the fluid be heterogeneous, shew that all surfaces of equal pressure are of uniform density.

8. If any number of bodies be acted upon by their mutual attractions, their centre of gravity will either be at rest, or move uniformly in a straight line.

9. Explain the method of finding the longitude of a place by the observed distance of the moon from the sun or a fixed star, and deduce formulæ adapted to logarithmic computation.

10. Find the attraction of a spherical shell on a particle without it, the law of attraction being $\frac{1}{(\text{dist.})^2}$.

TUESDAY AFTERNOON, 1 o'clock to 3.

First Class.

1. Explain the connection between the Differential Calculus, as depending on the Theory of Limits, and the Calculus of Differences; and deduce Taylor's Theorem from the equation

$$u_{x+n} = u_x + \frac{n}{1} \Delta u_x + \frac{n \cdot (n-1)}{1 \cdot 2} \Delta^2 u_x + \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.$$

2. Having given

$$nt = \theta - 2\epsilon \sin \theta + \frac{3\epsilon^2}{4} \sin 2\theta - \frac{\epsilon^3}{3} \sin 3\theta,$$

apply Lagrange's Theorem to prove that

$$\theta = nt + \left(2\epsilon - \frac{\epsilon}{4}\right) \sin nt + \frac{5\epsilon^2}{4} \sin 2nt + \frac{13\epsilon^3}{12} \sin 3nt,$$

neglecting higher powers of ϵ than the cube.

3. Having given a value which satisfies a differential equation of the first order, find whether it is included in the complete integral.

Prove that either of the two first integrals of a differential equation of the second order will give the same particular solution.

4. A body which is symmetrical with respect to a vertical plane passing through the centre of gravity revolves about a horizontal axis, the body being acted on by gravity only: find the pressure on the axis.

5. The centre of gravity of the earth and moon describes about the sun, very nearly, an ellipse in one plane, and the

area passed over by its radius vector is very nearly proportional to the time.

6. Determine the aberration of a pencil of parallel rays refracted directly through a double convex lens; and if the index of refraction be $\frac{3}{2}$, find the ratio of the radii when the aberration is a minimum.

7. State and prove the principle of conservation of vis viva; and when the vis viva is a maximum shew that the body passes through a position of stable equilibrium, and when a minimum through a position of unstable equilibrium.

8. Let any particle of a mass of incompressible fluid, the co-ordinates of which at any time t are x, y, z , be acted upon by the forces X, Y, Z , in the directions of the axes of co-ordinates; and let u, v, w , be its velocities, in the same directions; then if $u dx + v dy + w dz = d\phi$, a complete differential, the pressure and motion are determined by the equations

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

$$p = \int (X dx + Y dy + Z dz) - \frac{d\phi}{dt} - \frac{1}{2} \left(\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2} \right);$$

TUESDAY AFTERNOON, 1 o'clock to 3.

Second and Third Classes.

1. Trace the curve whose equation is $ay^2 = x^3 - bx^2$.
2. Determine the five first terms of Σx^n .
3. Let x and y be functions of a third variable t : it is required to determine what substitution must be made for

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3},$$

in an expression in which x is the independent variable, to change it to one in which t is the independent variable.

4. Determine the relation between the focal length of two lenses which shall achromatize each other when separated by a given interval.

5. A cylindrical body unrolls itself from a vertical string, the other end of which passes over a fixed pulley and supports a weight: it is required to determine the motion.

6. The resistance of a medium being $Q \cdot v^2$, and the force tending to a fixed centre being P , the differential equation of the trajectory is

$$\frac{d^2u}{dv^2} + u - \frac{Pc^{2/3}Qds}{h^2u^2} = 0.$$

7. Shew how the time, duration, and magnitude of a lunar eclipse may be computed.

8. Find the mean annual motion of the moon's nodes. **NEWTON. Book III. Prop. 32.**

9. Having given the three edges of a parallelipedon, and the angles they make with each other, find its solidity.

TUESDAY MORNING, 9 o'clock to 11.

Fourth Class.

1. If a solid angle be contained by three plane angles, any two of them are greater than the third.

2. If λ be the angle which the normal to any point in an ellipse makes with the axis major, the length of the normal

$$= \frac{b^2}{a(1 - e^2 \sin^2 \lambda)^{\frac{3}{2}}}.$$

3. Find the polar equation to the rectangular hyperbola, the centre of the hyperbola being the pole, and thence determine the area of the sector.

4. Integrate $\frac{dx}{\sqrt{a + bx + cx^2}}, \frac{dx}{\cos x},$
and $dy + Pydx = Qdx.$

5. Graduate the common steel-yard.

6. If the force of gravity be considered constant, and altitudes from the earth's surface be taken in arithmetical progression, the corresponding densities of the air will decrease in geometrical progression.

7. A ray of light, which passes through a prism in a plane perpendicular to the axis, is turned towards the thicker part of the prism, if it be denser than the surrounding medium.

8. In what positions of the apse of the lunar orbit is the eccentricity respectively a maximum and minimum? (**NEWTON, Prop. LXVI. Cor. 9.**)

9. Distinguish between a sidereal, a solar, and a mean solar day; and define the equation of time, stating the two causes from which it arises.

TUESDAY AFTERNOON, 1 o'clock to 3.

Fourth Class.

1. If a straight line be drawn from a given point perpendicular to a given straight line, find the co-ordinates of the point of intersection, and the length of the perpendicular:

2. Give a general method of determining the value of a

fraction when the numerator and denominator vanish ; and apply it to find the value of

$$\frac{(x^2 + a^2)(x^2 - a^2)^{\frac{3}{2}}}{(3x^2 - a^2x)(x^2 - a^2)^{\frac{3}{2}}}, \text{ when } x = a.$$

3. Find the centre of gravity of a semi-parabola.
4. A second's pendulum is carried to the top of a mountain, and there loses 48".6 in a day : determine the height of the mountain, supposing the earth's radius to be 4000 miles.
5. If a luminous point be placed between two plane mirrors inclined to each other, prove that there will be a series of images which lie in the circumference of a circle : determine also the number of images.
6. Explain Bramah's Hydrostatic Press. What is the limit to its practical application ?
7. A body descends in a straight line from a given point towards a centre of force, the force varying as $\frac{1}{D^2}$: find the time of describing any space, and the time to the centre.
8. The mean disturbing force of the sun on the moon in the direction of the radius vector is ablative and equal to half the mean additive force, supposing the orbits to be in the same plane.
9. Construct a horizontal dial for a given latitude, and determine the limits beyond which it is unnecessary to graduate it.

EXAMINATION PAPER FOR SMITH'S PRIZES.

PROFESSOR AIRY.

1. Give Huygen's construction for determining the course of the ordinary and extraordinary ray produced by the double refraction of Iceland spar.

Shew that the extraordinary ray is not generally in the plane of incidence.

Shew that in one case the tangents of the two angles of refraction are in a constant ratio : and that in another case the sines of the two angles are in a constant ratio.

Describe the construction of the doubly-refracting prism : and mention some of its uses.

2. Find the pressure of a revolving body on its axis of rotation, supposed to pass through its centre of gravity : (the effect of gravity being neglected.)

Define the *principal axes*, and find the pressure when the axis of rotation is a principal axis.

A rigid solid of revolution moves freely in space, and is not acted on by any external force; investigate the circumstances of its motion.

3. Describe the different methods of ascertaining the difference of longitude of two places, their latitudes being known. And explain the calculations which are necessary when the occultation of a fixed star by the moon has been observed at both places.

4. A wave of air diverges from a centre symmetrically on all sides: find the expression for the motion of a particle of air in any place: and deduce the value of the intensity and velocity of sound.

Calculate the tone of an organ-pipe of given length, both ends of which are stopped,

5. Find the size of the stone which must be put under the wheel of a coach of given dimensions in order to prevent it from running down a hill of given inclination.

6. Explain the terms *astigmatism* and *focal lines*.

The luminous point is 1 foot above the surface of stagnant water: a conical pencil of rays, whose radius at the distance 1 foot from the luminous point is $\frac{1}{16}$ inch, meets the surface of the water 10 feet from the foot of the perpendicular let fall from the luminous point: find the position of the focal lines: and give the form of sections of the pencil perpendicular to its axis at the distances 1, 2, 10, and 20 feet from the point of refraction.

Investigate the position of the focal lines when the surface is curved.

7. The A. R. of the moon's centre is

$64^{\circ} 51' 18'', 77$ (in space :) its N. P. D. $73^{\circ} 28' 56'', 65$:
the A. R. of a fixed star is

$65^{\circ} 7' 14'', 48$: its N. P. D. $73^{\circ} 24' 45'', 20$:

Calculate numerically the distance between the moon's centre and star.

Investigate different formulæ for the case above: and shew why that which you have used is preferable to the others.

8. Describe the phenomena of Newton's coloured rings: give Newton's explanation, and also the explanation on the theory of undulations. Explain why the rings, which at first are only black and white, become vividly coloured, and at last, after many alternations of pink and green, disappear entirely.

9. State whether, and how far, you consider the foundation of Statics to be experimental. And if it appears at all experimental, state the difference between the nature of these experiments and the nature of those on which Dynamics is founded.

Investigate the power of the toothed wheel.

10. Integrate the equation

$$\frac{d^2z}{dx^2} - \frac{2y}{x} \cdot \frac{d^2z}{dx \cdot dy} + \frac{y^2}{x^2} \cdot \frac{d^2z}{dy^2} = \phi(x, y) = 1 + \frac{y^2}{x^2}.$$

11. Give the theory of the achromatic object-glass: and shew the effect of a separation of the lenses on the focal length and on the chromatic correction.

Give the theory of the achromatic eye-piece of two lenses.

The focal length of the object-glass is 10 feet: that of the eye-glasses 3 inches and 1 inch: their distance 2 inches. Find the distance between the object-glass and the first eye-glass: and calculate the power of the telescope.

12. Shew that a mountain near the earth's pole would diminish precession: that a chain of mountains on the equator would increase it: and that a chain of mountains in latitude 45° would make no alteration.

Distinguish between *luni-solar precession* and *general precession*: shew the difference of effects which they produce on the A. R. and N. P. D. of stars: and mention the observations proper for determining the magnitude of each.

13. Explain Fresnel's method of calculating the intensity of light near the shadow of a body. Shew that the expressions are very much simplified if a lens be applied to the body whose focal length is such that rays diverging from the luminous point are made to converge exactly on the screen.

The body is a plate of metal with a square hole $\frac{1}{10}$ inch in diameter: the distance of the luminous point and that of the screen are each 10 feet: a lens of 5 feet focal length covers the hole: find the intensity of the light on various points on the screen

14. Investigate Lagrange's theorem, and apply it to this example:

$$pt = \theta + A \sin \overline{c\theta - \alpha} + B \sin \overline{2c\theta - 2\alpha} + C \cdot \sin \overline{2g\theta - 2\gamma} \\ + D \cdot \sin \overline{(2 - 2m)\theta + 2\beta} + E \sin \overline{(2 - 2m - c)\theta + 2\beta + \alpha} \\ + F \sin \overline{m\theta - \beta - \zeta},$$

where A is a small quantity of the first order, and B, C, D, E, F , are of the second order: to find θ in terms of t to the second order.

15. The equation to a curved surface is

$$(a^2x^2 + b^2y^2 + c^2z^2)(x^2 + y^2 + z^2) - a^2(b^2 + c^2)x^2 - b^2(a^2 + c^2)y^2 \\ - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0:$$

discuss this equation, and find the nature of the surface.

Examine the case where $c = b$.

16. If f, g, h , be the co-ordinates of an attracted point in the directions of x, y, z , and $c(1 + e), c(1 - e)$, and c the axes

of a spheroid in the same directions, and the point be without the spheroid, the attractions are

$$\begin{aligned} \text{in } f, & \frac{4\pi}{3} \epsilon \cdot \frac{c^3}{(f^2 + g^2 + h^2)^{\frac{3}{2}}} \cdot \left\{ 1 + 2\epsilon - \epsilon \cdot \frac{c^2(12h^2 - 3f^2 - 3g^2)}{5(f^2 + g^2 + h^2)^2} \right\} \cdot f, \\ \text{in } g, & \frac{4\pi}{3} \epsilon \cdot \frac{c^3}{(f^2 + g^2 + h^2)^{\frac{3}{2}}} \cdot \left\{ 1 + 2\epsilon - \epsilon \cdot \frac{c^2(12h^2 - 3f^2 - 3g^2)}{5(f^2 + g^2 + h^2)^2} \right\} \cdot g, \\ \text{in } h, & \frac{4\pi}{3} \epsilon \cdot \frac{c^3}{(f^2 + g^2 + h^2)^{\frac{3}{2}}} \cdot \left\{ 1 + 2\epsilon - \epsilon \cdot \frac{c^2(6h^2 - 9f^2 - 9g^2)}{5(f^2 + g^2 + h^2)^2} \right\} \cdot h. \end{aligned}$$

If the point be within the spheroid, the attractions are

$$\begin{aligned} \frac{4\pi}{3} \rho \cdot f \cdot \left(1 - \frac{2}{5} \epsilon\right), \quad \frac{4\pi}{3} \rho \cdot g \cdot \left(1 - \frac{2}{5} \epsilon\right), \\ \text{and } \frac{4\pi}{3} \rho \cdot h \cdot \left(1 + \frac{4}{5} \epsilon\right). \end{aligned}$$

Assume these expressions, and deduce Clairaut's theorem.

1830.	Moon's A. R.	Moon's decl.
17. Apr. 20.. 0 ^h	354°.59'.30".3:	2°.43'.21".7..S.
.... 20.. 12 ^h	2. 5.53.3:	0.22. 9.3..S.
.. 21.. 0 ^h	9.15.42.0:	2. 0.38.2..N.
.... 21.. 12 ^h	16.29.23.0:	4.22.26.3..N.
.... 22.. 0 ^h	23.47.12.0:	6.40.32.3..N.

Find when the moon was in the equator: and her A.R. at that time.

18. Investigate the equation to a *couche de niveau*, and the pressure at any point.

If the elastic force of air be proportional to its density, shew that the extent of the atmosphere must be infinite: and investigate its density at the surface of Jupiter.

Investigate the limits of the atmosphere on other suppositions.

19. Explain analytically the *variation of parameters*.

Explain its geometrical meaning, and apply it to the following instances: To find the motion of a pendulum disturbed by a small force: To find the alteration in the elements of a planet's orbit produced by a resisting medium.

20. Describe the terms in Physical Astronomy which are much increased by integration.

The moon's parallax being expressed (nearly) by

$$\begin{aligned} P \left\{ 1 + \epsilon \cos c\theta - \alpha + m^2 \cos (2 - 2m)\theta + 2\beta \right. \\ \left. + \frac{15}{8} m\epsilon \cdot \cos (2 - 2m - c)\theta + 2\beta + \alpha \right\}, \end{aligned}$$

explain the meaning of each term.

Explain particularly the effect of the last term.

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